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THE MODULE STRUCTURE OF EXT (F, T) OVER THE ENDOMORPHISM RING OF T

S. A. KHABBAZ AND E. H. TOUBASSI

§ 0. **Introduction.** Section 1 of the paper deals with the structure of the left End (T)-module Ext (F, T) where F is a torsion-free group and T a torsion group. The main theorem states that if the rank of F is n then the rank of Ext (F, T) over End (T) is at most n. It is also shown that Ext (F, T) is a torsion End (T)-module.

In § 2 the case when F is a torsion-free divisible group of finite rank is considered. It is shown that $\mathscr{F}_m$, the elements of the End (T)-End (F) bimodule Ext (F, T) whose splitting length is at most m, are a sub-bimodule. Moreover $\mathscr{F} = \bigcup_{i=1}^{\infty} \mathscr{F}_i$ as well as $\mathscr{F}_m$ are not finitely generated sub-bimodules. However, when T is the direct sum of cyclic groups, every finite number of elements of $\mathscr{F}_m$ can be embedded in a submodule generated by at most rank F elements belonging to $\mathscr{F}_m$.

We introduce some notation. The word "group" will always mean abelian group. Q will stand for the additive group of rational numbers, N the natural numbers, and N_0 the nonnegative integers. Let G be a group, then $T(G)$ is the maximal torsion subgroup of G. For $x \in G$ then $h_p(x)$ will denote its generalized p-height in G, $o(x)$ its order, and $\langle x \rangle$ the cyclic subgroup generated by x. For a torsion-free element $x \in G$, $U(x)$, the Ulm matrix of x, will refer to the matrix $h_p(p^i x)$ over all $i \in N_0$ and primes p [9]. For the module structure of Ext (F, T) over End (T)-End (F) we refer the reader to [8]. Following [8] for $\alpha \in$ End (T) [$\gamma \in$ End (F)] and $E \in$ Ext (F, T) by αE [$E\gamma$] we mean the extension arising from the pushout [pullback] diagram of E along α [γ].

§ 1. We begin by proving

THEOREM 1.1. *For torsion groups T and torsion-free groups F,* Ext (F, T) *is a torsion module over* End (T).

Proof. Without loss of generality we may assume T is an unbounded, reduced torsion group. Hence we may write $T = T_1 \oplus T_2$ where T_1 is a cyclic summand. Let $E \in$ Ext (F, T), $E: 0 \rightarrow T \rightarrow X \rightarrow F \rightarrow 0$. Define $\alpha \in$ End (T) by $\alpha|_{T_1} = 1$ and $\alpha|_{T_2} = 0$. Consider the middle group of αE, namely, $(T \oplus X)/N$ where $N = \{(\alpha(t), -t) \mid t \in T\}$. Observe that the torsion subgroup of $(X + N)/N$ is bounded by the order of T_1. Thus $(X + N)/N$ splits into the direct sum of a torsion and a torsion-free group and the same holds for $(T \oplus X)/N$. Hence αE is the trivial extension.

We now define our concept of rank.

DEFINITION. Let M be a left R-module. We shall say that M has rank at most n if and only if every finitely generated submodule of M is contained in one generated by n elements. It has rank n if in addition n is the least integer for which this holds.

We remark that our definition of rank is the one used by Prüfer in his fundamental work [10] on torsion groups. This definition is given again in [5] p. 49. It is equivalent to the usual definition for torsion-free and p-groups, and to that of reduced rank as given in [1] p. 34. We now list two properties of this notion of rank:

1. The rank of a homomorphic image of M does not exceed that of M.

2. If M is a finitely generated Z-module, where Z is the ring of integers, then the rank of M coincides with the number of elements in a canonical basis.

Note that in verifying that the rank of M is at most n it suffices to check if every $n+1$ elements of M can be embedded in a submodule generated by n elements.

We now state a special case of our main theorem.

