

ON LEVI FACTORS OF DERIVATION ALGEBRAS AND THE RADICAL EMBEDDING PROBLEM

FRANCIS J. FLANIGAN

Problem: given a finite-dimensional nilpotent associative k -algebra N , find all unital associative k -algebras A such that $\text{rad } A = N$. **An approach:** which subalgebras of $\text{Der}_k N$ are images of Lie homomorphisms $A/N \rightarrow \text{Der}_k N$? Here the author constructs N over very general fields k such that the “Levi factor” of $\text{Der}_k N$ is a direct sum of orthogonal Lie algebras $\mathfrak{o}(V, b)$ of arbitrarily prescribed symmetric and alternate bilinear spaces. In particular, if k is algebraically closed of characteristic zero, then every direct sum of classical simple Lie algebras A_n, B_n, C_n, D_n is Levi factor of some $\text{Der}_k N$.

1. Some questions. We ask initially: What are the possible Levi factors (or semisimple semidirect summands or, more generally, semisimple subalgebras) of the Lie algebras $\text{Der}_k N$ of all k -derivations $N \rightarrow N$, where N is required to be a nilpotent associative algebra finite dimensional over a field k (not necessarily of characteristic zero, and possibly finite)?

This rather general question, to be sharpened below, was prompted by a certain approach to the following *radical embedding problem* posed in Marshall Hall’s [4] and in our [2], [3]:

1.1. Given a nilpotent N as above, describe the set of unital associative k -algebras A satisfying $\text{rad } A = N$ (together with a certain nondegeneracy condition [2]).

The approach referred to above is this: If the scalar field k is perfect, then each solution A to 1.1 admits a semidirect Wedderburn decomposition $A = S + N$, with S a separable semisimple subalgebra. Since N is a Lie ideal in A_{Lie} , the usual bracketing induces a Lie homomorphism $S_{\text{Lie}} \rightarrow \text{Der}_k N$ with “small” kernel (thanks to the nondegeneracy condition). Moreover, for reasonable fields k the algebra S_{Lie} is a direct sum of Lie ideals each of the form $\mathfrak{sl}(r_i, k) \oplus k$ for various ranks r_i . We conclude that a solution A to $\text{rad } A = N$ in (1.1) tends to force $\text{Der}_k N$ to contain various copies of the familiar special linear Lie algebra [3].

It was this last observation that led us ask the easily stated but far too general “survey” question of the first paragraph above. (Thus in characteristic $p > 0$, there need not be a Levi-Malcev decomposition, and moreover the theory of semisimple Lie algebras is

far from complete, even over algebraically closed fields.) In the present paper, we offer answers, adequate for our purposes, to these sharpened versions of the original question:

1.2. Is every special linear Lie subalgebra of $\text{Der}_k N$ induced in the above manner by embedding N in an associative $A = S + N$?

1.3. Are there nilpotent N which admit orthogonal or symplectic or other "classical" Lie algebras of derivations, apparently unrelated to the radical embeddings of N ?

2. **Results and consequences.** The point of the theorems below is the existence of nilpotent N whose derivation algebras have semi-direct sum decompositions with certain prescribed "wild" summands. In particular, the answers to the questions above are: to (1.2), *no* (in Theorem 2.1, obtain $sl(2, k)$ as the orthogonal Lie algebra of the alternate form $b((X_1, X_2), (Y_1, Y_2)) = X_1 Y_2 - X_2 Y_1$) and, to question 1.3, *yes*.

