PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

ON THE EQUIVALENCE OF TWO TYPES OF OSCILLATION FOR ELLIPTIC OPERATORS

W. ALLEGRETTO

The strong oscillation of a second order symmetric elliptic operator is shown to be equivalent to the oscillation of all solutions of the associated homogeneous equation. Extensions of a nonoscillation theorem and of an existence theorem are obtained as applications.

Introduction. Let $\{E^n, \infty\}$ denote the topological space formed by the standard one point compactification of n-dimensional Euclidean space E^n. A real valued function u with domain in E^n is said to be oscillatory (at ∞) iff ∞ belongs to the closure (in the topology of $\{E^n, \infty\}$) of the set $\{x \mid x \in E^n \text{ and } u(x) = 0\}$. Let L denote a second order symmetric elliptic operator with coefficients defined in an unbounded domain Ω of E^n. Following I. Glazman [6], we define L to be *strongly oscillatory* (at ∞) iff L has a nodal domain in $N \cap \Omega$ for any given neighborhood N of ∞. That is: Given any neighborhood N of ∞, there exists a *bounded* domain $D \subset N \cap \Omega$ for which zero is the smallest eigenvalue for L (corresponding to Dirichlet boundary conditions). Since the classical Sturm-Kneser theorem can be extended to partial differential equations by means of the Swanson-Picone identity, [14, p. 187], [2], it follows that if L is strongly oscillatory then every C^2 function v which is a solution of the equation $Lu = 0$ in $N \cap \Omega$, for some neighbourhood N of ∞, must be oscillatory. This connection between the strong oscillation of L and the oscillation of the solutions to the equation $Lu = 0$ has been noted for some time, beginning with results of K. Kreith [8], for special cases of L; Headley and Swanson [7], for the general case; and, more recently, several other authors. Further extensions of these concepts have also been made to the case of elliptic systems [1], [16] and eigenvalue problem [2], [3]. We refer the reader to the recent book by K. Kreith [9], where these ideas are discussed and an extensive bibliography is given.

It is our main purpose to show that the strong oscillation of a second order elliptic operator L is *equivalent* to the oscillation of all solutions u of the equation $Lu = 0$ in neighborhoods of ∞, if the coefficients of L and Ω are reasonably regular. This extends a result which is obviously true for ordinary differential equations. As applications of our results, a nonoscillation theorem of C. A. Swanson is strengthened and some related results of L. M. Kuks are clarified and extended.

We shall restrict our discussion to the case where Ω is an unbounded domain and L has regular coefficients. The analogues of our results for the case where Ω is bounded but the coefficients of L are singular at some boundary point of Ω will be obvious from the presentation.

2. Assumptions and main results. As is usual, points of E^n will be denoted by $x = (x_1, \cdots, x_n)$ and differentiation with respect to x_i by D_i for $i = 1, \cdots, n$. Let Ω denote an unbounded domain of E^n. We shall use the following notation throughout:

$$\Omega_{\rho_1,\rho} = \Omega \cap \{x \mid x \in E^n \text{ and } \rho_1 < |x| < \rho\} ,$$
$$\Omega_{\rho,\infty} = \Omega \cap \{x \mid x \in E^n \text{ and } \rho < |x|\} ,$$

where $0 < \rho_1 < \rho < \infty$.

Let L denote the elliptic operator formally given by:

$$Lu = -\sum_{i,j=1}^{n} D_i[a_{ij}D_ju] + cu , \quad a_{ij} = a_{ji} .$$

The coefficients a_{ij} are assumed of class C^{m+1} and c is assumed of class C^m where $m = 3[[3 + n/2]/2]$ in the closure of any bounded subdomain of Ω, where $[q]$ denotes the largest integer not exceeding q. L is assumed uniformly elliptic in any bounded subdomain of Ω. These assumptions are more restrictive than what is needed for many of the results, however they lead to a unified and simple presentation. About Ω we shall only assume that $\Omega_{\rho_1,\rho}$ is a domain for all ρ_1, ρ with $\rho_0 < \rho_1 < \rho < \infty$ for some $\rho_0 > 0$. Unlike most results for unbounded domains, no other restrictions are placed on Ω or on the coefficients of L at ∞ or on the sign of c.