THEOREM 1.2. *Let T be a torsion group which is the direct sum of cyclic groups. Then the rank of* Ext (Q, T) *over* End (T) *is at most one.*

Since the proof of the theorem is rather lengthy we defer it to later. Assuming for the moment that it holds we show how to extend it to arbitrary torsion groups.

THEOREM 1.3. *For an arbitrary torsion group T, the rank of* Ext (Q, T) *over* End (T) *is at most one.*

Proof. Without loss of generality we may assume T is an unbounded reduced torsion group. By the observation following our definition of rank it suffices to show that for $E_1, E_2 \in \operatorname{Ext}(Q, T)$, E_i: $0 \to T \to X_i \to Q \to 0$, there exists an $E \in \operatorname{Ext}(Q, T)$ and $\alpha_i \in \operatorname{End}(T)$ such that $\alpha_i E = E_i$, $i = 1, 2$.

Consider the exact sequence $0 \to B \xrightarrow{\rho} T \to T/B \to 0$ with B basic in T. From that we get the exact sequence $\operatorname{Ext}(Q, B) \xrightarrow{\rho_*} \operatorname{Ext}(Q, T) \to \operatorname{Ext}(Q, T/B) = 0$ with the last term zero since T/B is injective. Let $E_i' \in \operatorname{Ext}(Q, B)$ such that $\rho_*(E_i') = \rho E_i' = E_i$, $i = 1, 2$. By Theorem 1.2

there exists an $E' \in \operatorname{Ext}(Q, B)$ and $\alpha'_i \in \operatorname{End}(B)$ such that $\alpha'_i E' = E'_i$, $i = 1, 2$. But by a theorem of Szele there exists a surjection $\omega: T \to B$. Hence $\operatorname{Ext}(Q, T) \xrightarrow{\omega*} \operatorname{Ext}(Q, B) \to 0$ is exact. Let $E \in \operatorname{Ext}(Q, T)$ such that $\omega_*(E) = \omega E = E'$. Set $\alpha_i = \rho\alpha'_i\omega$, $i = 1, 2$. Now $\alpha_i E = \rho\alpha'_i\omega E = \rho\alpha'_i E' = \rho E'_i = E_i$ as desired.

We are now in a position to prove the general theorem.[1]

THEOREM 1.4. *Let F be a torsion-free group of finite rank n and T an arbitrary torsion group. Then* $\operatorname{Ext}(F, T)$ *is of rank at most n over* $\operatorname{End}(T)$.

Proof. To prove our assertion it suffices to show that for any $E_1, \cdots, E_{n+1} \in \operatorname{Ext}(F, T)$ there exist $\Gamma_1, \cdots, \Gamma_n \in \operatorname{Ext}(F, T)$ and $\alpha_{ij} \in \operatorname{End}(T)$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n+1$ such that $\sum_{i=1}^n \alpha_{ij}\Gamma_i = E_j$ for all j. Let h be the embedding of F in $Q_1 \oplus \cdots \oplus Q_n$ where $Q_i = Q$ for $1 \leqq i \leqq n$. Therefore, the sequence $\bigoplus_{i=1}^n \operatorname{Ext}(Q_i, T) = \operatorname{Ext}(Q_1 \oplus \cdots \oplus Q_n, T) \xrightarrow{h^*} \operatorname{Ext}(F, T) \to 0$ is exact. Let $\bar{E}_j \in \bigoplus_{i=1}^n \operatorname{Ext}(Q_i, T)$ such that $h^*(\bar{E}_j) = E_j$, $j = 1, \cdots, n+1$. Write $\bar{E}_j = \sum_{i=1}^n \bar{E}_{ij}$ where $\bar{E}_{ij} \in \operatorname{Ext}(Q_i, T)$ for all $1 \leqq j \leqq n+1$. By applying Theorem 1.3 to $\bar{E}_{i1}, \cdots, \bar{E}_{in+1} \in \operatorname{Ext}(Q_i, T)$ we conclude that there exists a $\bar{\Gamma}_i \in \operatorname{Ext}(Q_i, T)$ and $\alpha_{i1}, \cdots, \alpha_{in+1} \in \operatorname{End}(T)$ such that

$$\alpha_{i1}\bar{\Gamma}_i = \bar{E}_{i1}, \cdots, \alpha_{in+1}\bar{\Gamma}_i = \bar{E}_{in+1} \quad \text{for all} \quad i = 1, \cdots, n\,.$$