THEOREM 2.1. *Given any field k of characteristic not 2 and any array $(V_1, b_1), \dots, (V_m, b_m)$ of finite-dimensional nondegenerate symmetric or alternate bilinear k -spaces, there exists a finite-dimensional nilpotent associative k -algebra N such that*

- (i) *N is directly indecomposable,*
- (ii) *the derivation algebra $\text{Der}_k N$ is the semidirect sum $A + \Omega$ of a Lie subalgebra A and a nilpotent ideal Ω ,*
- (iii) *$A \cong o(V_1, b_1) \oplus \dots \oplus o(V_m, b_m)$,*
- (iv) *the ideal Ω consists of nilpotent derivations.*

In the above statement $o(V, b)$ is the orthogonal Lie algebra of the bilinear space (V, b) , that is,

$$o(V, b) = \{f \in \text{End}_k V \mid b(f(x), y) + b(x, f(y)) = 0, \text{ all } x, y \in V\}.$$

Note that we do not assert that $o(V, b)$ is always simple.

THEOREM 2.2. *Let k and (V_i, b_i) be as in (2.1), and suppose given integers $r_1, \dots, r_n \geq 2$, none divisible by the characteristic of k . Then there exists a finite-dimensional nilpotent associative k -algebra N such that*

- (i) *the derivation algebra $\text{Der}_k N$ is the semidirect sum $A + \Omega$ of a Lie subalgebra A and a solvable ideal Ω ,*
- (ii) *$A \cong (\bigoplus_{i=1}^m o(V_i, b_i)) \oplus (\bigoplus_{j=1}^n sl(r_j, k))$,*
- (iii) *a maximal toroidal subalgebra of Ω has dimension $n + 1$.*

COROLLARY 2.3. *Let k be an algebraically closed field of charac-*

teristic zero. Given any finite direct sum Σ of classical simple special linear, orthogonal, and symplectic Lie algebras (types A_n, B_n, C_n, D_n) over k , there is a finite-dimensional nilpotent associative k -algebra N such that the Levi factor of $\text{Der}_k N$ is isomorphic to Σ .

We will construct these N in §§3-5. Here are some further comments.

2.4. In the language of [3], Theorem 2.1 produces nilpotent N of genus zero, that is, nilpotents to which one can adjoin a unity element in the familiar way (cf. $A = k \cdot 1 + N$), but cannot adjoin any more complicated semisimple S . The direct indecomposability is essential here. These N of genus zero are noteworthy in that their derivation algebras are far from nilpotent. Compare Theorem 4.1 and the examples of §5 of [3].

2.5. Analogous constructions may be carried out for nilpotent Lie algebras N . In this case one may begin with the characteristically nilpotent Lie algebra of Dixmier-Lister [1] in place of the algebra W (§3) all of whose derivations are nilpotent. A special case of such a construction, involving one split simple three-dimensional Lie algebra, was effected by Leger and Luks in [6, Proposition 6.5].

We are pleased to acknowledge a helpful correspondence with George Leger on these questions, as well as our indebtedness to certain ideas in §6 of [6].

3. **Lemmas on nilpotent derivation algebras.** Let the arbitrary not necessarily associative k -algebra $B = I_1 \oplus \cdots \oplus I_n$ be a direct sum of (two-sided) ideals. We will now relate the derivation algebra of B to those of the I_i . Define the subspaces

$$\Delta_{ii} = \{D \in \text{Der}_k B \mid D(I_i) \subset I_i, D(I_h) = 0 \text{ for } h \neq i\},$$

and, for $i \neq j$,

$$\Delta_{ij} = \{D \in \text{Der}_k B \mid D(I_h) = D(I_j^2) = (0), h \neq j; D(I_j) \subset \text{Ann } I_i\},$$

where $\text{Ann } I_i$ is the two-sided annihilator of I_i in I_i .

The following is standard. Statement (ii) is Exercise 19, page 30 of Jacobson [5].

LEMMA 3.1. *Let $B = I_1 \oplus \cdots \oplus I_n$ as above. Then*

- (i) $\text{Der}_k B = \bigoplus_{i,j} \Delta_{ij}$ as a k -space;
- (ii) if each $I_i^2 = I_i$, then $\text{Der}_k B = \bigoplus_i \Delta_{ii}$, an ideal direct sum;
- (iii) if each $I_i^2 \supset \text{Ann } I_i$, then Δ_{ij} is an abelian ideal of $\text{Der}_k B$

when $i \neq j$;

(iv) if each $I_i^2 \supset \text{Ann } I_i$ and if each Δ_{ii} is a nilpotent Lie algebra, then $\text{Der}_k B$ is a nilpotent Lie algebra;

(v) in particular, if all derivations of I_i are nilpotent, then $I_i^2 \supset \text{Ann } I_i$, and all derivations of B are nilpotent.

