PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

LATTICES OF HAUSDORFF COMPACTIFICATIONS OF A LOCALLY COMPACT SPACE

V. KANNAN AND T. THRIVIKRAMAN

This paper gives a lattice theoretic characterization of (complete) lattices which are lattices of Hausdorff compactifications of locally compact spaces. This is accomplished via a characterization of the lattices of closed equivalence relations on T_1 spaces.

NOTATIONS. L is a complete dually atomic lattice.
D is the set of all dual atoms of L.

1. DEFINITION. Let $p \in L$. Then the set $H(p) = \{d \in D | d \geqq p\}$ is called the hull of p.

NOTE. $H(1) = \varnothing$.

2. DEFINITION. Let $E \subset D$. The kernel of E denoted by Ker E is defined as $\bigwedge_{d \in E} d$.

3. DEFINITION. Let $p \in L$. p is said to be a primary element if whenever q and r are two elements in $H(p)$ such that Card. $H(q \wedge r) \neq 3$ and $s \in D$ is such that Card. $H(s \wedge q) = 3 =$ Card. $H(s \wedge r)$, it is true that $s \in H(p)$.

NOTE. Trivially, 1 as well as any dual atom is primary.

4. DEFINITION. A star of L is defined as a subset S of D which is maximal with respect to the following property: $d, d' \in S \Rightarrow d \wedge d'$ is a primary element and if (d, d') and (d_1, d_2) are distinct pairs of elements of S, then $d \wedge d' \neq d_1 \wedge d_2$.

5. DEFINITION. Let p, q be primary elements. Then a primary rectangle is defined as $H(p, q) = \{(S_1, S_2) | S_1, S_2$ are stars such that $S_1 \cap H(p) \neq \varnothing \neq S_2 \cap H(q)\}$.

6. DEFINITION. Let $A \subset D$. $\alpha(A)$ is defined as the set of all unordered pairs (S, S') of stars such that $S \cap S' \cap A \neq \varnothing$.

7. THEOREM. *A complete, dually atomic lattice L is isomorphic to the lattice of closed equivalence relations of a T_1 space X if and*

only if the following are true:

(i) $\bigvee_{i \in J} p_i$ *is a primary element for any collection* $\{p_i | i \in J\}$ *of primary element in* L.

(ii) (a) *If* $d \in D$, *then* d *belongs to exactly two stars.*

(b) *Any two stars intersect in a singleton.*

(iii) $H \subset D$ *is a hull if and only if*

(a) *if* $d_1, d_2 \in H$ *and if* $d \in D$ *such that* $d \geqq d_1 \wedge d_2$, *then* $d \in H$.

(b) $\alpha(H)$ *is an intersection of finite unions of primary rectangles.*

(iv) $a = \mathrm{Ker}\,(H(a))$ *for every* $a \in L$.

Proof (Necessity). Easily checked, bearing in mind the discussion in §1 of [3].

(Sufficiency). Let X be the set of all stars in L. From (ii) (a) and (ii) (b), there exists a bijection θ between the set D of all dual atoms of L and the set of all unordered pairs of distinct stars.

From (i), and noticing that $0 \in L$ is a primary element, it follows that primary elements of L form a complete lattice P under the same order. Now D is precisely the set of all dual atoms of P. We can form the hull-kernel topology for D in the lattice. This topology can then be translated to X as follows:

A set $A \subset X$ is closed if and only if $\theta^{-1}(A \times A)$ is a hull of a member of P. We show now that this defines a T_1 topology on X. Clearly $\theta^{-1}(\varnothing \times \varnothing) = \varnothing$ and $\theta^{-1}(X \times X) = D$ so that $\varnothing$ and X are closed. If $S \in X$, then $\theta^{-1}(\{S\} \times \{S\}) = \varnothing = H(1)$ so that every singleton is closed. If $S \neq S'$, $S, S' \subset X$, $\theta^{-1}(\{S\} \times \{S'\})$ is a singleton. So any two-element set is closed.

