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ON CONJUGATION COBORDISM

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An almost-complex manifold supports an involution if there is a differentiable self-map on the manifold of period two. The differential of the map acts on the coset space of the almost-complex structures on M by inner automorphism. This action is also of period two. If the almost-complex structure is sent to its conjugate, the manifold with structure, together with the given involution is called a conjugation. Any linear involution of Euclidean space may be used to stabilize this situation, giving a cobordism theory of exotic conjugations. The question considered here is: What is the image in complex cobordism of the functor which forgets equivariance. The result shown in the next section is: If a stably almost-complex manifold supports an exotic conjugation, every characteristic number is even.

The first cobordism results on conjugations are due to Conner and Floyd [3] (§ 24). In [4], Landweber established the equivariant analogues of the Thom theorems. Certain examples have been considered by Landweber, [5] (§ 3), and together with the result here the image of the forgetful functor can be seen to be maximal, in some cases.

2. *Proof of the theorem.* It is well-known from the work of Thom and Milnor that the unoriented bordism ring $\mathscr{N}_*$, with spectrum $\boldsymbol{MO}$, is a polynomial ring over $\boldsymbol{Z}_2$ on manifold classes n_t, $t+1$ any positive integer not a power of two (t nondyadic). Also $\mathscr{U}_*$, the complex bordism ring with spectrum $\boldsymbol{MU}$, is a polynomial ring over $\boldsymbol{Z}$ on manifold classes u_t, $t = 0, 1, \cdots$. Representatives for the dyadic generators u_t, $t+1 = 2^j$, may be chosen so that every normal characteristic number is even. The principal ideal in $\mathscr{U}_*$ generated by dyadic generators is the graded Milnor ideal associated to 2, I. $I_{2k} = I \cap \mathscr{U}_{2k}$.

If a partition of k contains a dyadic integer the partition will be called dyadic. Let $d(k)$ denote the dyadic partitions of k, $n(k)$ the nondyadic partitions of k. If $\alpha = a_1 a_2 \cdots a_r$ is a partition of k then the group generator $u_{a_1} \cdots u_{a_r} \in \mathscr{U}_{2k}$ will be denoted u_α. Similarly for $n_\alpha \in \mathscr{N}_k$.

If $MU(n)$ is given the involution defined in [4] then it is a G-complex, $G = \boldsymbol{Z}_2$, in the sense of Bredon. Note that $\tilde{\omega}_0(MU(n)) = \tilde{\omega}_1(MU(n)) = 0$. The construction given in the next section produces, for each partition of k, α, and sufficiently large n, an equivariant inclusion and a G-complex $e^\alpha\colon MU(n) \to Y^\alpha$ such that

(c i) $\tilde{\omega}_{n+k}(Y^\alpha) = \begin{cases}(Z_2 \to 0) & \text{if } \alpha \in n(k)\\ 0 & \text{if } \alpha \in d(k)\end{cases}$

(c ii) $\tilde{\omega}_{2n+2k}(Y^\alpha) = (0 \to \{Z, (-1)^{n+k}\})$

(c iii) $\omega_t(Y^\alpha) = 0$ if $t \neq n + k,\ 2n + 2k$

(c iv) $e^\alpha\left(\frac{G}{e}\right)_\#\colon \tilde{\omega}_{2n+2k}(MU(n))\left(\frac{G}{e}\right) \cong \mathscr{U}_{2k} \to \tilde{\omega}_{2n+2k}(Y^\alpha)\left(\frac{G}{e}\right) \cong Z$ maps u_α to an odd multiple of the generator $\alpha \in n(k)$.

Let the $r + s$ sphere with the orthogonal involution fixing an equatorial s-sphere be denoted $S^{r,s}$. The G-complex formed by attaching the cone over $S^{0,s}$ in $S^{r,s}$ will be denoted $S^{r,s}/S^{0,s}$. Let the equivariant homotopy groups

$$\left|\left[\frac{S^{n+a,n+b}}{S^{0,n+b}}, MU(n)\right]\right| \quad \text{and} \quad \left|\left[\frac{S^{n+a,n+b}}{S^{0,n+b}}, Y^\alpha\right]\right|$$

be denoted $\lambda\mathscr{U}_{a,b}$ and $\lambda Y_{a,b}$ respectively. It is understood that $a + b$ is much less than n whenever this is used.

