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CENTRAL EMBEDDINGS IN SEMI-SIMPLE RINGS

S. A. AMITSUR

A ring S is a central extension of a subring R if $S = RC$ and C is the centralizer of R in S, i.e., $C = \{s \in S;\ sr = rs\}$ for every $r \in R$. We shall also say that R is centrally embedded in S.

We have shown that if a ring R is centrally embedded in a simple artinian ring then R is a prime Öre ring and its quotient ring Q is the minimal central extension of R which is a simple artinian ring; furthermore, the centralizer of R can be characterized. In the present note we extend these results and show that rings which can be centrally embedded in semi-simple artinian rings are semi-prime Öre rings with a finite number of minimal primes and their rings of quotients are the minimal central extension of this type.

2. The Ring $Q_0(R)$. We recall some definitions and results of [1].

Let R be an associative ring (not necessarily with a unit) and let $L_0(R)$ be the set of all (two-sided) ideals A of R with the property:

(A) "$\forall x \in R,\ Ax = 0 \Rightarrow x = 0$".

The set $L_0(R)$ is a filter. That is: closed under finite intersection and inclusion. We shall also assume henceforth that $R \in L_0(R)$ i.e. $Rx = 0 \Rightarrow x = 0$.

Consider every $A \in L_0(R)$ as left R-module and define the ring

$Q_0(R) = \varinjlim \mathrm{Hom}_R(A, R)$, where A ranges over all $A \in L_0(R)$. A more detailed description of $Q_0(R)$ is as follows: Let $U = \cup\ \mathrm{Hom}_R(A, R)$, $A \in L_0(R)$, and in U we define an equivalence relation, addition and multiplication as follows:

For $\alpha: A \to R$, $\beta: B \to R$ and $A, B \in L_0(R)$ we put:

(i) $\alpha + \beta: A \cap B \to R$ defined by $x(\alpha + \beta) = x\alpha + x\beta$ for $x \in A \cap B$.

(ii) $\alpha\beta: BA \to R$ by: $(\Sigma ba)\alpha\beta = \Sigma[b(a\alpha)]\beta$ for $b \in B$, $a \in A$.

(iii) $\alpha \equiv \beta$ if there exists $C \subseteq A \cap B$, $C \in L_0(R)$ for which $c\alpha = c\beta$ for every $c \in C$.

The ring $Q_0(R)$ is the ring of equivalence classes of U with respect to preceding definitions. Furthermore, R is canonically mapped into $Q_0(R)$ by identifying R with the right multiplications on R.

The center $\Gamma = \Gamma(R)$ of $Q_0(R)$ can be characterized as the set of all $\bar{\gamma} \in Q_0(R)$ which have a representative $\gamma \in \mathrm{Hom}(A, R)$ such that γ is in fact a bi-R-module homomorphism of A into R, i.e. it satisfies $(ax)\gamma = (a\gamma)x$ and $(xa)\gamma = x(a\gamma)$ for $a \in A$, $x \in R$. Also $\bar{\gamma} \in \Gamma$ if and only if it commutes with the element of R.

From the results of **[1]** we quote the following:

If R is semi-simple artinian, then R is both a right and left Öre ring and its quotient ring is $Q_0(R) = R\Gamma$. [**1**, Theorem 6].

If $S = RC$ is a simple artinian central extension of R then $\Gamma \subseteq C$, $S = R\Gamma \otimes_\Gamma C$ and $R\Gamma = Q_0(R)$ is also simple artinian [**1**, Theorem 18].

The ring $R\Gamma$ is semi-simple artinian if and only if the number of minimal primes P of R is finite, and for each P, $(R/P)\Gamma(R/P)$ is simple artinian. [**1**, Corollary 13].

It follows also from the proofs of [**1**, Theorem 10] that the number of simple components of R equals the number of minimal primes of R.

3. The main result. Let $S = S_1 \oplus \cdots \oplus S_m$ a direct sum of a finite number of simple rings S_i with units ϵ_i, and $1 = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$. The ring S will be said an extension of *minimal length* of a subring R if for every i there exist $0 \neq r \in R$ such that $r\epsilon_j = 0$ for all $j \neq i$, or equivalently $r(1-\epsilon_i) = 0$. This means that for no subring $S(1-\epsilon_i) = S_1 \oplus \cdots \oplus S_{i-1} \oplus S_{i+1} \oplus \cdots \oplus S_m$ the subring $R(1-\epsilon_i)$ is isomorphic with R.