We now go on to construct nilpotent associative algebras which admit only nilpotent derivations. Choose integers $\alpha, \beta \geq 3$ with $\alpha + \beta - \alpha\beta$ not divisible by $\text{char } k$. Let W be the k -algebra (without unity) on two generators u, v satisfying the relations

$$uv = 0, \quad u^\alpha = vu = v^\beta.$$

LEMMA 3.2. *The k -algebra W is finite-dimensional nilpotent associative with $\text{Ann } W \subset W^2$ and with all derivations nilpotent.*

Proof. Only the nilpotence of $D \in \text{Der}_k W$ needs to be checked. Since W is nilpotent, it suffices to prove $DW \subset W^2$. We have, for $a, b, c, d \in k$,

$$Du \equiv au + bv(\text{mod } W^2), \quad Dv \equiv cu + dv(\text{mod } W^2).$$

Now one checks that $0 = D(uv)$ forces $b = c = 0$. Having this, one checks that $Du^\alpha = D(vu) = Dv^\beta$ forces $\alpha a = a + d = \beta d$, whence $(\alpha + \beta - \alpha\beta)a = 0$ in k . It follows that $a = d = 0$, so that D is nilpotent as asserted. Done.

In one part of his thesis, James Malley pushes these matters further by examining the structural consequences of the hypothesis $\text{Der}_k N$ nilpotent [7, Chapter 5].

4. *Proof of Theorem 2.1.* First we construct N . Define $V_0 = W_1 \oplus \cdots \oplus W_t$, where each W_i is a copy of the algebra W of Lemma 3.2 and $t = s$ if $s = 1, 2$ and $t = s - 1$ if $s \geq 3$. (These choices will be justified below.) The underlying k -space of N is now defined to be $V_0 \oplus V_1 \oplus \cdots \oplus V_s$, and from now on we identify V_i with the corresponding subspace of N .

We multiply in N as follows: products in V_0 are as before, $V_i V_j = (0)$ for i, j distinct and, for x, y in V_i with $i \geq 1$, define $xy = b_i(x, y)z_i$ where z_i, \dots, z_s will now be chosen. Let w_j be a nonzero element of the (one-dimensional) $\text{Ann } W_j$. If $s = t = 1$, define $z_1 = w_1$. If $s = t = 2$, define $z_1 = w_1 + w_2$ and $z_2 = w_1 - w_2$. If $s \geq 3$, define $z_1 = w_1, \dots, z_{s-1} = w_{s-1}$, and $z_s = w_1 + \cdots + w_{s-1}$. We observe that N is nilpotent, that $N^2 = (V_0)^2$, and that $\text{Ann } N$ has k -basis w_1, \dots, w_t .

4.1. Now we check that N is indecomposable (assertion (i)). If $s = 1$, this is immediate because every nonzero ideal contains z_1 . Assume in general that N is an ideal direct sum $P \oplus Q$. On the one hand, we have $\text{Ann } N = P_0 \oplus Q_0$, where $P_0 = P \cap \text{Ann } N$ and likewise for Q_0 . On the other hand, the nondegeneracy of the bilinear form b_i implies that z_i is in either P_0 or Q_0 . (In the anomalous case $s = t = 2$, one also readily verifies that each w_i is in either P_0 or Q_0 .) Having this, and noting that any t or fewer of the z_i are linearly independent, one sees that the equation $t = \dim \text{Ann } N = \dim P_0 + \dim Q_0$ forces Q_0 (say) to be zero, whence $Q = (0)$. Thus N is indecomposable, as asserted.

