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COVERING THEOREMS FOR FINITE NONABELIAN SIMPLE GROUPS. V.

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In the alternating group A_n, $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has the property that CC covers the group. For $n = 16k$ there is a class C of period $n/4$ in A_n such that CC covers A_n; C is the class of type $(4k)^4$.

1. Introduction. It was shown by E. Bertram **[1]** that for $n \geqq 5$ every permutation in A_n is the product of two l-cycles, for any l satisfying $[3n/4] \leqq l \leqq n$. Hence A_n can be covered by products of two n-cycles and also by products of two $(n-1)$-cycles. But if n is odd the n-cycles in A_n fall into two conjugate classes C, C', and similarly for the $(n-1)$-cycles if n is even, so that the quoted result does not decide whether

$$CC = A_n. \tag{1}$$

The question was decided affirmatively for $n = 4k + 2$ and negatively for $n = 4k$, $4k - 1$ in **[2]**. The question is now decided affirmatively in the remaining case $n = 4k + 1$, $n \neq 5$.

THEOREM 1. *For $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has property* (1).

The proof is in §§2–4.

Regarding the product CC', it was shown in **[2]** that CC' covers A_n $(n \geqq 5)$ if $n = 4k$, $4k - 1$, while if $n = 4k + 1$, $4k + 2$, CC' contains all of A_n but the identity.

By an argument quite similar to the proof of Theorem 1, we have proved

THEOREM 2. *For $n = 16k$, the class C of type $(4k)^4$ in A_n has property* (1).

The proof and some related matters are discussed in §5. Note that the class in Theorem 2 has period $n/4$.

2. The case $n = 9$. Let $a = (123456789)$. For every class in A_9, a conjugate b of a can be found such that ab represents (lies in) that class. This assertion is the substance of the table below.

b	ab
a^{-1}	1
(193248765)	(14) (38)
(176235894)	(13) (25) (48) (79)
(132987654)	(193)
(134765289)	(18) (24) (379)
(132798465)	(174) (369)
(184523796)	(135) (274) (698)
(137259486)	(15) (276) (3849)
(123794865)	(1384) (2769)
(132798654)	(17693)
(189623574)	(13) (25) (47986)
(132869745)	(18764) (359)
(132845697)	(18746) (359)
(159348726)	(162495) (38)
(186974532)	(3598764)
a	(135792468) $\sim a$
(125678934)	(315792468)

3. A lemma. In §3 and §4, C will denote the class of the cycle $a = (12 \cdots n)$ in A_n.

LEMMA. *If $n = 4k + 1 > 5$, then CC contains the type $2^{2k}1^1$.*

Proof. If $n \equiv 1 \pmod 8$, then $x =$

$$(n\ n-3\ n-2\ n-1,\ n-4\ n-7\ n-6\ n-5; \cdots ; 9678,\ 5234;\ 1)$$

is conjugate to a and

$$ax = (1\ 3)(2\ 4)(5\ 7)(6\ 8) \cdots (n-4\ n-2)(n-3\ n-1).$$

If $n \equiv 5 \pmod 8$, $n > 13$, then $y =$

$$(n\ n-3\ n-2\ n-1,\ n-4\ n-7\ n-6\ n-5; \cdots ; 21\ 18\ 19\ 20,$$
$$17\ 14\ 15\ 16;\ 13\ 96\ 10,\ 12\ 78\ 11;\ 5234,\ 1)$$

is conjugate to a and

$$ay = (1\ 3)(2\ 4)(5\ 10)(68)(7\ 11)(9\ 12)(13\ 15)(14\ 16) \cdots$$
$$(n-4\ n-2)(n-3\ n-1).$$

If $n = 13$ use the last 13 letters of the above y. (The pattern of y differs from that of x only in the last block of 8 letters between semi-colons, $13\ 9 \cdots 11$, in which the number of reversals is odd, whereas in every other such block of 8 letters in either x or y, the number of reversals is even.)

4. The induction. The induction proceeds from $n-4$ to $n = 4k+1$. The induction hypothesis is: For every permutation T in A_{n-4}, there are two $(n-4)$-cycles d_1 and d_2, both in the class of the $(n-4)$-cycle $(1\ 2 \cdots n-6\ n-5\ n-4)$, and also two other $(n-4)$-cycles d_1' and d_2', both in the class of $(1\ 2 \cdots n-6\ n-4\ n-5)$, such that $T = d_1 d_2 = d_1' d_2'$.