Observe that $\bar{E}_j = \sum_{i=1}^n \alpha_{ij}\bar{\Gamma}_i$ for all $j = 1, \cdots, n+1$. Set $\Gamma_i = h^*(\bar{\Gamma}_i)$. Now $\sum_{i=1}^n \alpha_{ij}\Gamma_i = \sum_{i=1}^n \alpha_{ij}h^*(\bar{\Gamma}_i) = \sum_{i=1}^n h^*(\alpha_{ij}\bar{\Gamma}_i) = h^*(\sum_{i=1}^n \alpha_{ij}\bar{\Gamma}_i) = h^*(\bar{E}_j) = E_j$ for all $j = 1, \cdots, n+1$, using the fact that h^* is an $\operatorname{End}(T)$-homomorphism. This concludes our proof.

Before we give the proof of Theorem 1.2 we state a fact which we will use and the proof of which is omitted.

Let A be an abelian group which is the quotient of a free group on the generators $a, a_0, a_1, \cdots$, modulo a subgroup generated by relations in one of the forms described below:

I. $p^{k_i}a - p^{h_i}a_i$ $i = 0, 1, 2, \cdots$, and where $k_0 = 0$, $k_i, h_i \in N_0$, $k_{i+1} > k_i$, and $h_{i+1} > h_i + k_{i+1} - k_i$ for all i.

II. $p^{k_i}a - p^{h_i}a_i$ for $i = 0, 1, \cdots, n$ and $p^{k_{n+1}}a - p^{\gamma_j}a_{n+j}$ for $j = 1, 2, \cdots$, such that $k_0 = 0$, $k_i, h_i, \gamma_j \in N_0$, and $k_{i+1} > k_i$, $h_{i+1} > h_i + k_{i+1} - k_i$ for $i = 1, 2, \cdots, n-1$, $\gamma_1 > h_n + k_{n+1} - k_n$ and $\gamma_{j+1} > \gamma_j$ for all j.

III. $p^{k_i}a - p^{h_i}a_i$ for $i = 1, \cdots, n$, $a_{n+j} - pa_{n+j+1}$ for $j = 0, 1, 2, \cdots$, such that $k_0 = 0$, $k_i, h_i \in N_0$, $k_{i+1} > k_i$ and $h_{i+1} > h_i + k_{i+1} - k_i$ for $i = 1, \cdots, n-1$.

[1] This result has since been generalized by the authors in a paper which is to appear in the Proc. Amer. Math. Soc.

Then for relations of type I, $T(A) = \bigoplus_{i=0}^{\infty} \langle a_i - p^{h_{i+1}-h_i-k_{i+1}+k_i} a_{i+1} \rangle$ type II, $T(A) = (\bigoplus_{i=0}^{n-1} \langle a_i - p^{h_{i+1}-h_i-k_{i+1}+k_i} a_{i+1} \rangle) \oplus \langle a_n - p^{\gamma_1 - h_n - k_{n+1} + k_n} a_{n+1} \rangle \oplus (\bigoplus_{j=1}^{\infty} \langle a_{n+j} - p^{\gamma_{j+1}-\gamma_j} a_{n+j+1} \rangle)$ and type III,

$$T(A) = \bigoplus_{i=1}^{n-1} \langle a_i - p^{h_{i+1}-h_i-k_{i+1}+k_i} a_{i+1} \rangle$$

and in all cases $A/T(A) \simeq Q_p$ where $Q_p = \{n/p^i \in Q \mid n, i \in N\}$.