LEMMA 1. *Let $S = RC$ be a central extension of R, and let $S = S_1 \oplus \cdots \oplus S_m$ be a direct sum of simple rings S_i with units ϵ_i. Then:*

(1) *For every central idempotent ϵ, $S\epsilon$ is a central extension of $R\epsilon$; and it is also a direct sum of simple rings with a unit.*

(2) *There exists a direct summand $S\epsilon$ of S such that $R \cong R\epsilon$, and $S\epsilon$ is a central extension of R of minimal length.*

Proof. A central idempotent ϵ of S is of the form $\epsilon = \epsilon_{i_1} + \cdots + \epsilon_{i_r}$ and hence $S\epsilon = S_{i_1} \oplus S_{i_2} \oplus \cdots \oplus S_{i_r}$. Furthermore $S = RC$ yields that $S\epsilon = (RC)\epsilon = (R\epsilon)\,(C\epsilon)$ and the elements of $C\epsilon$ commute with the elements of $C\epsilon$, which readily implies that $S\epsilon$ is a central extension of $R\epsilon$.

To prove the second part, we consider the set of all central idempotents ϵ of S with the property: "$r\epsilon = 0$, $r \in R \Rightarrow r = 0$". Clearly for such ϵ, $R \cong R\epsilon$ by corresponding: $r \to r\epsilon$. The set of these idempotents is not empty since the unit 1 has this property. Each of the central idempotent ϵ has the form $\epsilon = \epsilon_{i_1} + \cdots \epsilon_{i_\rho}$, $i_1 < i_2 < \cdots < i_r$. So choose ϵ of this set with minimal ρ. Then $S\epsilon$ is a central extension of $R\epsilon$ of minimal length, since the minimality of ρ implies that for any $1 \leqq \lambda \leqq \rho$, there exists $r \neq 0$ such that $r(\epsilon - \epsilon_{i_\lambda}) = 0$.

The preceding lemma shows that if a ring R has a central extension

S of the type described above, then replacing S by a direct summand we get a central extension of minimal length of a ring isomorphic with R. We can, therefore, restrict ourselves to the study of central extension of minimal length. Our results is the following.

THEOREM A. *Let $S = RC$ be a central extension of R of minimal length then R is semi-prime and we can embed $\Gamma \subseteq C$. Furthermore, $R\Gamma$ is also a central extension of R of the same type with the same number of components.*

THEOREM B. *Let $S = RC$ be a semi-simple artinian ring and a central extension of R of minimal length then $R = Q_0(R)$ is also semi-simple artinian and $S = R\Gamma \otimes_\Gamma C$.*

In view of the results quoted from [1] we deduce that:

COROLLARY C. *If R has a central extension which is a semi-simple artinian ring, then R is a semi-prime* (right and left) *Öre ring with a finite number of minimal primes. Its ring of quotient is $Q_0(R)$ and it is a minimal semi-simple artinian central extension of R.*

4. *Proofs.* Before proceeding with the proof we need a few lemmas.

LEMMA 2. *Let $S = RC$ be a central extension of R of minimal length, then an ideal A in R belongs to $L_0(R)$ if and only if $AC = S$.*

Indeed, let $S = S_1 \oplus \cdots \oplus S_n$, S_i simple with a unit ϵ_i. If $AC = S$ and $Ax = 0$ for some $x \in R$, then $Sx = (AC)x = (Ax)C = 0$ but S has a unit and so $x = 0$, i.e. $A \in L_0(R)$. Conversely, it suffices to show that $AC \cap S_i \neq 0$, since then $AC \cap S_i$ is a nonzero ideal in the simple ring implies that $S_i = AC \cap S_i$. This in turn yields that $AC \supset S_i$ and, therefore $AC \supset S_1 \oplus \cdots \oplus S_n = S$. To prove that $AC \cap S_i \neq 0$, we note that if $AC \cap S_i = 0$ then $A\epsilon_i \subseteq AS_i \subseteq ARC \cap S_i \subseteq AC \cap S_i = 0$. Let $P = \{r, r\epsilon_i = 0\}$ and $Q = \{r \in R,\ r(1-\epsilon_i) = 0\}$. Since S is of minimal length it follows that $P \cap Q = 0$, $Q \neq 0$ and $P \supseteq A$. Thus $AQ \subseteq P \cap Q = 0$ which contradicts the assumption that $A \in L_0(R)$ (i.e., A satisfies (A) of §2).

We can follow now the proofs of [1] Lemma 14 and show:

LEMMA 3. *If S is as above then there is an embedding of Γ into the center of S which contains C.*

Proof. Let $\alpha: A \to R$, $A \in L_0(R)$ be a representative of an element $\bar{\alpha} \in \Gamma$. First we show that there is a unique element $c_\alpha \in C$

depending on $\bar{\alpha}$ (and not on the representative α) such that $a\alpha = ac_\alpha$ for every $a \in A$. Next we prove that the correspondence: $\bar{\alpha} \to \delta_\alpha$ is the required embedding. The proof follows the proof of [1] Lemma 14.

