PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

GROUPS OF $*$-AUTOMORPHISMS AND INVARIANT MAPS OF VON NEUMANN ALGEBRAS

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Let M be a von Neumann algebra and let G be a group acting on M by $*$-automorphisms of M. M is G-finite if for every nonnegative element a in M with $a \neq 0$, there exists a G-invariant normal state ϕ such that $\phi(a) \neq 0$. The main result in this paper asserts that M is G-finite if and only if for every weakly relatively compact subset K of the predual of M, the orbit of K under G is also weakly relatively compact.

Given a noncommutative dynamical system, that is, pairs (M, G) where M is a von Neumann algebra and G is a group of $*$-automorphisms of M, one can ask whether or not there are sufficiently many G-invariant normal states (we call such a case that (M, G) is G-finite [9])?

First result along these lines is due to I. Kovacs and J. Szücs [9] who obtained that (M, G) is G-finite if and only if there is a G-invariant faithful normal projection of norm one from M onto the fixed subalgebra M^G under G (see also [11, 14]).

Recently, using results of Akemann [1] and Takesaki [15] concerning the predul of a von Neumann algebra, together with the Ryll-Nardzewski fixed point theorem ([5, 10]), F. J. Yeadon gave an elegant proof of the existence of a trace in a finite von Neumann algebra [16].

In this paper, we will give a Banach space like characterization of the G-finiteness of (M, G) using weakly relatively compact subsets of the predual M_* of M which is a noncommutative extension of a theorem of Hajian and Kakutani ([7, 8]) and in case where G is the inner automorphisms of M, includes the result of F. J. Yeadon (see also [16]). The result in this paper can be easily extended to groups of identity preserving isometries of M.

2. Notations and a statement of a theorem. Let (M, G) be a noncommutative dynamical system and M_* be the predual of M, that is, the Banach space of all bounded normal (or σ-weakly continuous) linear functionals on M([3, 12]). Let $(T_g\varphi)(a) = \varphi(a^g)$, $a \in M$, $g \in G$ and $\varphi \in M_*$, then T_g is a linear isometry of M_* onto M_*. We say that (M, G) is G-finite if M has sufficiently many normal states in the sense that for every nonnegative element a in M with $a \neq 0$, there exists a G-invariant normal state ϕ(that is, $T_g\phi = \phi$, $g \in G$) such that $\phi(a) \neq 0$.

Now we state our main theorem.

THEOREM. *Let (M, G) be a noncommutative dynamical system, then (M, G) is G-finite if and only if for every weakly relatively compact* (w.r.c.) *subset K of M_*, the orbit of K under G, that is, the set $\{T_g\phi; g \in G, \phi \in K\}$ is also* w.r.c.

3. **Proof of Theorem.** "If" part of Theorem is valid under a weaker assumption, more precisely to say that if for every ϕ in M_* with $\phi \geqq 0$, $\{T_g\phi; g \in G\}$ is w. r. c., then (M, G) is G-finite. However, this is an easy consequence of lemma in [14] (see also [11]). To prove the converse, we need the following lemma which concerns with the continuity of the map $(\Phi, \omega) \rightarrow \omega\circ\Phi$ from $L_*(M) \times M_* \rightarrow M_*$ where $L_*(M)$ is the σ-weakly continuous bounded linear maps of M into M equipped with the weak operator topology and M_* has the W^*-topology. For the later discussions, we state it in the following form.

LEMMA 1. *Let N be a von Neumann algebra with a set H of normal $*$-homomorphisms of N into N. Suppose that for every ϕ N_** (the predual of N) *with $\phi \geqq 0$, and every sequence $\{b_n\}$ in the nonnegative part of the unit sphere S of N such that $b_n \rightarrow 0$* (σ-weakly), *$\phi(\Phi(b_n)) \rightarrow 0 (n \rightarrow \infty)$ uniformly for $\Phi \in H$. Let $\{\phi_n\}$ be a sequence in N_* which converges weakly to some ϕ_0 in N_* and $\{a_n\}$ be a sequence of self-adjoint element in S which converges strongly to 0, then $\phi_j(\Phi(a_n)) \rightarrow 0 (n \rightarrow \infty)$ uniformly not only for $\Phi \in H$ but also for j.*

Proof. Observe first that the σ-weak topology restricted on S is a compact Hausdorff topology with the neighborhood basis which consists of all possible sets $\{a; a \in S, |\psi_i(a) - \psi_i(a_0)| < \varepsilon, i = 1, 2, \cdots, n\}$ with $a_0 \in S$, $\varepsilon > 0$ (real number) and $\psi_i \in N_*(\psi_i \geqq 0)$. Let $H_i = \{a \in S; |(\phi_j - \phi_0)(a)| \leqq \varepsilon$ for all $j \geqq i\}$, then H_i is σ-weakly closed subset of S for each i and $S = \bigcup_{i=1}^\infty H_i$. Now Baire's category theorem says that there are a natural numbers $i(0)$, m, an element a_0 in S and $\psi_i (i = 1, 2, \cdots, m)$ in N_* with $\psi_i \geqq 0$ for all i such that

$$\bigcap_{i=1}^{m} \{a; a \in S; |\psi_i(a) - \psi_i(a_0)| < 1\} \subset H_{i(0)} .$$