Although in the proof that follows it may seem that we are laboring unnecessarily hard, the details of our construction are tailored to the next section where we prove results on certain submodules of Ext.

Proof of Theorem 1.2. It suffices to show that for any $E_1, E_2 \in \operatorname{Ext}(Q, T)$ there exists an $E \in \operatorname{Ext}(Q, T)$ and $\alpha_i \in \operatorname{End}(T)$ such that $\alpha_i E = E_i$, $i = 1, 2$. Let $E_1\colon 0 \to T \to X \xrightarrow{\sigma_1} Q \to 0$ and $E_2\colon 0 \to T \to Y \xrightarrow{\sigma_2} Q \to 0$. Choose $x \in X$, $y \in Y$ such that $\sigma_1(x) = 1$, $\sigma_2(y) = 1$. We now proceed to construct a group G whose direct sum with a torsion group will constitute the middle group of the desired extension E. [The object of the construction is to keep the height slope and thus the splitting length of G compatible with those of X and Y, see § 2 for the definitions.] The group G will be the quotient of a free group on a countable set of generators modulo a subgroup generated by certain relations. We shall call the set of free generators S and the set of relations B. Let $g \in S$. The remaining elements of S and the elements of B will be defined inductively below. For every prime p we proceed as follows. Define

$$m_i = \begin{cases} \infty & \text{if} \quad h_p(p^i x) = \infty = h_p(p^i y) \\ \omega & \text{if} \quad \omega \leqq \min(h_p(p^i x), h_p(p^i y)) < \infty \\ \min(h_p(p^i x), h_p(p^i y)) & \text{otherwise} . \end{cases}$$

Consider m_0. We have the following cases.

I. $m_0 = \infty$. Let S include the generators $g_0, g_1, g_2, \cdots$, and B the relations $g - pg_0$, $g_i - pg_{i+1}$, $i = 0, 1, 2, \cdots$. Note that the relations added to B produce no torsion elements in G. We do not introduce any further generators or relations relevant to this particular prime and proceed to the next one.

II. $m_0 = \omega$. Add to S the generators $g_0, g_1, g_2, \cdots$, and to B the relations of the form $g - p^{\gamma_i} g_i$ where $\gamma_{i+1} > \gamma_i$ and p^{γ_i} is the order of a cyclic generator in T for $i = 0, 1, 2, \cdots$. T contains such generators since $\omega \leqq \min(h_p(x), h_p(y)) < \infty$ implies that the Ulm invariants of the p-component of T do not vanish cofinitely often. Here again we do not define any more generators or relations for that prime.

III. $m_0 < \omega$. We put $g_0 \in S$ and the relation $g - p^{m_0}g_0$ in B. Observe that this does not introduce any torsion in G.

Suppose inductively we have taken care of $m_0, \cdots, m_n$. Consider m_{n+1}. Observe that if $m_i = \omega$ or ∞ for $0 \leqq i \leqq n$, then all the generators and relations relevant to that prime had been added to S and B respectively. Hence we may assume that $m_i < \omega$ for $0 \leqq i \leqq n$. We have three cases.

I. $m_{n+1} = \infty$. Adjoin to S the generators g_{n+i}, $i = 1, 2, \cdots$, and to B the relations $p^{n+1}g - p^{m_{n+1}}g_{n+1}$ and $g_{n+i} - pg_{n+i+1}$ for $i = 1, 2, \cdots$. These relations introduce a torsion element in G generated by $\langle g_n - g_{n+1}\rangle$ and whose order is p^{m_n+1}. T contains a summand with this order since there is a gap either between $h_p(p^n x)$ and $h_p(p^{n+1}x)$ or between $h_p(p^n y)$ and $h_p(p^{n+1}y)$. This will end the definition of generators and relations relative to this particular prime.