Now let $A, B \subset X$ be closed, each containing at least two elements. Then $A \cup B$ is closed. For, let p and q be the primary elements determined by A and B respectively. Then consider the primary rectanglls $H(p, p)$, $H(q, q)$, $H(p, q)$, $H(q, p)$. Now if $C = \theta^{-1}((A \cup B) \times (A \cup B))$, then $\alpha(C)$ is the union of these primary rectangles. For, if $(S, S') \in \alpha(C)$, then there exists $d \in C$ such that $S \cap S' = \{d\}$; now $\theta(d) = (S, S') \in ((A \cup B) \times (A \cup B))$ so that $S, S' \in A \cup B$; it is easy to check that if S, S' both belong to A (respectively B), then $(S, S') \in H(p, p)$ (respectively $H(q, q)$). If $S \in A$ and $S' \in B$, then $(S, S') \in H(p, q)$ and if $S \in B$ and $S' \in A$, then $(S, S') \in H(q, p)$. On the other hand, that all these four primary rectangles are subsets of $\alpha(C)$ is easily verified.

Hence by condition (iii), C is a hull set. Note that condition (iii) (a) is satisfied here, by the maximality in the definition of stars.

Let K be the kernel of C. It can be seen that K is primary. It follows that $A \cup B$ is closed.

Let $A_i \subset X$ be closed for every $i \in J$ and let $A = \bigcap A_i$. Then

$\theta^{-1}(A \times A)$ is the intersection $\cap\, \theta^{-1}(A_i \times A_i)$ and so is a hull set. For, by condition (iii), intersection of hull sets is again a hull set. Let r be its kernel. Then by using condition (iv), it can be proved that $r = \vee p_i$ where p_i is the primary element corresponding to A_i. So it follows from condition (i) that r is primary. So A is closed. Thus X is a T_1 space.

Next we show that L is isomorphic to $L(X)$, the lattice of closed equivalence relations for this space X. Let us define a map $\eta: L \rightarrow L(X)$ as follows: If $x \in L$, $\eta(x)$ is the relation defined on X as below: $S_1\eta(x)S_2$ if and only if either $S_1 = S_2$ or $S_1 \cap S_2 \cap H(x) \neq \emptyset$. This is an equivalence relation. For, let $S_1\eta(x)S_2$ and $S_2\eta(x)S_3$; also let $S_1 \cap S_2 = \{d_3\}$, $S_2 \cap S_3 = \{d_1\}$, and $S_3 \cap S_1 = \{d_2\}$; then by the maximality in the definition of stars, it can be proved that $d_2 \geqq d_1 \wedge d_3$ and so by (iii) (a), $d_2 \in H(x)$ since $H(x)$ is a hull set. So $S_1\eta(x)S_3$. Thus transitivity is proved. Reflexivity and symmetry of $\eta(x)$ are trivial.

Now we show that $\eta(x) \in L(X)$. That is, $\eta(x)$ is a closed equivalence relation on X. The relation $\eta(x)$ viewed as a subset of $X \times X$ is precisely $\alpha(H(x))$. So by (iii)(b), it is an intersection of finite union of primary rectangles. Each primary rectangle is a closed subset of $X \times X$ since $H(p, q) = A \times B$ where A and B are the closed subsets of X determined by p and q respectively. So $\eta(x)$ is a closed subset of $X \times X$.

Thus $\eta: L \rightarrow L(X)$ is well defined.

Now η is injective. For, let $x \neq y$. Then by condition (iv) we get that $H(x) \neq H(y)$. Let $d \in H(x) - H(y)$ (say). Let S, S' be the two stars containing d. Then $(S, S') \in \alpha(H(x))$ but $\notin \alpha(H(y))$. So $\eta(x) \neq \eta(y)$. Futher η is surjective. For, let R be a closed equivalence relation on X. Then R is an intersection of finite unions of closed rectangles, whereas each closed rectangle is a primary rectangle. Let $H = \{d \in D \mid \text{the pair } (S_1, S_2) \text{ of stars containing } d \text{ belongs to } R\}$. Then H satisfies conditions (iii) (a) and (iii) (b) and hence by (iii), H is a hull set. If x is its kernel, it follows that $\eta(x) = R$.