Since $A \in L_0(R)$, it follows by Lemma 2 that $AC = S$ and hence $1 = \Sigma a_i c_i$ for some $a_i \in A$ and $c_i \in C$. Set $c_\alpha = \Sigma(a_i\alpha)c_i$. Since $\bar{\alpha} \in \Gamma$, α is a bi-R hence for every $a \in A$:

$$a\alpha = (a\alpha)1 = \Sigma(a\alpha)a_i c_i = \Sigma(aa_i)\alpha c_i = a\Sigma(a_i\alpha)c_i = ac_\alpha.$$

To prove that $c_\alpha \in C$, we observe that for every $a \in A$ and $x \in R$: $(ax)c_\alpha = (ax)\alpha = (a\alpha)x = ac_\alpha x$. Hence, $a(xc_\alpha - c_\alpha x) = 0$. Consequently, $S(xc_\alpha - c_\alpha x) = (CA)\ (xc_\alpha - c_\alpha x) = 0$ and since $1 \in S$ it follows that $xc_\alpha - c_\alpha x = 0$ for every $x \in R$, i.e. $c_\alpha \in C$.

The element c_α which belongs to C, actually commutes also with the elements of R and hence belongs to the center of S. Indeed, let $c \in C$ and $a \in A$ then since C centralizes A we have $(a\alpha)c = c(a\alpha)$ as $a\alpha \in R$. Also $a\alpha = ac_\alpha = c_\alpha a$ and, therefore:

$$c_\alpha(ac) = (ac_\alpha)c = (a\alpha)c = c(a\alpha) = (ca)c_\alpha = (ac)c_\alpha.$$

That is, c_α commutes with all the elements of $AC = S$, and this means that c_α is in the center of S.

Next we show that c_α depends only on $\bar{\alpha} \in F$: let $\beta: B \to R$ be another representative of $\bar{\alpha}$ then $\alpha = \beta$ on some $D \subseteq A \cap B$ which belongs to $L_0(R)$. Hence for $d \in D$: $dc_\alpha = d\alpha = d\beta = dc_\beta$, which implies that $D(c_\alpha - c_\beta) = 0$ and therefore $S(c_\alpha - c_\beta) = (CD)\ (c_\alpha - c_\beta) = 0$ which yields $c_\alpha - c_\beta = 0$.

Finally $c_{\alpha+\beta} = c_\alpha + c_\beta$, $c_{\alpha\beta} = c_\alpha c_\beta$ since for some ideals in $L_0(R)$ we have the following relations for their elements:

$$xc_{\alpha+\beta} = x(\alpha+\beta) = x\alpha + x\beta = xc_\alpha + xc_\beta = x(c_\alpha + c_\beta)$$

$$yc_{\alpha\beta} = y(\alpha\beta) = (y\alpha)\beta = (y\alpha)c_\beta = y(c_\alpha c_\beta)$$

and as in preceding proofs this implies that $c_{\alpha+\beta} = c_\alpha + c_\beta$ and $c_{\alpha\beta} = c_\alpha c_\beta$.

We, henceforth, identify Γ with its image in C and thus we may assume that $\Gamma \subseteq C$.

LEMMA 4. *Let $S = RC = S_1 \oplus \cdots \oplus S_n$, S_i simple with unit ϵ_i, be a central extension of R of minimal type, then $\epsilon_i \in \Gamma$.*

For let $P = \{r \in R,\ r\epsilon_i = 0\}$ and $Q = \{r \in R,\ r(1-\epsilon_i) = 0\}$. Since S of minimal length, $P \neq 0$, $Q \neq 0$ and $P \cap Q - 0$. We first assert that $P + Q \in L_0(R)$ and, indeed, $(QC)\epsilon_i = (Q\epsilon_i)C = QC = QRC = QS =$

$Q \neq 0$ and so $QC \subseteq S_i$ but QC is and ideal in S and therefore, also in S_i which yields $QC = S_i$ since S_i is simple. A similar proof which uses the fact that $P\epsilon_j \neq 0$ for $j \neq i$ shows that $(PC)\epsilon_j = S_j$. Hence

$$(P+Q)C = \Sigma(P+Q)C_k = \Sigma S_k = S$$

and thus $P + Q \in L_0(R)$ by Lemma 1. Consider now the map ϵ: $P + Q \to Q$ given by $(p+q)\epsilon = q$. Clearly, this is a bi-R-homomorphism, hence $\bar{\alpha} \in \Gamma$ and so there exists $c_\epsilon \in C$ such that $(p+q)c_\epsilon = q$. Consequently, $(p+q)c_\epsilon = q = q\epsilon_i = (p+q)\epsilon_i$. By the uniqueness of c_ϵ it follows that $c_\epsilon = \epsilon_i$

We are now in position to prove the main theorems.

