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HOMOTOPY INVARIANCE OF CONTRAVARIANT FUNCTORS ACTING ON SMOOTH MANIFOLDS

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It is sometimes possible to prove that a functor is homotopy invariant using only a knowledge of the domain and range categories of the functor. It is known, for example, that every covariant or contravariant functor from the category of simplicial complexes (with continuous mappings) to the category of countable groups is homotopy invariant. This result has been extended to covariant, but not contravariant, functors with domain the category of smooth manifolds. In the contravariant case, the proof breaks down because certain mappings are not differentiable. This fault will be corrected in this paper. Among other results, it will be shown that every contravariant functor from the category of smooth manifolds to the category of countable groups is homotopy invariant.

The results mentioned above are proved in [4]. As in [4], we will use the word "cofunctor" to mean a contravariant functor. $\mathscr{C}$ will denote any full subcategory of the category of smooth manifolds which contains the real line $\boldsymbol{R}$ and is closed under the operation product-with-$\boldsymbol{R}$. $\mathscr{G}$ will denote any subcategory of the category of sets in which every object is countable. Let $C^\infty(\boldsymbol{R}, \boldsymbol{R})$ denote the monoid of smooth mappings from $\boldsymbol{R}$ to $\boldsymbol{R}$ under composition. Let D denote the monoid dual to $C^\infty(\boldsymbol{R}, \boldsymbol{R})$. In light of paragraphs 15 and 16 of [4], Theorem 11 of [4] may be restated as follows:

THEOREM 2. *If D cannot act faithfully on any countable set, then every cofunctor $\Delta\colon \mathscr{C} \to \mathscr{G}$ is homotopy invariant.*

The revised approach.

3. Suppose that D acts faithfully on a set B. We will prove that B is uncountable. Let I denote the closed interval $[0, 1]$. For each $x \in I$, let P_x be the set of all $p \in D$ such that the following two conditions are satisfied:

4. $w \in (x, 1) \Rightarrow p(w) \in (x, 1)$
5. $w \notin (x, 1) \Rightarrow p(w) = w$.

It is easy to verify that:

6. If $p' \in P_w$, $p \in P_x$, and $w \leqq x$, then $pp' \in P_w$.
7. Observe that every subset of I has a greatest lower bound in I. Hence we may define, for each $b \in B$, a number $\lambda(b) \in I$ which is

the greatest lower bound of $\{x \in I \mid p \in P_x \Rightarrow pb = b\}$.

THEOREM 8. *Consider $x \in I$ and $b \in B$ with $x \neq \lambda(b)$. Then $\lambda(b) < x \Leftrightarrow pb = b$ for all $p \in P_x$.*

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) If $\lambda(b) < x$, there exists $w \in I$ such that $\lambda(b) \leqq w < x$ and $p'b = b$ for all $p' \in P_w$. Consider any $p \in P_x$. By 6, we have $pp' \in P_w$. Hence $pp'b = b$, and so $pb = b$.

9. Let F denote the set of all $f \in D$ such that $w \notin (0, 1)$ implies $f(w) = w$ and such that f has an inverse in D. Recall that the order of composition is reversed in D. Given $p \in P_x$ and $f \in F$, it is easy to verify that $f^{-1}pf \in P_{f(x)}$. Hence, for fixed $f \in F$, we have a mapping $p \mapsto f^{-1}pf$ from P_x to $P_{f(x)}$. This mapping has an inverse, namely $p \mapsto fpf^{-1}$. So:

10. Given $x \in I$ and $f \in F$, a one-to-one correspondence between P_x and $P_{f(x)}$ is provided by $p \mapsto f^{-1}pf$.

THEOREM 11. *For any $f \in F$ and $b \in B$, $f^{-1}(\lambda(b)) = \lambda(fb)$.*

Proof. Consider $x \in I$ such that x is not equal to $f^{-1}(\lambda(b))$ or $\lambda(fb)$. It suffices to prove that $f^{-1}(\lambda(b)) < x \Leftrightarrow \lambda(fb) < x$. Our approach is as follows:

$$\begin{gathered}
f^{-1}(\lambda(b)) < x \\
\Big\Updownarrow \text{(i)} \\
\lambda(b) < f(x) \\
\Big\Updownarrow \text{(ii)} \\
p'b = b, \text{ for all } p' \in P_{f(x)} \\
\Big\Updownarrow \text{(iii)} \\
f^{-1}pfb = b, \text{ for all } p \in P_x \\
\Big\Updownarrow \text{(iv)} \\
pfb = fb, \text{ for all } p \in P_x \\
\Big\Updownarrow \text{(v)} \\
\lambda(fb) < x
\end{gathered}$$

(i) follows from the fact that f and f^{-1} preserve order. Note that since $x \neq f^{-1}(\lambda(b))$, $f(x) \neq \lambda(b)$. So (ii) follows from Theorem 8. (iii) follows from 10. (iv) is obvious, and (v) follows from Theorem 8.