LEMMA 0. *Let β be a positive constant and x_0 a point in Ω so that $\{x \mid |x - x_0| \leqq \beta\} \subseteqq \Omega$. Define the $C_0^{2l-1}(\Omega)$ function Ψ as*

$$\Psi(x) = \begin{cases} -(|x - x_0|^2 - \beta^2)^{2l} , & \text{if } |x - x_0| \leqq \beta \\ 0 , & \text{if } |x - x_0| \geqq \beta . \end{cases}$$

Then given ε, $0 < \varepsilon < 1/2$, there is a $l_0 = l_0(x_0, \beta, \varepsilon)$ such that for each positive integer $l \geqq l_0$, the following hold:

(i) $L\Psi(x) \geqq 0$ *for* $\varepsilon\beta \leqq |x - x_0|$

(ii) $L\Psi(x) > 0$ *for* $\varepsilon\beta \leqq |x - x_0| \leqq (1 - \varepsilon)\beta$.

The proof of Lemma 0 follows easily from the locally uniform ellipticity of L and the local boundedness of the coefficients of L.

As an immediate consequence of Lemma 0, we note that given any arbitrarily large $q > 0$ and any sufficiently small $\varepsilon > 0$ we can construct a function $\phi_0 \in C_0^2(\Omega_{q,q+2\varepsilon})$ such that $L(\phi_0) \geqq 0(\not\equiv 0)$ in $\Omega_{q+\varepsilon,q+2\varepsilon}$.

Given any bounded subdomain D of Ω we introduce the space $C^1(\bar{D})$ of all continuously differentiable functions in $\bar{D}$ and the space $H^1(D)$ which is the completion of $C^1(\bar{D})$ in the norm:

$$||u||_1^2 = \int_D \left\{\sum_{i=1}^n (D_i u)^2 + u^2\right\} d\Omega .$$

By $H_0^1(D)$ we denote the closure in the $|| \ ||_1$-norm of the space $C_0^\infty(D)$, of all infinitely differentiable functions with compact support in D.

We shall not distinguish in notation between the equivalence classes which form the elements of $H^1(D)$ and functions chosen from various equivalence classes.

Given any function $u \in H_0^1(D)$, we define u to be nonnegative in the H_0^1 sense iff there exists a sequence of nonnegative $C_0^1(D)$ functions which converges to u in the $|| \ ||_1$-norm. If $u, v \in H_0^1(D)$, we write $u \geqq v$ iff $u - v \geqq 0$ in the H_0^1 sense. Let G denote the function from R^1 to R^1 given by:

$$G(x) = \begin{cases} x & x \geqq 0 \\ 0 & x < 0 . \end{cases}$$

For any $u \in H_0^1(D)$, we set $u^+ = G \circ u$, $u^- = G \circ (-u)$ and define $|u|$ by $|u| = u^+ + u^-$.

The form $B(u, \phi)$ given by:

$$B(u, \phi) = \int_D \sum_{i,j=1}^n a_{ij} D_i u D_j \phi + cu\phi$$

is naturally associated with the operator L and subdomain D. A function $u \in H^1(D)$ is called subsolution with respect to L in D iff $B(u, \phi) \leqq 0$ for $\phi \in C_0^\infty(D)$, $\phi \geqq 0$. A generalized solution of $Lu = f$ is a function u in $H^1(D)$ such that $B(u, \phi) = (f, \phi)$ for all $\phi \in C_0^\infty(D)$ where, as usual, we set $(f, \phi) = \int_D f\phi$.