R is semi-prime, for if $A^2 = 0$then $(AC)^2 =$ in S, but S is semi-prime and so $AC = 0$ which implies that $A = 0$.

Let $S = RC = S_1 \oplus \cdots \oplus S_n$ be a central extension of R of minimal length, with ϵ_i the units of S_i. Put $P = \{r \in R, r\epsilon_1 = 0\}$, and consider R as a subring of $Q_0(R)$. Then we readily have, since $\epsilon_1 \in \Gamma \subseteq Q_0(R)$ that $P = R \cap Q_0(R)\ (1 - \epsilon_1)$. Furthermore, P is a prime ideal: indeed let $AB \subseteq P$ with A, B ideals in R containing P, then since $B \not\subseteq P$, $B\epsilon_1 \neq 0$ and, therefore, $(BC)\epsilon_1$ is a nonzero ideal in S_1 which implies that $BC\epsilon_1 = S_1$. Thus:

$$0 = (CP)\epsilon_1 \supseteq (CAB)\epsilon_1 = A(CB)\epsilon_1 = AS_1.$$

This yields that $A\epsilon_1 = 0$ and so $A \subseteq P$. We can now apply **[1]** Theorem 8, which in our case means that $Q_0(R/P) \cong Q_0(R)\epsilon_1$ and $\Gamma(R/P) \cong \Gamma(R)\epsilon_1 = \Gamma\epsilon_1$.

Denote, $R_1 = R\epsilon_1$ (which isomorphic with R/P) and $c_1 = c\epsilon_1$ then $RC\epsilon_1 = R_1C_1 = S_1$ which shows that R_1 is a prime ring with a central extension which is a simple ring S_1 with a unit. It follows, therefore, by **[1]** Theorem 18 that $R_1\Gamma(R_1)$ is simple with a unit. Now $\Gamma(R_1) = \Gamma(R/P) = \Gamma\epsilon_1$ by the preceding result. So $R_1(\Gamma\epsilon_1)$ is simple with a unit and note also that $R_1\Gamma\epsilon_1 = (R\Gamma)\epsilon_1$. The same follows for all the other idempotents ϵ_i and so we get that $R\Gamma = R\Gamma\epsilon_1 + R\Gamma\epsilon_2 + \cdots + R\Gamma\epsilon_n$ is a direct sum of simple rings with units, which completes the proof of Theorem A.

The proof of Theorem B follows the same lines by applying the second part of **[1]** Theorem 18 which was quoted in the present note (§2). Namely, if S is semi-simple artinian then each summand S_i is simple artinian and hence, by **[1]** Theorem 18 $R_1\Gamma_1 = (R\epsilon_1)(\Gamma\epsilon_1) = (R\Gamma)\epsilon_1$ is simple artinian. Furthermore, we also have $R_1\Gamma_1 = Q_0(R/P) = Q_0(R)\epsilon_1$ by (iii) of **[1]** Theorem B. Thus, $Q_0(R) = \Sigma Q_0(R)\epsilon_i = \Sigma R_i\Gamma_i = R\Gamma$.

Finally, $(RC)\epsilon_i = R_i\Gamma_i \otimes_{\Gamma_i} C\epsilon_i$ for every i, from which it follows that:

$$RC = \Sigma RC\epsilon_i = \Sigma R\Gamma_i \underset{\Gamma_i}{\otimes} C\epsilon_i \cong R\Gamma \underset{\Gamma}{\otimes} C$$

since $\Gamma = \Sigma\Gamma\epsilon_i$ and the elements ϵ_i belong to the center of $S = RC$. The last isomorphism is given by the mappings $r\alpha \otimes c \to \Sigma(r\alpha)\epsilon_i \otimes_{\Gamma_i} c\epsilon_i$; $r\alpha_i \otimes_{\Gamma_i} c\epsilon_i \to r\alpha_i \otimes_{\Gamma} c\epsilon_i$.

Corollary C follows now immediately by Theorem 6 and Corollary 13 of **[1]**.

We finish with an immediate corollary of the fact that $\Gamma \subseteq$ Cents S, and Cent $S \subseteq C$:

COROLLARY D. *If RC is a central embedding of R in a direct sum of simple rings of minimal length, then so is* R(Cent C).

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THE HEBREW UNIVERSITY OF JERUSALEM