η preserves arbitrary unions. For, let $x = \bigvee_\alpha x_\alpha$. Now $\cap\, H(x_\alpha)$ is a hull set by (iii). It ought to contain $H(x)$. Let y be its kernel. Then $y \geqq x_\alpha$ for each α. So $y \geqq x$. Therefore, $H(y) \subset H(x)$. So $\eta(y) = \eta(x)$. But $\eta(y) = \bigvee_\alpha \eta(x_\alpha)$ since it can be easily seen that $S_1 \bigvee_\eta(x_\alpha) S_2$ if and only if $S_1 \cap S_2 \cap (\cap\, H(x_\alpha)) \neq \emptyset$, while $\cap\, H(x_\alpha) = H(y)$.

η preserves intersections. This can be seen as in the case of unions, by considering the union of the hulls.

Thus η is an isomorphism.

Thus sufficiency is proved.

8. THEOREM. *Let L be a complete, dually atomic lattice. Then L is the lattice of T_2 compactifications of a locally compact space if*

and only if it satisfies conditions (i) *through* (iv) *above and also*:

(v) *Given any two primary elements* $p_1, p_2 \in L$, *there exist primary elements* $q_1, q_2 \in L$ *such that* $p_1 \vee q_1 = p_2 \vee q_2 = 1$ *and* $q_1 \wedge q_2 = 0$.

(vi) *Given any collection of primary elements* $\{p_\alpha\}_{\alpha \in J}$ *such that* $\bigwedge_{\alpha \in K} p_\alpha$ *is a primary element for any finite subset* K *of* J, *then* $\bigwedge_{\alpha \in J} p_\alpha$ *is also a primary element.*

Proof. Notice that (v) is equivalent to saying that the space X uniquely specified by L, is normal.

Also (vi) is equivalent to compactness.

It can be proved that the lattice of all closed equivalence relations of $\beta X - X$ is isomorphic to the lattice of all Hausdorff compactifications of X, when X is locally compact and Hausdorff.

Thus the theorem.

9. REMARK. When X is not locally compact, the upper semilattice $K(X)$ of all Hausdorff compactifications of X is not necessarily a lattice (cf. [5]). Now the problem arises whether the semilattice $K(X)$ determines the space $\beta X - X$. When X is locally compact, the answer is in the affirmative (cf. [1]). A method to construct the space $\beta X - X$ from $K(X)$ is given in [2]. But when x is not locally compact, $K(X)$ does not determine $\beta X - X$ (cf. [2]). A study in this direction forms a part of [4].

REFERENCES

1. K. D. Magill, Jr., *The lattice of compactifications of a locally compact space*, Proc. London Math. Soc., **18** (1968), 231-244.

2. T. Thrivikraman, *On compactifications of Tychonoff spaces*, Yokohama Math. J., **20** (1972), 99-106.

3. ———, *On Hausdorff quotients of spaces and Magill's theorem*, Monat. für-Math., **76** (1972), 345-355.

4. ———, *Studies in Compactifications and Other Extensions*, Doctoral Thesis submitted to Madurai University, July, 1971.

5. J. Visliseni and J. Flachsmeyer, *The power and structure of the lattice of all compactifications of a completely regular space*, Doklady Nauk SSSR (N. S.), **165** (1965), 1423-1425.

Received July 16, 1973 and in revised form April 30, 1974.

MADURAI UNIVERSITY, MADURAI-21
AND
MAR ATHANASTUS COLLEGE, KOTHAMANGALAM, KERALA, INDIA