The above terminology was introduced in [13] where the following two results were established (in much greater generality):

LEMMA 1 [13, p. 18]. *If u is a member of $H_0^1(D)$ then so are u^+ and u^- and*:

$$D_i u^+ = \begin{cases} 0 & u \leqq 0 \\ D_i u & u \geqq 0 \end{cases} a.e. D; \qquad D_i u^- = \begin{cases} 0 & u \geqq 0 \\ -D_i u & u \leqq 0 \end{cases} a.e. D .$$

LEMMA 2 [13, p. 75]. *If u, v are subsolutions in D and the form B is coercive over $H_0^1(D)$ then* $\max(u, v)$ *is also a subsolution.*

If $u \in H_0^1(D)$ then clearly u^+ and u^- are nonnegative in the $H_0^1(D)$ sense.

The next two lemmas show that if L has no nodal domains in $\Omega_{m,\infty}$ then there exists a positive C^2 function u such that $Lu(x) = 0$ for all $|x|$ sufficiently large. Such a function u is obtained as the limit of solutions of suitable boundary value problems involving bounded subdomains of $\Omega_{m,\infty}$.

LEMMA 3. *Let L have no nodal domain in $\Omega_{m,\infty}$ for some integer $m > \rho_0$ and let $\phi_0 \in C_0^2(\Omega_{m,m+2\varepsilon})$ be a function constructed by the above procedures (for some $\varepsilon > 0$) such that $L\phi_0 \geqq 0 (\not\equiv 0)$ in $\Omega_{m+\varepsilon,m+2\varepsilon}$. Then for each integer $k > m + 2\varepsilon$ there exists a function u_k in $H_0^1(\Omega_{m+\varepsilon,k})$ such that $Lu_k = L\phi_0$ in a generalized sense. Furthermore, if $k_1 > k_2 > m + 2\varepsilon$ then $u_{k_1} \geqq u_{k_2} \geqq 0$.*

Proof. Since L has no nodal domains in $\Omega_{m,\infty}$ then for each integer k, $k > m + 2\varepsilon$, there exists a positive constant γ such that for all functions $\phi \in C_0^\infty(\Omega_{m+\varepsilon,k})$ we have:

$$(L\phi, \phi) \geqq \gamma(\phi, \phi) .$$

Consequently L is uniformly positive definite in $C_0^\infty(\Omega_{m+\varepsilon,k})$ and we can form the Friedrichs extension of L (also denoted by L) whose domain is contained in the completion of $C_0^\infty(\Omega_{m+\varepsilon,k})$ in the $\|\ \|_L$-norm, where $\|\phi\|_L^2 = (\phi, L\phi)$. We note that if $\phi \in C_0^\infty(\Omega_{m+\varepsilon,k})$ then $\|\phi\|_L^2 \leqq M\|\phi\|_1^2$ for some constant M which depends on the coefficients of L, and, conversely,

$$\|\phi\|_1^2 \leqq \left[\frac{1}{\lambda}\left(1 + \frac{N}{\gamma}\right) + \frac{1}{\gamma}\right]\|\phi\|_L^2$$

where λ denotes a positive lower bound on the smallest eigenvalue of $(a_{ij}(x))$ and $N = \sup |c(x)|$ for $x \in \Omega_{m+\varepsilon,k}$. Consequently, the $\|\ \|_L$-norm and $\|\ \|_1$-norm are equivalent for $C_0^\infty(\Omega_{m+\varepsilon,k})$ and the completion of $C_0^\infty(\Omega_{m+\varepsilon,k})$ in the $\|\ \|_L$-norm is $H_0^1(\Omega_{m+\varepsilon,k})$. By the results stated, for example, in [12, Chapter 1] there exists a unique function u_k in $H_0^1(\Omega_{m+\varepsilon,k})$ which is a generalized solution of $Lu_k = L\phi_0$ in $\Omega_{m+\varepsilon,k}$ and is further characterized as the function which minimizes the functional:

$$J(\phi) = \|\phi\|_L^2 - 2(\phi, L\phi_0)$$

over the space $H_0^1(\Omega_{m+\varepsilon,k})$. By Lemma 1, it follows that $|u_k| \in H_0^1(\Omega_{m+\varepsilon,k})$, $(D_i|u_k|)^2 = (D_iu_k)^2$, and $|u_k|^2 = u_k^2$ a.e. $\Omega_{m+\varepsilon,k}$. Consequently,

$$J(|u_k|) - J(u_k) = 2\int_{\Omega_{m+\varepsilon,k}} (u_k - |u_k|)L\phi_0 \leqq 0$$

since $L\phi_0 \geqq 0$ in $\Omega_{m+\varepsilon,\infty}$. By the uniqueness of the minimizing function u_k it follows that $u_k = |u_k|$ a.e. and consequently $u_k \geqq 0$. If $k_1 > k_2 > m + 2\varepsilon$ then $u_{k_2} - u_{k_1}$ is a solution in $H^1(\Omega_{m+\varepsilon,k_2})$ of $Lu = 0$ and since the the constant function $v \equiv 0$ is also a solution of $Lv = 0$ then $\max(u_{k_2} - u_{k_1}, 0) = \{u_{k_2} - u_{k_1}\}^+$ is a subsolution (in $H_0^1(\Omega_{m+\varepsilon,k_2})$) by Lemma 2. Consequently,

$$B(\{u_{k_2} - u_{k_1}\}^+, \phi) \leqq 0 ,$$

for all nonnegative ϕ in $C_0^\infty(\Omega_{m+\varepsilon,k_2})$ and by continuity:

$$B(\{u_{k_2} - u_{k_1}\}^+, \{u_{k_2} - u_{k_1}\}^+) \leqq 0 .$$

It follows that $\{u_{k_2} - u_{k_1}\}^+ = 0$ and hence $u_{k_1} \geqq u_{k_2}$.

LEMMA 4. *Let the conditions of Lemma 3 hold. Then there exists a positive function $u \in C^2(\Omega_{m+2\varepsilon,\infty})$ such that $Lu(x) = 0$ for $x \in \Omega_{m+2\varepsilon,\infty}$.*

Proof. Consider the sequence $\{u_i\}_{i>m+2\varepsilon}$ with u_i set identically equal to zero outside $\Omega_{m+\varepsilon,i}$. This is a minimizing sequence for the functional J in the space formed by completing $C_0^\infty(\Omega_{m+\varepsilon,\infty})$ in the $\|\ \|_L$-norm since if $\phi \in C_0^\infty(\Omega_{m+\varepsilon,\infty})$ then $J(\phi) \geqq J(u_j)$ for any j chosen so that $\operatorname{supp}\phi \subset \Omega_{m+\varepsilon,j}$. We note that the expression $\|\phi\|_L^2 = (\phi, L\phi)$ defines a norm even for $\phi \in C_0^\infty(\Omega_{m,\infty})$ since L has no nodal domains in $\Omega_{m,\infty}$. Consequently, the Cauchy-Schwartz inequality shows that the map $\phi \to (\phi, L\phi_0)$ is a bounded linear functional on the space $C_0^\infty(\Omega_{m,\infty})$ (and hence on $C_0^\infty(\Omega_{m+\varepsilon,\infty})$), with respect to the $\|\ \|_L$-norm. By means of the Riesz representation theorem, we conclude that the minimum of J is achieved in the completion of $C_0^\infty(\Omega_{m+\varepsilon,\infty})$ (with respect to the $\|\ \|_L$-norm) and it follows that the sequence $\{u_i\}$ converges in the $\|\ \|_L$-norm, [12, Chapter 1]. If ϕ denotes any function in $C_0^2(\Omega_{m,\infty})$ we again employ the Cauchy-Schwartz inequality to conclude that $\{(u_i, L\phi)\}$ converges. Let x denote any point of $\Omega_{m+\varepsilon,\infty}$. Since $\Omega_{m+\varepsilon,\infty}$ is connected, it is possible to find a finite number of spheres $\{S_i\}_{i=1}^k$ such that: $L\phi_0 > 0$ in $\bar{S}_1$; the center of S_{i+1} belongs to S_i; $x \in S_k$; and $\bigcup_{i=1}^k \bar{S}_i \subset \Omega_{m+\varepsilon,\infty}$. As noted above, the sequence $\{(u_i, L\phi_0)\}$ converges. Since $L\phi_0 \geqq 0$ in $\Omega_{m+\varepsilon,\infty}$ then $\{u_i\}$ is L^1 cauchy in S_1. Let ϕ_1 be a $C_0^2(\Omega_{m,\infty})$ function constructed by the above procedures such that $L\phi_1 > 0$ in $\bar{S}_2 - S_1$ and $L\phi_1 \geqq 0$ in the complement of S_1. Since:

$$(u_i, L\phi_1) = \int_{S_1} u_i L\phi_1 + \int_{\Omega - S_1} u_i L\phi_1 ,$$

we conclude that $\{u_i\}$ is $L^1(S_2)$ cauchy. By induction, it follows that $\{u_i\}$ is cauchy in $L^1(S_k)$. We set $u = \sup u_i$ and conclude that u is of class $L^1_{loc}(\Omega_{m+\varepsilon,\infty})$ and that for all $\phi \in C_0^\infty(\Omega_{m+\varepsilon,\infty})$ we have $(u, L\phi) = (\phi, L\phi_0)$. Clearly if $\operatorname{supp}\phi$ does not intersect $\Omega_{m+\varepsilon,m+2\varepsilon}$ then $(u, L\phi) = 0$. Let K denote a regular subdomain with $\bar{K}$ a compact subset of $\Omega_{m+\varepsilon,\infty}$ which does not intersect $\Omega_{m+\varepsilon,m+2\varepsilon}$. Following a standard regularity argument (see, for example, [4, p. 195]), we employ the Sobolev embedding theorem as follows: Let $\phi \in C_0^\infty(K)$ then:

$$|(u, \phi)| \leqq \sup_{x\in K} |\phi(x)| \cdot \int_K |u| \leqq C||\phi||_{[n/2]+1} \cdot \int_K |u| \ .$$

It follows that the map $\phi \rightarrow (u, \phi)$ is a bounded linear functional on the space $H_0^{[n/2]+1}(K)$. By the Riesz representation theorem we have:

$$(u, \phi) = (u_0, (-1)^{[n/2]+1}\Delta^{[n/2]+1}\phi) \ ,$$

for some u_0 in $H_0^{[n/2]+1}$, where $H_0^{[n/2]+1}(K)$ is now considered as the completion of $C_0^\infty(K)$ in the (equivalent) norm:

$$||\phi||^2 = (\phi, (-1)^{[n/2]+1}\Delta^{[n/2]+1}\phi) \ .$$

It follows that:

$$(u, \phi) = (\psi_0, (-1)^t\Delta^t\phi) \ ,$$

and, consequently,

$$(u, L\phi) = (\psi_0, (-1)^t\Delta^tL\phi) = 0 \ ,$$

for some ψ_0 in $L^2(K)$ and all ϕ in $C_0^\infty(K)$, where $t = [1/2[(n + 6)/2]]$. Since the coefficients of L are of class C^{3t+1} and C^{3t} respectively, we conclude by a classical result (see, for example, [5, p. 56]) that $\psi_0 \in C^{2t+2}(K)$ and consequently that $u \in C^2(K)$. Hence u is a classical solution of $Lu = 0$ in $\Omega_{m+2\varepsilon,\infty}$. Since u is obviously nonnegative, then u must be positive [10].