It is easy to see, from the cochain complex, [1] I § 6, of $S^{r,s}/S^{0,s}$ that if $\tilde{\omega}$ is any generic coefficient system with a G-action g on $\tilde{\omega}\left(\frac{G}{e}\right)$ then

$$H^k_G\left(\frac{S^{r,s}}{S^{0,s}}; \tilde{\omega}\right) \cong \begin{cases} 0 \quad \text{if } 0 < k \leqq s \quad \text{or } r + s < k \\ \dfrac{\operatorname{Ker}(1 + (-1)^{k-s}g)}{\operatorname{Im}(1 + (-1)^{k-s-1}g)} \quad \text{if } s < k < r \\ \dfrac{\tilde{\omega}\left(\frac{G}{e}\right)}{\operatorname{Im}(1 + (-1)^{r+s}g)} \quad \text{if } k = r + s\,. \end{cases}$$

Note that the groups $\lambda Y_{a,b}$ are the same for all partitions α of k. I.e., by Bredon's classification theorem [1] II (2.11)

$$\lambda Y_{k+q,k-q} \cong \frac{Z}{(1 + (-1)^{q+1})Z}$$

$$\lambda Y_{k+q+t,k-q} \cong \begin{cases} 0 & q \ \text{even} \\ Z_2 & q \ \text{odd} \end{cases} \quad t \geqq 1$$

$$\lambda Y_{l,m} = 0 \quad l + m < 2k\,.$$

From this computation the main result may now be deduced. Let ψ denote the forgetful functor.

THEOREM. $u_\alpha \in \operatorname{Image}\{\psi\colon \lambda U_{k+q,k-q} \to \mathscr{U}_{2k}\}$ *only if* $\alpha \in d(k)$.

Proof. Suppose u_α is in the image of ψ. Consider the commutative diagram with exact row (see [3], p. 286 for definitions of α, β, and ψ):

$$\begin{array}{ccccc} & \lambda\mathscr{U}_{k+q,k-q} & \xrightarrow{\psi} & \mathscr{U}_{2k} & \\ & e^{\alpha}\left(\frac{G}{G}\right)_{\#}\Big\downarrow & & \Big\downarrow e^{\alpha}\left(\frac{G}{e}\right)_{\#} & \\ \cdots \longrightarrow \lambda Y_{k+q+1,k-q} \xrightarrow[\beta]{} & \lambda Y_{k+q,k-q} & \xrightarrow[\psi]{} & \pi_{2n+2k}(Y^{\alpha}) & \xrightarrow[\alpha]{} \lambda Y_{k+q+1,k-q-1} \end{array} \tag{2.1}$$
$$\xrightarrow[\beta]{} \lambda Y_{k+q,k-q-1} \cdots .$$

If q were odd, the lower ψ is zero. By (c iv) the upper ψ is zero and $u_\alpha = 0$, a contradiction. Now suppose q is even. The exact row then is $0 \to Z \to Z \to Z_2 \to 0$ so that $e^{\alpha}\left(\frac{G}{e}\right)_{\#}$ maps u_α to an even multiple of the generator and by (c iv), $\alpha \in d(k)$.

COROLLARY. Image $\psi \subseteqq I$.

Proof. By ([4], (4.1)), $2u_\alpha \in$ Image ψ for every α.

Then if $w \in$ Image ψ, subtract off even multiples of group generators until we have $w = 2w' + u_{\alpha_1} + u_{\alpha_2} + \cdots + u_{\alpha_l}$. Now construct diagram (2.1) for α successively equal to $\alpha_1, \cdots, \alpha_l$. This shows that $\alpha_1 \in d(k), \cdots, \alpha_l \in d(k)$, and the corollary is proved.