Since $a_n \rightarrow 0 (n \rightarrow \infty)$ strongly, by the spectral theorem, for any given positive number ε, there is a sequence $\{e_n\}$ of projections in M such that $e_n \rightarrow 1$ (strongly) and $||a_ne_n|| \leqq \varepsilon/6$ for each n. By the uniform boundedness theorem, we may assume that $\text{Sup}_j\{||\phi_j||, ||\phi_0||\} = 1$ without loss of generality. For each $\Phi \in H$, we have $||\Phi(e_na_ne_n)|| \leqq ||a_ne_n|| \leqq \varepsilon/6$, $||\Phi(e_na_n(1 - e_n))|| \leqq ||a_ne_n|| \leqq \varepsilon/6$ and $||\Phi((1 - e_n)a_ne_n))|| \leqq$

$\| a_n e_n \| \leqq \varepsilon/6$ for each n. Thus we have

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(a_n))| &\leqq |(\phi_j - \phi_0)(\Phi(e_n a_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi(e_n a_n (1 - e_n)))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n (1 - e_n)))| \\ &\leqq \varepsilon + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n (1 - e_n)))| \,. \end{aligned}$$

Put $b_n(\Phi) = \Phi((1 - e_n)a_n(1 - e_n)) + \Phi(e_n)a_0\Phi(e_n)$, then, since $b_n(\Phi) - a_0 = (1 - \Phi(e_n))\Phi(a_n)(1 - \Phi(e_n)) - (1 - \Phi(e_n))a_0\Phi(e_n) - \Phi(e_n)a_0(1 - \Phi(e_n)) - (1 - \Phi(e_n))a_0(1 - \Phi(e_n))$, we have, by Schwarz' inequality,

$$|\psi_j(b_n(\Phi) - a_0)| \leqq \psi_i(\Phi(1 - e_n)) + 3 \| \psi_i \| \psi_i(\Phi(1 - e_n))^{1/2} \,.$$

Similarly, we have

$$|\psi_j(\Phi(e_n)a_0\Phi(e_n) - a_0)| \leqq \psi_i(\Phi(1 - e_n)) + 2 \| \psi_i \| \psi_i(\Phi(1 - e_n))^{1/2} \,.$$

Since, by the assumption, $\psi_i(\Phi(1 - e_n)) \to 0 (n \to \infty)$ uniformly for $\Phi \in H$ and $i = 1, 2, \cdots, m$, we can choose a natural number $n(\varepsilon)$ (depends only on ε) such that $b_n(\Phi), \Phi(e_n)a_0\Phi(e_n) \in H_{i(0)}$ for all $n \geqq n(\varepsilon)$. Thus, we have

$$|(\phi_j - \phi_0)(\Phi((1 - e_n)a_n(1 - e_n)))| < 2\varepsilon$$

for all $j \geqq i(0)$, all $\Phi \in H$ and all $n \geqq n(\varepsilon)$. Since, for each $j (j = 1, 2, \cdots, i(0) - 1)$

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(a_n))| &= | \, |\phi_j - \phi_0| (\Phi(a_n)v_j)| \\ &\leqq \{|\phi_j - \phi_0| (\{\Phi(a_n)\}^2)\}^{1/2} \| \phi_j - \phi_0 \|^{1/2} \\ &\leqq 2^{1/2}\{|\phi_j - \phi_0| (\Phi(a_n^2))\}^{1/2} \end{aligned}$$

and

$$|\phi_0(\Phi(a_n))| = | \, |\phi_0| (\Phi(a_n)v)| \leqq \{|\phi_0| (\Phi(a_n^2))\}^{1/2}$$

where $\phi_j - \phi_0 = R_{v_j} |\phi_j - \phi_0|$ (resp. $\phi_0 = R_v |\phi_0|$) is the polar decomposition of $\phi_j - \phi_0$ (resp. ϕ_0) ([12]), $a_n^2 \to 0$ weakly implies, by the assumption, that there is a positive integer $n(\varepsilon)'$ (depending only on ε) such that $|(\phi_j - \phi_0)(\Phi(a_n))| < \varepsilon$ and $|\phi_0(\Phi(a_n))| < \varepsilon$ for all $\Phi \in H$, $j = 1, 2, \cdots, i(0) - 1$ and all $n \geqq n(\varepsilon)'$.

Combining the above estimations, we have

$$|\phi_j(\Phi(a_n))| < 4\varepsilon \quad \text{for all} \quad n \geqq \max(n(\varepsilon), n(\varepsilon)') \,, \quad \text{all} \quad j$$

and all $\Phi \in H$. This completes the proof of Lemma 1.