It is interesting to note that the conclusion of Lemma 4 cannot be strengthened to read: "there exists a function u solution of $Lu = 0$ in Ω and positive in $\Omega_{m+\varepsilon,\infty}$", as the following counterexample shows. Let $\Omega = E^2$ and let c denote any regular nonpositive function with support in $\{x \mid |x| < 1\}$ such that the operator formally defined by $Lu = -\Delta u + cu$ has no nodal domains in $\{x \mid |x| < 2\}$. L has no nodal domains in $\Omega_{1,\infty}$ since if $\phi \in C_0^\infty(\Omega_{1,\infty})$ then $L\phi = -\Delta\phi$. Assume that there exists a function v such that $Lv = 0$ in E^2 and $v > 0$ in $\Omega_{1+\varepsilon,\infty}$ with $0 < \varepsilon < 1$. If v vanishes at some point of E^2 then it must change sign [10]. By Lemma 1, $(-v)^+ \in H_0^1(\{x \mid |x| < 2\})$ and by Lemma 2 $(-v)^+$ is a subsolution. Consequently, we have

$$B((-v)^+, (-v)^+) \leqq 0 \ .$$

If $(-v)^+ \neq 0$ then we have a contradiction to the assumption that L has no nodal domains in $\{x \mid |x| < 2\}$. It follows that v is positive in E^2 and therefore $\Delta v = cv \leqq 0$. But, by Liouville's theorem, any function which is bounded below and superharmonic in the whole of E^2 is a constant. If c is nontrivial, this is a contradiction.

THEOREM 1. *L is strongly oscillatory iff every C^2 function u, solution of $Lu = 0$ in some neighborhood of ∞, is oscillatory.*

Proof. If L is strongly oscillatory then, given any neighborhood N of ∞ it follows by the standard theory of eigenvalue problems that there exists a bounded domain $D \subset N \cap \Omega$ and a function $\omega \in H_0^1(D)$ such that $(\omega, L\omega) < 0$. By arguments involving the Swanson-Picone identity (see, for example, [14, p. 205], [2]) we conclude that all solutions u of $Lu = 0$ in D must change sign in D. Consequently, all solutions of the equation $Lu = 0$ in some neighborhood of ∞ are oscillatory. Conversely, if L is not strongly oscillatory, then, by Lemma 4, there exists a positive solution to the equation $Lu = 0$ in some neighborhood of ∞.

The proofs of the lemmas and of Theorem 1 would be even simpler if the fact that L had no nodal domains in $\Omega_{m,\infty}$ implied that, for $\varepsilon > 0$, the eigenvalues of L in the bounded subdomains of $\Omega_{m+\varepsilon,\infty}$ were uniformly bounded below by a positive constant. Simple examples can be constructed to show that this is, in general, false.

As an application of Theorem 1 it is possible to give strengthened versions of known nonoscillation theorems. As an example we give the following corollary which strengthens a result of C. A. Swanson [15], which is itself an extension of a result of Glazman.

COROLLARY 1. *Assume that Ω is the complement of a sphere in E^n, let $\lambda_0 > 0$ denote the ellipticity constant of L* (i.e. $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqq \lambda_0 \sum_{i=1}^n \xi_i^2$ *for all* $(x, \xi_1, \cdots, \xi_n)$ *in* $\Omega \times E^n$), *and let $g(r)$ denote the minimum of $c(x)$ on $\{x \mid |x| = r\}$. If*

$$\liminf_{r\to\infty} r^2 g(r) > \frac{-(n-2)^2\lambda_0}{4}, \tag{1}$$

then there exists a positive solution in a neighborhood of ∞ to the equation $Lu = 0$.

Proof. It is shown in [15] that condition (1) is sufficient for L to have no nodal domains in a neighborhood at ∞. The conclusion then follows from Theorem 1.

It is obviously possible to obtain other such results by using known nonoscillation criteria, but we do not pursue this point.