Comment. Our choice of t was complicated by the possibility that the scalar field k is finite and "small". For k infinite and $s \geq 2$, taking $t = 2$ suffices.

Now we analyze $\text{Der}_k N$. For $i = 0, 1, \dots, s$, let $\pi_i: N \rightarrow N$ be the usual projection on the subspace V_i corresponding to the decomposition $N = V_0 \oplus \dots \oplus V_s$. In what follows, we take D in $\text{Der}_k N$ and define $D_{ij} = \pi_i \circ D \circ \pi_j$, a k -linear map (not *a priori* a derivation) $N \rightarrow N$, so that $D = \sum_{0 \leq i, j \leq s} D_{ij}$.

4.2. For $1 \leq i \leq s$, we prove $D_{i0}(N^2) = (0)$. Since $N^2 = (V_0)^2$, it suffices to consider $x, y \in V_0$. Then $D_{i0}(xy) = D(xy) - \sum_{j \neq i} D_{j0}(xy)$. But the left hand side here is in V_i and $D(xy)$ is in V_0 . Thus the right hand side has zero projection into V_i . Thus both sides are zero.

4.3. We prove that the restriction of D_{00} to V_0 is a derivation (and therefore nilpotent by (3.1) and (3.2)). For let $x, y \in V_0$. Then $D_{00}(xy) = D(xy) - \sum_{i \geq 1} D_{i0}(xy)$, and the result follows from (4.2) and the fact that $V_0 V_i = V_i V_0 = (0)$ for $i \neq 0$.

4.4. We prove that if $1 \leq i < j \leq s$, then $D_{ij} = D_{ji} = 0$. For let $x \in V_i, y \in V_j$. Then

$$0 = D(0) = D(xy) = (D_{ji}x)y + xD_{ij}y = b_j(D_{ji}x, y)z_j + b_i(x, D_{ij}y)z_i.$$

Since the z_i are pairwise linearly independent (consider our definition for $s = 2$), we have $b_j(D_{ji}x, y) = b_i(x, D_{ij}y) = 0$. But if $D_{ji}x \neq 0$, then the nondegeneracy of b_j implies there exists y in V_j such that $b_j(D_{ji}x, y) \neq 0$. Thus $D_{ji} = 0$, and likewise for D_{ij} by symmetry.

4.5. We prove that if $1 \leq i \leq s$, then $b_i(D_{ii}x, y) + b_i(x, D_{ii}y) = 0$ for all x, y in V_i . For consider $D(xy) = b_i(x, y)Dz_i = b_i(x, y)D_{00}z_i$, by (4.2). On the other hand,

$$D(xy) = x(D_{ii}y) + (D_{ii}x)y = \{b_i(x, D_{ii}y) + b_i(D_{ii}x, y)\}z_i.$$

If this last coefficient is nonzero, then z_i is an eigenvector for D_{00} whose corresponding eigenvalue is nonzero, an impossibility by (4.3).

4.6. We prove that, for $1 \leq i \leq s$, D_{ii} is a derivation of N . For if $x, y \in N$, then on the one hand $xy \in (V_0)^2$, so that $D_{ii}(xy) = 0$. On the other hand, $x(D_{ii}y) + (D_{ii}x)y = \pi_i(x)D_{ii}(\pi_i y) + (D_{ii}\pi_i(x))\pi_i(y) = 0$ by (4.5). This settles (4.6).

4.7. For $D \in \text{Der}_k N$, we define $D^* = D - \sum_{1 \leq i \leq s} D_{ii}$, a derivation of N by (4.6). Thus we have a vector space decomposition $\text{Der}_k N = A_1 \oplus \cdots \oplus A_s \oplus \Omega$, where $A_i = \{D \in \text{Der}_k N \mid D = D_{ii}\}$ and $\Omega = \{D \in \text{Der}_k N \mid D = D^*\}$. It is immediate that A_i is a Lie subalgebra of $\text{Der}_k N$.