II. $m_{n+1} = \omega$. Add to S the generators g_{n+i} and to B the relations $p^{n+1}g - p^{\gamma_i}g_{n+i}$ for $i = 1, 2, \cdots$, such that $\gamma_1 > m_n + 1$, $\gamma_{i+1} > \gamma_i$ and p^{γ_i} is the order of a cyclic generator in T for $i = 1, 2, \cdots$. The relations introduced in B produce the following torsion in G, $\langle g_n - p^{\gamma_1 - m_n - 1}g_{n+1}\rangle \oplus (\bigoplus_{i=1}^{\infty} \langle g_{n+i} - p^{\gamma_{i+1}-\gamma_i}g_{n+i+1}\rangle)$ with $o(g_n - p^{\gamma_1 - m_n - 1}g_{n+1}) = p^{m_n+1}$ and $o(g_{n+i} - p^{\gamma_{i+1}-\gamma_i}g_{n+i+1}) = p^{\gamma_i}, i = 1, 2, \cdots$. Since $\min(h_p(p^n x), h_p(p^n y)) < \omega$ and $\omega \leqq \min(h_p(p^{n+1}x), h_p(p^{n+1}y)) < \infty$ we can conclude that T contains a summand isomorphic to this subgroup. Here again this will end the definition of the generators and relations for that prime.

III. $m_{n+1} < \omega$. We adjoin g_{n+1} to S. If $m_{n+1} = m_n + 1$ then add the relation $g_n - g_{n+1}$ to B. If $m_{n+1} > m_n + 1$ then we introduce the relation $p^{n+1}g - p^{m_{n+1}}g_{n+1}$. Observe that no torsion arises if $m_{n+1} = m_n + 1$ while in the case $m_{n+1} > m_n + 1$ we get a cyclic group generated by $g_n - p^{m_{n+1}-m_n-1}g_{n+1}$ and of order p^{m_n+1}. T contains such a summand since there is a gap either between $h_p(p^n x)$ and $h_p(p^{n+1}x)$ or between $h_p(p^n y)$ and $h_p(p^{n+1}y)$.

The group G is defined inductively on introducing generators and relations for all primes p. It is easy to see that $G/T(G) \simeq Q$. Recall that at each step of the construction of G we made sure that any torsion we introduced was isomorphic to a summand of T. Moreover, no overlap occurs at the various steps and for the various primes. Thus we can conclude that $T(G)$ is indeed a summand of T.

Let $T = T(G) \oplus T_1$. We are ready to define $E \in \mathrm{Ext}(Q, T)$ by $E: 0 \rightarrow T \rightarrow T_1 \oplus G \xrightarrow{\sigma} Q \rightarrow 0$ where $\sigma(g) = 1$. It remains to produce $\alpha_i \in \mathrm{End}(T)$, $i = 1, 2$. To get $\alpha_1 \in \mathrm{End}(T)$ we first define a homomorphism $f: G \rightarrow X$. We begin by defining f on S, the set of generators defining G. Set $f(g) = x$. Consider the generators on a fixed prime p. We have to define f on the generators arising from three cases.

I. The generators arising from the case $m_n < \omega$ and $m_{n+1} = \infty$.

From the definition of m_{n+1} we have $h_p(p^{n+1}x) = \infty$. Hence we may write $p^{n+1}x = p^{m_n+1}x_{n+1}$, $x_{n+i} = px_{n+i+1}$ for some $x_{n+i} \in X$, $i = 1, 2, \cdots$. Set $f(g_{n+i}) = x_{n+i}$.

II. The generators arising from $m_n < \omega$ and $m_{n+1} = \omega$. Again from the definition of m_{n+1}, $h_p(p^{n+1}x) \geqq \omega$. Hence there exist $x_{n+i} \in X$ related to $p^{n+1}x$ in the same manner as the g_{n+i} are related to $p^{n+1}g$ in G. We define $f(g_{n+i}) = x_{n+i}$, $i = 1, 2, \cdots$.