As a corollary of the construction in [5] § 3 there are free exotic conjugations on representatives u_t, $t = 2^j - 1$, showing that Image $\{\psi: \lambda\mathscr{U}_{t+q,t-q} \to \mathscr{U}_{2t}\}$ contains u_t provided q divisible by $2^{\phi(t+2)}$. Since the image of a forgetful functor is an ideal in $\mathscr{U}_*$ this shows:

COROLLARY. *Image* $\{\psi: \lambda U_{k+q,k-q} \to \mathscr{U}_{2k}\} = I_{2k}$ *if* $t = 2^j - 1 \leqq k < 2^{j+1} - 1$ *and* q *divisible by* $2^{\phi(t+2)}$. $\phi(m)$ *is the familiar number equal to the number of integers* s, $0 < s < m$ *with* $s \equiv 0, 1, 2, 4 \pmod 8$.

3. The construction. Recall Bredon's procedure for killing the homotopy groups of a G-space X, with $\tilde{\omega}_0(X, x_0) = \tilde{\omega}_1(X, x_0) = 0$. Let T be some G-set and $F(T)$ the free abelian G-module on T such that Hom $(F(T), \tilde{\omega}_r(X))$ contains an epimorphism A_r. By use of [2], Chapter II, (2.11), take a representative $a_r: S^r(T^+) \to X$ and define X_{r+1} by the equivariant Puppe sequence,

$$S^r(T^+) \xrightarrow{a_r} X \xrightarrow{j} X_{r+1} \longrightarrow S^{r+1}(T^+) \longrightarrow \cdots .$$

Bredon shows, [2], (6.6), that

$$j_\#: \tilde{\omega}_t(X) \longrightarrow \tilde{\omega}_t(X_{r+1}) \text{ is an isomorphism for } 0 \leqq t \leqq r-1 \text{ and } \tilde{\omega}_r(X_{r+1}) = 0 .$$

In this construction of Y^α there are at most two r where A_r is not taken to be an epimorphism. To begin, let α be a partition of $k \geqq 0$ and take $n > 2k - 1$ so that $\pi_{n+k}(MO(n)) = \tilde{\omega}_{n+k}(MU(n))\left(\frac{G}{G}\right) \cong \mathscr{N}_k$ and $\pi_{2n+2k}(MU(n)) = \tilde{\omega}_{2n+2k}(MU(n))\left(\frac{G}{e}\right) \cong \mathscr{U}_{2k}$. If α is dyadic let $n_\alpha \in \mathscr{N}_k$ denote the zero element. Regard n_α and u_α as elements of $\tilde{\omega}_*(MU(n))$.

Let $Y_0 = MU(n)$ and let all A_r be epimorphisms $0 < r < n + k$. Denote the composition of the inclusions by $E_r\colon MU(n) = Y_0 \subset \cdots \subset Y_r$. If α is dyadic, let A_r be epimorphisms $0 < r < 2n + 2k$; if not let A_{n+k} be defined as follows. Let T_{n+k} be the G-set of all elements in $\tilde{\omega}_{n+k}(Y_{n+k-1})\left(\frac{G}{G}\right)$ except $E_{n+k\#}(n_\alpha)$ and all elements in $\tilde{\omega}_{n+k}(Y_{n+k-1}) \times \left(\frac{G}{e}\right)$. Take A_{n+k} to be the natural homomorphism defined by extending the G-set inclusion $T_{n+k} \subseteqq \tilde{\omega}_{n+k}(Y_{n+k-1})$. Now let A_r, $n + k < r < 2n + 2k$, be epimorphisms. Let the free cyclic summand containing $E_{2n+2k-1\#}(u_\alpha)$ in $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})\left(\frac{G}{e}\right)$ be denoted F. Define T_{2n+2k} to be the G-set of elements in the union of the sets $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})\left(\frac{G}{G}\right)$ and $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})\left(\frac{G}{e}\right) - F$, and define A_{2n+2k} to be the natural induced homomorphism. To define Y_r, $2n + 2k < r$, let A_r be epimorphisms. This defines Y^α as a limit of G-complexes $MU(n) = Y_0 \subset Y_1 \subset \cdots$. Let $e^\alpha\colon MU(n) \to Y^\alpha$ be the inclusion.