4.8. Now we prove that $A_i \cong o(V_i, b_i)$ for $1 \leq i \leq s$ (statement (iii)). By (4.5) we have an embedding of A_i into $o(V_i, b_i)$. To see that this embedding is surjective, let $f \in o(V_i, b_i)$ and extend f to $\hat{f}: N \rightarrow N$ by defining $\hat{f}(V_j) = 0$ for $j \neq i$. One readily checks that \hat{f} is a derivation of N , whence $A_i \cong o(V_i, b_i)$.

4.9. We give a direct proof that Ω is a Lie ideal in $\text{Der}_k N$. Let $D \in \Omega$, $E \in \text{Der}_k N$. We will show $(DE)_{ii} = (ED)_{ii} = 0$ for each $i \geq 1$. Now $(DE)_{ii} = \sum_{0 \leq j \leq s} D_{ij}E_{ji} = D_{i0}E_{0i}$ and, likewise, $(ED)_{ii} = E_{i0}D_{0i}$. Since $D_{i0}(N^2) = E_{i0}(N^2) = (0)$ by (4.2), we are done if we can prove that for any derivation E (say), $E_{0i}(N) \subset N^2$. Now note that $N^2 = (V_0)^2 \supset (\text{Ann } N: N) = \{u \in N \mid uN, Nu \subset \text{Ann } N\}$. Thus, one takes $x_i \in V_i$, $x_0 \in V_0$ and readily checks that $x_0E_{0i}(x_i)$ and $E_{0i}(x_i)x_0$ are in $\text{Ann } N$.

4.10. To prove statement (iv) that $D \in \Omega$ is nilpotent (whence the ideal Ω is nilpotent by Engel's theorem), write $D = \sum_i D_{i0} + D_{00} + \sum_i D_{0i}$ with $1 \leq i \leq s$ and simplify the iterates D^2, D^3, \dots by applying these facts: D_{00} is nilpotent, D_{00} stabilizes N^2 , $D_{i0}(N^2) = (0)$, in particular $D_{i0}D_{0i} = 0$, and $D_{0i}(N) \subset N^2$, as in (4.9). One readily sees that some power of D vanishes. This completes the proof of Theorem 2.1.

5. *Proof of Theorem 2.2.* Define N to be $P \oplus Q$ as follows. Use Theorem 2.1 to form a nilpotent P such that $\text{Der}_k P = A_p + \Omega_p$ and A_p is a direct sum of the given algebras $o(V_i, b_i)$. The algebra Q will be a "block strictly upper triangular" matrix algebra $T_0(\rho, k)$ which we will now discuss.

Let $\rho = (r_1, \dots, r_m)$ be a vector whose entries are positive integers, and let $r = r_1 + \dots + r_m$. Form the usual full r by r matrix algebra $M(r, k)$ and imagine each matrix partitioned into rectangular blocks of size r_i by r_j as usual, so that reading down the “block main diagonal” one sees blocks of size r_1 by r_1, r_2 by r_2, \dots, r_m by r_m . The “block upper triangular” subalgebra $T(\rho, k)$ of $M(r, k)$ consists of all matrices with only zero blocks below the block main diagonal, while the “block strictly upper triangular” algebra $T_0(\rho, k)$ is the nilpotent subalgebra of $T(\rho, k)$ consisting of all matrices with only zero blocks both below and on the block main diagonal. One checks that $T_0(\rho, k) = \text{rad } T(\rho, k)$ and that permuting the entries of the vector ρ may yield nonisomorphic algebras.

The content of the following routine exercise is that all derivations of $T_0(\rho, k)$ are induced by bracketing with elements from $T(\rho, k)$.

LEMMA 5.1. *There is a short exact sequence of Lie algebra homomorphisms*

$$0 \longrightarrow k \cdot 1 + \text{Ann } T_0(\rho, k) \xrightarrow{\iota} T(\rho, k)_{\text{Lie}} \xrightarrow{\delta} \text{Der}_k T_0(\rho, k) \longrightarrow 0,$$

where $\iota = \text{inclusion}$ and $(\delta c)c_0 = [c, c_0]$.