Let S ($\neq 1$) be a permutation in A_n. To show that CC contains S we consider several cases. In each case we find a conjugate S_1 of S, and a certain permutation g in A_n, such that $T = S_1 g^{-1}$ fixes the letters $n, n-1, n-2, n-3$ and thus its restriction to $1, 2, \cdots, n-4$ lies in A_{n-4}.

Case 1. S contains a cycle with 5 or more letters: take

$$g = (n\ n-1\ n-2\ n-3\ n-4).$$

Case 2. S contains no cycle with 5 or more letters, but S contains at least one cycle with 4 letters: take

$$g = (n\ n-1\ n-2\ n-3)(n-4\ n-5).$$

Case 3. S contains no cycle with more than 3 letters, but S does contain two 3-cycles: take

$$g = (n\ n-1\ n-2)(n-3\ n-4\ n-5).$$

Case 4. S is of type $3^1\, 2^{2k-2}\, 1^2$: take

$$g = (n\ n-1\ n-2).$$

Now, if S contains no cycle longer than a transposition, either S is of type $2^{2k}\, 1^1$, whence CC contains S by the lemma, or we have

Case 5. S fixes 5 or more letters: take $g = 1$.

The argument in Case 5 is quite simple. Since S fixes 5 or more letters, S has a conjugate S_1 that fixes $n, n-1, n-2, n-3$. Hence by the induction hypothesis $S_1 = d_1 d_2$, where d_1 and d_2 both fix $n, n-1, n-2, n-3$, and can be expressed

$$d_1 = (a_1\, a_2 \cdots a_{n-5}\; n-4), \qquad d_2 = (b_1\, b_2 \cdots b_{n-5}\; n-4),$$

where the permutation $a_i \to b_i$ is an even permutation of the letters $1, 2, \cdots, n-5$. Then $S_1 = d_3 d_4$, with

$$d_3 = (a_1\, a_2 \cdots a_{n-5}\; n\; n-1\; n-2\; n-3\; n-4),$$
$$d_4 = (b_1\, b_2 \cdots b_{n-5}\; n-4\; n-3\; n-2\; n-1\; n),$$

and d_3, d_4 belong to the same class, be it C or C'. If the other part of the induction hypothesis is used in a similar fashion, the assertion that CC contains S follows.

The details for Case 1 are as follows. Since $T = S_1 g^{-1}$ moves at most the first $n-4$ letters, we have by the induction hypothesis $T = d_1 d_2 = d_1' d_2'$ where d_1, d_2 $[d_1', d_2']$ are from the same class in A_{n-4}. Writing

$$d_1 = (a_1\, a_2 \cdots a_{n-5}\; n-4), \qquad d_2 = (b_1\, b_2 \cdots b_{n-5}\; n-4),$$

the permutation $a_i \to b_i$ is an even permutation of $1, 2, \cdots, n-5$. Now $S_1 = Tg = d_3 d_4$, with $g = (n\; n-1\; n-2\; n-3\; n-4)$ and

$$d_3 = (a_1 \cdots a_{n-5}\; n-2\; n\; n-3\; n-1\; n-4),$$
$$d_4 = (b_1 \cdots b_{n-5}\; n\; n-3\; n-1\; n-4\; n-2).$$

Note that d_3 and d_4 are in the same class, be it C or C', in A_n. By again using d_1' and d_2' in place of d_1 and d_2, the proof is completed in this case.

In Case 2, S has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes at least 5 letters. Hence without loss of generality the factors d_1, d_2 $[d_1', d_2']$ can be chosen so that $T = d_1 d_2 = d_1' d_2'$ with

$$d_1 = (a_1 \cdots a_{n-6}\; n-5\; n-4), \qquad d_1' = (a_1' \cdots a_{n-6}'\; n-5\; n-4)$$
$$d_2 = (b_1 \cdots b_{n-6}\; n-4\; n-5), \qquad d_2' = (b_1' \cdots b_{n-6}'\; n-4\; n-5)$$

and where $a_i \to b_i$ $[a_i' \to b_i']$ is an *odd* permutation of the letters $1, 2, \cdots, n-6$. Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = (a_1 \cdots a_{n-6}\; n-1\; n-5\; n-3\; n-2\; n\; n-4),$$
$$d_4 = (b_1 \cdots b_{n-6}\; n-5\; n-2\; n\; n-3\; n-4\; n-1).$$

The permutations d_3 and d_4 belong to the same class in A_n. Priming the a_i and b_i completes the proof in this case.