It is clear that (c i) and (iii) are satisfied by this construction. To check the others some notation will be required. Let $g\colon S^{2n+2k} \to MU(n)$ be some representative for u_α, transverse regular on $BU(n) \subset MU(n)$ and let $M_\alpha = g^{-1}(BU(n))$. Let $v_n \in \tilde{H}^{2n}(MU(n); \boldsymbol{Z})$ denote the universal Thom class and $s_\alpha \in H^{2k}(BU(n); \boldsymbol{Z})$ the symmetric function associated to α in the universal Chern classes $c_1, c_2, \cdots$. Let $f\colon MU(n) \to K(\boldsymbol{Z}, 2n + 2k)$ represent $s_\alpha \cup v_n \in \tilde{H}^{2n+2k}(MU(n); \boldsymbol{Z})$. It is well-known that the degree defined by $f \circ g$ is the normal characteristic number of M_α, $s_\alpha(u_\alpha)$.

The G-action of conjugation sends c_1 to $-c_1$, so by the splitting principle c_n is sent to $(-1)^n_{c_n}$, v_n to $(-1)^n v_n$ and $s_\alpha \cup v_n$ to $(-1)^{n+k} s_\alpha \cup v_n$. However, this determines the G-action on homology which, through the Hurewicz isomorphism, gives the G-action on $\pi_{2n+2k}(MU(n))$. To check the remainder of (c ii) we attempt to extend the map f to a map $h\colon Y^\alpha \to K(\boldsymbol{Z}, 2n + 2k)$.

The preceding construction shows that an extension of f to $f''\colon Y_{2n+2k-1} \to K(\boldsymbol{Z}, 2n + 2k)$ exists for dimensional reasons. Thus there is an integer, $N \neq 0$, such that $N \cdot f''_\#(E_{2n+2k-1\#}(u_\alpha)) = f_\#(u_\alpha)$ in $\pi_{2n+2k}(K(\boldsymbol{Z}, 2n + 2k))$. Note thatth is justifies the preceding claim that $E_{2n+2k-1\#}(u_\alpha)$ lies in an infinite cyclic summand in $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(G/(e,$

F. Since $n + k$ may be taken odd, F has only one fixed point, 0. Thus, in the construction, Image A_{2n+2k} and F have only 0 in common. But $f''_\#$ lives on F, so an extension f': $Y_{2n+2k}K(\boldsymbol{Z}, 2n + 2k)$ exists. The desired extension, h, exists now by dimensional considerations and the following homotopy diagram commutes.

$$\begin{array}{ccc} \pi_{2n+2k}(S^{2n+2k}) & \xrightarrow{(e^\alpha \circ g)_\#} & \pi_{2n+2k}(Y^a) \\ {\scriptstyle g_\#}\downarrow & \nearrow {\scriptstyle e^\alpha_\#} & \downarrow {\scriptstyle h_\#} \\ \pi_{2n+2k}(MU(n)) & \xrightarrow[f_\#]{} & \pi_{2n+2k}(K(\boldsymbol{Z}, 2n + 2k)) \end{array}$$

Since $f_\#$ carries a generator to nonzero multiple of the generator, $s_\alpha(u_\alpha)\cdot g$, we see that $\pi_{2n+2k}(Y^\alpha)$ cannot be finite. By construction, it is cyclic on one generator and this completes the verification of (c ii).

From this diagram, note that $e^\alpha_\#$ carries u_α to some multiple of the generator, y, of $\pi_{2n+2k}(Y^\alpha)$, $e^\alpha_\#(u_\alpha) = My$. By commutativity, M divides $s_\alpha(u_\alpha)$. But if $\alpha \in n(k)$, $s_\alpha(u_\alpha)$ is odd; thus M is odd and (c iv) is verified.

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