We note some consequences for the derivations of $T_0(\rho, k)$. It is immediate that $T(\rho, k)_{\text{Lie}} \cong (\bigoplus_i M(r_i, k)_{\text{Lie}}) + T_0(\rho, k)_{\text{Lie}}$, a semidirect sum. Moreover, if the characteristic of k does not divide r_i , then each $M(r_i, k)_{\text{Lie}} = \text{sl}(r_i, k) \oplus k e_i$ where e_i is the identity of $M(r_i, k)$, $\text{sl}(r_i, k)$ consists of those matrices of trace zero, and this sum is Lie direct. Applying Lemma 5.1 we conclude, under the hypothesis on the characteristic of k , that the solvable radical of $\text{Der}_k T_0(\rho, k)$ is itself a semidirect sum of the toroidal subalgebra $\sum_i k \delta e_i$ with the nilpotent ideal $\delta T_0(\rho, k)_{\text{Lie}}$. Note too that $\sum_i \delta e_i = \delta(1) = 0$.

Having this general analysis of $T_0(\rho, k)$, we may complete the proof of Theorem 2.2. Given r_1, \dots, r_n , in the statement, we define $\rho = (r_1, \dots, [r_n, 1, 1])$ and $Q = T_0(\rho, k)$. Note that $Q^2 \supset \text{Ann } Q$. It follows from the first paragraph of this section and from Lemma 3.1 that $\text{Der}_k N = \text{Der}_k (P \oplus Q) = \text{Der}_k P \oplus \text{Der}_k Q \oplus \Delta_{PQ} \oplus \Delta_{QP}$ as a vector space, with Δ_{PQ} and Δ_{QP} abelian ideals consisting of nilpotent derivations as in (3.1). Note $[\Delta_{PQ}, \Delta_{QP}] = (0)$.

From Theorem 2.1, $\text{Der}_k P = \Delta_P + \Omega_P$ with Δ_P a direct sum of specified $\mathfrak{o}(V_i b_i)$ and Ω_P an ideal of nilpotent derivations. Likewise, from Lemma 5.1, $\text{Der}_k Q = \Delta_Q + \Omega_Q$ where Δ_Q is a direct sum of specified $\text{sl}(r_i, k)$ and $\Omega_Q = (\sum_{i=1}^{i=n+2} k \delta e_i) + \delta T_0(\rho, k)$. Note that the toroidal algebra spanned by the δe_i has dimension $n + 1$ (cf. statement (iii)). Theorem 2.2 follows by putting $\Delta = \Delta_P \oplus \Delta_Q$ (cf. statement

(ii)) and $\Omega = \Omega_P + \Omega_Q + A_{PQ} + A_{QP}$.

REFERENCES

1. J. Dixmier and W. Lister, *Derivations of nilpotent Lie algebras*, Proc. Amer. Math. Soc., **8** (1957), 155-158.
2. F. J. Flanigan, *Radical behavior and the Wedderburn family*, Bull. Amer. Math. Soc., **79** (1973), 66-70.
3. ———, *Radical embedding, genus, and toroidal derivations of nilpotent associative algebras*, Bull. Amer. Math. Soc., **80** (1974), 986-990.
4. M. Hall, Jr., *The position of the radical in an algebra*, Trans. Amer. Math. Soc., **48** (1940), 381-404.
5. N. Jacobson, *Lie Algebras*, Interscience, New York, 1962.
6. G. Leger and E. Luks, *On derivations and holomorphs of nilpotent Lie algebras*, Nagoya Math. J., **44** (1971), 39-50.
7. J. Malley, *Automorphisms and derivations of associative algebras*, Ph. D. thesis, University of California, San Diego, 1975.

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SAN DIEGO STATE UNIVERSITY