Before going into the proof of theorem, we prepare the following:

LEMMA 2. *Keep the notations in theorem. If (M, G) is G-finite, then, for every sequence $\{a_n\}$ of nonnegative elements in the unit sphere S of M which converges weakly to 0, and every ϕ in M_*, $(T_g\phi)(a_n) \to 0$ uniformly for $g \in G$.*

Proof. If not, there exists a positive number ε_0 such that for each positive integer n, we can choose a positive integer $k(k(n) \uparrow \infty)$ and $g(n) \in G$ such that

$$(*) \qquad | T_{g(n)}\phi(a_{k(n)}) | \geqq \varepsilon_0 .$$

Put $a_{k(n)} = b(n)$ then since $\{(b(n)^{g(n)}\}$ is a σ-weakly relatively compact subset of $S \cap M^+$ (where M^+ is the positive portion of M), there is a σ-weakly cluster point $a(a \geqq 0)$ of $\{(b(n)^{g(n)}\}$. Thus for every positive number δ, every G-invariant normal state ρ and every positive integer n, there is a natural number $i(n)(i(n) > n$ and $i(n) \uparrow \infty)$ such that

$$| \rho(a) - \rho(b(n))^{g(i(n))}) | < \delta \qquad n = 1, 2, \cdots .$$

Since ρ is G-invariant, $\rho((b(i(n))^{g(i(n))}) = \rho(b(i(n))) \to 0(i(n) \to \infty)$. Thus $|\rho(a)| \leqq \delta$ for every δ and the G-finiteness of (M, G) implies $a = 0$. Hence this contradicts with the inequality $(*)$. Thus $(T_g\phi)(a_n) \to 0(n \to \infty)$ uniformly for $g \in G$ and the proof is completed.

Proof of Theorem. Suppose (M, G) is G-finite. We will prove that for every w.r.c. subset K of M_*, $\{T_g\phi; \phi \in Kg \in G\}$ is also w.r.c. To prove this, we have only to show that for every orthogonal sequence $\{e(n)\}$ of projections, $\lim_{n\to\infty} T_g\phi(e(n)) = 0$ uniformly for $g \in G$ and $\phi \in K$. If not, there is a positive number ε such that for each positive integer k, there are a natural number $n(k)(n(k) \uparrow \infty)$, $g(k) \in G$ and $\phi_k \in K$ such that

$$(**) \qquad | T_{g(k)}\phi_k(e(n(k))) | \geqq \varepsilon .$$

By Eberlein-Šmulian's theorem ([4]), there is a subsequence $\{\phi_{k(p)}\}$ of $\{\phi_k\}(k(p) \uparrow \infty)$ such that $\phi_{k(p)} \to \phi_0$ weakly $(p \to \infty)$ for some ϕ_0 in M_*. Now $e(n(k(p))) \to 0(p \to \infty)$ strongly, which implies by Lemma 2 and Lemma 1, that $| T_{g(k(p))}\phi_{k(p)}(e(n(k(p))) | \to 0(p \to \infty)$ and this contradicts with the inequality $(**)$. This completes the proof of theorem.

4. Remarks. Theorem is a generalization of [11]. We should remark that the result of theorem can be easily extended to groups of Jordan Automorphisms of M. [13] When G is a semi-group of normal Jordan homomorphisms ([13]) of M into M, by an easy modification of Lemma 1 and Lemma 2, "only if" part of theorem is valid,

however, as the following example shows, the converse assertion does not hold in general, even if G is a semi group of $*$-isomorphisms of M into M.

Let $M = L^\infty([0, 1))$ be the abelian von Neumann algebra of essentially bounded complex-valued functions on $[0, 1)$ with respect to Lebesque measure μ. Let us consider two measurable transformations g_1 and g_2 defined as follows ([2, 8]): $g_1(\omega) = 3\omega(\text{mod } 1)$, $\omega \in [0, 1)$, $g_2(\omega) = 2\omega + 1/3$(resp. $= (\omega - 1/3)/2$, $\omega \in [0, 1/3)$(resp. $\omega \in [1/3, 1)$). For each $f \in M$, let $(\Phi_1 f)(\omega) = f(g_1\omega)$, $\omega \in [0, 1)$ and $(\Phi_2 f)(\omega) = f(g_2\omega)$, $\omega \in [0, 1)$. Let H be the semi-group of normal $*$-homomorphisms of M into M generated by Φ_1 and Φ_2. Then by [2] and [8], we can easily check that for each $\phi \in M_*(= L^1([0, 1)))$, $\{\phi\circ\Phi, \Phi \in H\}$ is w. r. c.. Thus by [6] and Lemma 1, for every w. r. c. subset K of M_*, $\{\phi\circ\Phi, \Phi \in H, \phi \in K\}$ is also w. r. c. However, since g_1 is ergodic with respect to μ and μ is not invariant under g_2, (M, H) has no H-invariant functionals in M_*.

The above example implies that the Ryll-Nardzewski fixed point theorem is not valid in general without the assumption of distal action of H.

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Received March 4, 1974 and in revised form March 3, 1975. The author is partially supported by The Sakkokai Foundation.

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