Theorem 11, restated categorically as in [4], asserts that λ is a natural transformation.

THEOREM 12. *There exists $b \in B$ with $0 < \lambda(b) < 1$.*

Proof. There exists $r \in D$ such that $r(1/2) = 1/2$ and $r(w) = 3/4$ for all $w \in [3/4, 1]$. Note that $pr = r$ for all $p \in P_{3/4}$. Hence, for any $b \in B$, $prb = rb$ for all $p \in P_{3/4}$. So $\lambda(rb) \leqq 3/4$ for any $b \in B$.

There also exists $p' \in P_{1/4}$ such that $p'(1/2) = 3/4$. Since $r(p'(1/2)) = 3/4$ and $r(1/2) = 1/2$, we have $p'r \neq r$. Since D acts faithfully on B, there exists $b \in B$ such that $p'rb \neq rb$. And since $p' \in P_{1/4}$, $\lambda(rb) \geqq 1/4$ for this b.

In summary, we have found $b \in B$ such that $1/4 \leqq \lambda(rb) \leqq 3/4$.

THEOREM 13. *λ maps B onto* $(0, 1)$.

Proof. By Theorem 12 there exists $b \in B$ with $\lambda(b) \in (0, 1)$. Given any $x \in (0, 1)$, there exists $f \in F$ such that $f^{-1}(\lambda(b)) = x$. Then, by Theorem 11, $\lambda(fb) = x$.

COROLLARY 14. *B is uncountable.*

And by Theorem 2:

COROLLARY 15. *Every cofunctor $\Delta: \mathscr{C} \to \mathscr{G}$ is homotopy invariant.*

This proves, for example, that every cofunctor from smooth manifolds to countable groups is homotopy invariant.

Categories without $\boldsymbol{R}$.

16. Until now, we have been assuming that $\boldsymbol{R}$ was an object in $\mathscr{C}$. Hence we cannot at present apply Corollary 15 to the category of compact smooth manifolds. Let us correct this problem. As was noted in paragraphs 15 and 16 of [4], we may use the circle S^1 to take the place of $\boldsymbol{R}$. Viewing S^1 as the closed interval $[-1, 2]$ with end points identified, we may define P_x to be the set of all smooth mappings from S^1 to S^1 satisfying 4 and 5. Likewise, we may replace $\boldsymbol{R}$ by S^1 in 6 through 15 without problems. So Corollary 15 applies to any full subcategory $\mathscr{C}$ of the category of smooth manifolds which contains S^1 and is closed under the operation product-

with-S^1.

Other generalizations.

17. If we replace the word "smooth" by "continuous" everywhere in this paper, the proofs remain valid. We have spoken only of smooth structures because our results were already established for continuous structures [4]. Likewise, the proofs given here can be applied to functors as well as cofunctors, with only slight modification. Thus neither $\mathrm{Top}(I, I)$, $\mathrm{Top}(\boldsymbol{R}, \boldsymbol{R})$, $\mathrm{Top}(S^1, S^1)$, $C^\infty(\boldsymbol{R}, \boldsymbol{R})$, $C^\infty(S^1, S^1)$, nor their duals can act faithfully on a countable set.

Conclusion.

List A. simplicial complexes
topological manifolds
topological manifolds with boundary
compact topological manifolds
compact topological manifolds with boundary
smooth manifolds
smooth manifolds with boundary
compact smooth manifolds
compact smooth manifolds with boundary
pairs in any category above

List B. countable groups
countable rings
countable dimensional vector spaces over a field K
countable dimensional algebras over a field K

18. Putting together the results of this paper and [4], we have shown that every functor or cofunctor from a category in List A to a category in List B is homotopy invariant.

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