III. The generator arising when $m_{n+1} < \omega$. Again from the definition of m_{n+1}, $h_p(p^{n+1}x) \geqq m_{n+1}$. Hence there exists an x_{n+1} such that $p^{n+1}x = p^{m_{n+1}}x_{n+1}$, and we set $f(g_{n+1}) = x_{n+1}$.

We repeat this for all generators arising from all primes. Observe that the function f defined above on S respects all the relations in B and hence is indeed a homomorphism from $G \to X$. This induces $\bar{f}\colon T_1 \oplus G \to X$ defined by $\bar{f}|_{T_1} = 0$ and $\bar{f}|_G = f$. Observe that in the diagram

$$\begin{array}{ccccccccc} E\colon 0 & \longrightarrow & T & \longrightarrow & T_1 \oplus G & \xrightarrow{\sigma} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \bar{f}} & & \| & & \\ E_1\colon 0 & \longrightarrow & T & \longrightarrow & X & \xrightarrow{\sigma_1} & Q & \longrightarrow & 0 \end{array}$$

$\sigma_1\bar{f} = \sigma$. So by defining $\alpha_1 \in \operatorname{End}(T)$ by $\alpha_1 = \bar{f}|_T$ we ensure the commutativity of the diagram. Thus $\alpha_1 E = E_1$ as desired. Similarly we can define an $\alpha_2 \in \operatorname{End}(T)$ such that $\alpha_2 E = E_2$ and our proof is complete.

§ 2. **Applications.** In this section we generalize a theorem of [4] and apply our results to investigate the structure of the subbimodules $\mathscr{F}_m$ and $\mathscr{F}$ of $\operatorname{Ext}(D, T)$ where D is a torsion-free divisible group of finite rank and T a torsion group. But first we need to recall some definitions from [3], [4], and [7]. The splitting length of an abelian group A, written $l(A)$, is defined to be the least positive integer n, otherwise infinity, such that $A \otimes \cdots \otimes A$ (n factors) splits into a direct sum of a torsion and a torsion-free group [3]. The concept of splitting length is induced on $\operatorname{Ext}(D, T)$ as follows. For $E \in \operatorname{Ext}(D, T)$, $E\colon 0 \to T \to X \to D \to 0$, define $l(E) = l(X)$ [4].

Let A be an extension of T by D. In [7] the height slope of A, $h.\,s.\,A$, is defined by

$$h.\,s.\,A = \begin{cases} 0 \text{ if } h_p(a) = 0 \text{ for infinitely many primes } p \\ \quad \text{and some torsion-free element } a \in A \\ \inf\limits_{\substack{a \in A \\ o(a)=\infty}} \sup\limits_{n \in N} \inf\limits_{p} \inf\limits_{i} \left\{ \dfrac{h_p(p^i na)}{i} \right\} \text{ otherwise.} \end{cases}$$

We now state the main theorem in [7] relating height slope to splitting length.

THEOREM 2.1. *Let G be a nonsplitting mixed group having arbitrary Λ-primary torsion subgroup $T(G)$ for which $G/T(G)$ is Λ-divisible. Suppose $h.s.G = c \neq 0$. Then $l(G) = n$ if and only if one of the following holds:*

(i) c belongs to the open interval $(n/(n-1), (n-1)/(n-2))$, or

(ii) $c = n/(n-1)$ and for every torsion-free $g \in G$ there is an $n(g) \in N$ such that for every $p \in \Lambda$ there is a function $f_p: N_0 \rightarrow N_0$ nondecreasing to infinity such that $h_p(p^i n(g)g) > c(i + f_p(i))$ for all $i \in N_0$.

We now generalize a theorem of [4] where the result was proven for p-primary T.