In Case 3, S has at least two 3-cycles, and has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes the letters $n, n-1, n-2, n-3, n-4, n-5$. By the induction hypothesis permutations d_1 and d_2 exist such that $T = d_1 d_2$ with

$$d_1 = (n-4\ a_1 \cdots a_k\ n-5\ a_{k+1} \cdots a_{n-6}),$$
$$d_2 = (n-4\ b_1 \cdots b_l\ n-5\ b_{l+1} \cdots b_{n-6}),$$

and where d_1 and d_2 are in the same class in A_n. (We cannot assume that $n-4$ and $n-5$, which are fixed by T, are neighbors in d_1 and d_2, but it is possible that $k = 0$ and $l = n-6$ or that $k = n-6$ and $l = 0$.) Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = d_1 h, \quad d_4 = h^{-1} d_2 g,$$

with $h = (n-5\ n-3\ n-2)(n-4\ n-1\ n)$. Then d_3 and d_4 are both n-cycles. It has only to be checked that they are in the same class in A_n; to do this is tedious, but straightforward. To complete the proof in this case we observe that since S contains two 3-cycles and $S_1 = d_3 d_4$, the decomposition $S_1 = d_3' d_4'$ can be obtained by applying a certain outer automorphism of A_n.

In the only remaining case, S fixes 2 letters, and therefore has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes

$$n,\ n-1,\ n-2,\ n-3,\ n-4.$$

Again we have $T = d_1 d_2$, where we can write

$$d_1 = (a_1 \cdots a_{n-6}\ n-4\ n-5), \qquad d_2 = (b_1 \cdots b_{n-6}\ n-5\ n-4),$$

and where the permutation $a_i \to b_i$ is an odd permutation of the letters $1, 2, \cdots, n-6$. Then $S_1 = Tg = d_3 d_4$, with

$$d_3 = (a_1 \cdots a_{n-6}\ n-1\ n\ n-3\ n-2\ n-4\ n-5),$$
$$d_4 = (b_1 \cdots b_{n-6}\ n-5\ n-4\ n\ n-2\ n-3\ n-1),$$

and these belong to the same class. By priming we again conclude CC contains S, and the proof is complete in all cases. Hence Theorem 1.

5. Covering A_{16k}. By means of an almost identical argument we have shown that the class C of type $4l_1\ 4l_2\ 4l_3\ 4l_4$ ($l_i \geqq 1$) in A_n ($n = 4\Sigma l_i$) has the covering property (1). The lemma required is simpler: Let $m = 4l$, $b = (1\,2 \cdots m)$. Taking $x =$

$$(m\ m-3\ m-2\ m-1, m-4\ m-7\ m-6\ m-5, \cdots, 8\,5\,6\,7, 4\,1\,2\,3)$$

gives

$$bx = (1\ 3)(2\ m)(4\ 6)(5\ 7) \cdots (m-4\ m-2)(m-3\ m-1).$$

Hence if D is the class of type $4l_1\ 4l_2 \cdots 4l_r$ (r even) in A_n, then DD contains the type $2^{n/2}$.

In order to start the induction we had to prove that the class C of type 4^4 has the property $CC = A_{16}$. The calculations are too lengthy to be included. (A copy can be had from any of the authors.) This yields Theorem 2.

One can ask how small a period is possible for a class C with property (1). The first result in this direction was that of *Xu* **[4]** who found such a class with period $n-3$ if n is odd and period $n-2$ if n is even. From the result of Bertram quoted in the introduction, it follows that the smallest period of such C is $\leqq 3n/4$. While Theorem 2 does not give covering for all n, it nevertheless yields, among classes C in A_n satisfying (1),

$$\liminf_{n\to\infty} \frac{\text{period of } C}{n} \leqq \frac{1}{4}$$

as opposed to Bertram's 3/4.

From the other direction we have shown **[3]** that for $n > 6$ there is no class C in A_n having property (1) and period 2, and if $n = 12k+10$ there is no such class of period 3. There may be such a class of period 4, however. More precisely, we conjecture that for $n = 8k$, the class $C = 4^{2k}$ has the covering property (1).

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