In conclusion we note that L. M. Kuks has stated related results which, as given in [11], appear valid only under the implicit assumption that the (open) domain Ω and the coefficients of L are such that the standard existence and uniqueness theories apply to the whole of Ω. Specifically, it is stated in [11] that a necessary and sufficient condition for the unique solvability of the Dirichlet problem in the subdomains of Ω is that there exists a positive solution to the inequality $Lu \geqq 0$ (cf. Definition 2 and Theorem 3 of [11]). However, this is false for the (open) domain Ω even if Ω is bounded and regular, unless some restrictions are placed on the coefficients of L up to the boundary of Ω, as the following example indicates: Let $\Omega = (0, 1) \times (0, 1)$ and let L be formally given by:

$$Lu = \frac{\partial}{\partial x}\left(\frac{1}{x^2}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y^2}\frac{\partial u}{\partial y}\right) + 2\left[\frac{1}{x^4} + \frac{1}{y^4}\right]u \, .$$

Then $u(x, y) = (x^2 - x/n)(y^2 - y/n)$ solves $Lu = 0$ and vanishes on the boundary of $(0, 1/n) \times (0, 1/n)$. Consequently, the Dirichlet problem does not have unique solutions in the subdomains of $(0, 1) \times (0, 1)$. Yet the function $v(x, y) = xy$ satisfies $Lv = 0$ and is positive in Ω. Analogous examples are possible for unbounded domains. If the "domain" of [11] is not assumed open, i.e., if $\Omega = Q \cup P$ with Q some open set and P a nonempty subset of the boundary of Q, then clearly Theorem 3 of [11] is again false, for if the smallest eigenvalue for L in Q is zero and conditions are sufficiently regular then by the Swanson-Picone identity every solution of the inequality $Lu \geqq 0$ must vanish in Ω even though the Dirichlet problem has a unique solution in each (proper) subdomain of Ω. It follows that, as claimed above, Theorem 3 of [11] is valid (for open sets) only under global regularity assumptions and, consequently, it does not imply Theorem 1. We state a result which is, in form, a local version of Theorem 3 of [11].

COROLLARY 2. *There exists an integer m such that the Dirichlet problem for L has a unique (generalized) solution in any bounded subdomain of $\Omega_{m,\infty}$ iff there exists an integer m' such that the equation $Lu = 0$ has a positive solution in $\Omega_{m',\infty}$.*

Proof. If there exists a positive solution u to the equation $Lu = 0$ in $\Omega_{m',\infty}$ then by Theorem 1, L has no nodal domains in $\Omega_{m',\infty}$. Consequently, the Dirichlet problem for any bounded subdomain of $\Omega_{m',\infty}$ has a unique solution. Conversely, if the Dirichlet problem for L in the bounded subdomains of $\Omega_{m,\infty}$ has a unique solution then L has no nodal domains in $\Omega_{m,\infty}$. The existence of a positive solution in a neighborhood of ∞ follows from Theorem 1.

If Ω and the coefficients of L are such that, for some $\varepsilon > 0$, it is possible to regularly extend the coefficients of L to the open set $\Omega_{m-\varepsilon,\infty}$ of E^n in such a way that the extension has no nodal domains in $\Omega_{m-\varepsilon,\infty}$ then we choose $m' = m$ in Corollary 2 and in this case we have:

COROLLARY 3. *The Dirichlet problem for L has a unique solution in any bounded subdomain of* $\Omega_{m,\infty}$ *iff there exists a positive solution in* $\Omega_{m,\infty}$ *to the equation* $Lu = 0$.

Note that if $\bar{\Omega} \neq E^n$ we may replace "$\Omega_{m,\infty}$" by "Ω" in the statement of Corollary 3 and hence we have, for this special case, an analogue of Theorem 3 of [11] for our unbounded domain. Even though it is easy to give simple sufficient conditions for the exten- of L, as required in Corollary 3, to be possible, necessary and sufficient conditions for such an extension are not known to the author at this time.

Added in Proof. The significance to spectral theory of the equivalence of the two types of oscillation has been considered by J. Piepenbrink in his recent paper: "Nonoscillatory Elliptic Equations", J. Differential Equations, 15, 541–550 (1974). Theorem 1 answers in the affirmative the question posed by J. Piepenbrink at the end of his paper.

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Received April 9, 1974 and in revised form August 29, 1974.

UNIVERSITY OF ALBERTA