THEOREM 2.2. *Regard* Ext (D, T) *as an* End (D)-End (T)*-bimodule where D is torsion-free divisible and T an arbitrary torsion group. Then the subset $\mathscr{F}_m = \{E \in \text{Ext}(D, T) \mid l(E) \leqq m\}$ is a sub-bimodule.*

Proof. We give a proof based on the concept of height slope. Let $E \in \text{Ext}(D, T)$, $E: 0 \rightarrow T \rightarrow X \rightarrow D \rightarrow 0$ and $\alpha \in \text{End}(T)$, $\gamma \in \text{End}(D)$. It suffices to show that $l(\alpha E) \leqq l(E)$ and $l(E\gamma) \leqq l(E)$ where $\alpha \in \text{End}(T)$ and $\gamma \in \text{End}(D)$. The result now follows from the following two facts.

(1) If $(\alpha, f, 1): E \rightarrow \alpha E$ is the morphism given by the pushout diagram then for any torsion-free element $x \in X$, $U(x) \leqq U(f(x))$. This implies that $h.s.X \leqq$ the height slope of the middle group of the extension αE and so $l(\alpha E) \leqq l(E)$.

(2) Let $(1, g, \gamma): E\gamma \rightarrow E$ be the morphism given by the pullback diagram and Y the middle group of $E\gamma$. Then using the fact that D is torsion-free divisible it is easy to see that $U(y) = U(g(y))$ when $y \notin T(Y)$. Thus $h.s.X \leqq h.s.Y$ and $l(E\gamma) \leqq l(E)$.

COROLLARY 2.3. *For D and T as in the theorem then $\mathscr{F} = \bigcup_{m=1}^{\infty} \mathscr{F}_m$ is a sub-bimodule of* Ext (D, T).

We now turn our attention to the structure of the End (T)-submodules $\mathscr{F}_m$ and $\mathscr{F}$ when D is a torsion-free divisible group of finite rank and T an arbitrary torsion group. We raise the following questions: 1. Is $\mathscr{F}_m$, $\mathscr{F}$ a finitely generated module? 2. If the answer to 1 is negative, is $\mathscr{F}_m$, $\mathscr{F}$ of finite rank?

The answer to 1 is negative even for $D = Q$ and T the direct sum of cyclic p-groups of unbounded order. Suppose $E_1, \cdots, E_k$ generate $\mathscr{F}_m$ over End (T). Let $c_1, \cdots, c_k$ be the height slopes of the middle groups of $E_1, \cdots, E_k$. Choose $E \in \text{Ext}(Q, T)$ such that the height slope of its middle group is less than $\min\{c_1, \cdots, c_k\}$ but also lies in $(m/(m-1), (m-1)/(m-2))$. This is possible by [11]. By Theorem 2.1 $E \in \mathscr{F}_m$. Since the action of the ring End(T) does not decrease height slope, so $E \notin \langle E_1, \cdots, E_k \rangle$. This also shows that $\mathscr{F}$ is not finitely generated.

The answer to question 2 is affirmative when T is the direct sum of cyclic groups. We will show this for $D = Q$ as the general case follows by a proof similar to that of Theorem 1.4. Here we use the details of our construction in Theorem 1.2. Suppose $E_1, E_2 \in \mathscr{F}_m$ with middle groups X and Y respectively. Consider the extension E as defined in Theorem 1.2 and let G be its middle group. Observe that G was built with an Ulm matrix whose finite entries correspond to the minimum of the finite entries of the Ulm matrices of X and Y. This insures that $h.s.\,G = \min(h.s.\,X, h.s.\,Y)$ and by Theorem 2.1 $l(G) = \max(l(X), l(Y))$. Thus $E \in \mathscr{F}_m$, as desired. Question 2 is open for arbitrary T.

Note that although in the last two paragraphs $\mathscr{F}_m$ and $\mathscr{F}$ were viewed as End(T)-submodules of Ext(D, T), the restriction was needless. Our conclusions in fact hold for the sub-bimodule structure since endomorphisms of D do not increase the splitting-length.

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LEHIGH UNIVERSITY
AND
THE UNIVERSITY OF ARIZONA