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P-PRIMARY DECOMPOSITION OF MAPS INTO AN H-SPACE

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If Y is a finitely generated homotopy associative H-space[1] and X is finite CW then $[X, Y]$ is a nilpotent group. Using this it is easy to show that for any set of prime integers P, a localization map $\ell: Y \to Y_P$ induces $\ell_*[X, Y] \to [X, Y_P]$ with the order of $\ell_*^{-1}(\alpha)$ prime to P. (e.g. see [2]) Since there is no theory of the localization of algebraic loops the same technique does not apply if Y is not homotopy associative. The purpose of this paper is to show that the above theorem holds in this situation.

THEOREM A *Let X be finite CW, Y be a finitely generated H-space (or the localization of such a space) and let $\ell: Y \to Y_P$ be a localization map. Let $\alpha \in [X, Y_P]$; then the order of $\ell_*^{-1}(\alpha)$ is prime to P or is empty. Furthermore there is always a localization map $L: Y \to Y_P$ such that $L_*^{-1}(\alpha)$ is not empty.*

By [3], $[X, Y]$ is finite if and only if $[X, Y_P]$ is finite and in this situation $\ell_*: [X, Y] \to [X, Y_P]$ is onto for any ℓ. Thus from Theorem A we get the following result.

THEOREM B. *Let X and Y be as in* A *and let $[X, Y]$ be finite. Then $[X, Y] \cong \prod [X, Y_q]$ where q is a prime integer and the order of $[X, Y_q]$ is a power of q.*

The structure of this paper is as follows: in §2 we prove an algebraic lemma which we need and in §3 we prove the main theorem.

With reference to Theorem B it should be noted that $[X, Y]$ is a finite (centrally) nilpotent loop ([5]) which is a product of loops of prime power order. While every finite nilpotent group possesses this property it is known ([1], p. 98) that there exists finite nilpotent loops which are not direct products of loops of prime power order.

2. Recall that an algebraic loop G is a set with a binary operation with a unit which satisfies the cancellation laws and has left and right inverses.

Consider the following commuting diagram of algebraic loops and homomorphisms.

[1] By space we mean connected simple CW space.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{h} & D \end{array}$$

LEMMA 2.1. *Let $b \in B$ with $b \in \operatorname{Ker} g$. Assume that $f^{-1}(b)$ is a finite set of order n. Let $a \in f^{-1}(b)$ and a' the left inverse for a (i.e. $a'a = 1$). Then*

(1) $$\operatorname{Ker} f = a'f^{-1}(b) = \{a'\alpha \mid \alpha \in f^{-1}(b)\}$$

(2) $$\operatorname{Ker} k \cap f^{-1}(b)$$

is either empty or the order of $\operatorname{Ker} k \cap f^{-1}(b)$ *is equal to the order of* $\operatorname{Ker} k \cap \operatorname{Ker} f$ *and divides n.*

Proof. (1) Trivially there is a $1-1$ set map $\Phi: f^{-1}(b) \to \operatorname{Ker} f$ defined by $\Phi(\alpha) = a'\alpha$ similarly there is a $1-1$ map $\Psi: \operatorname{Ker} f \to f^{-1}(b)$ defined by $\Psi(\beta) = a\beta$. Since A is not associative Φ and Ψ are not necessarily inverses but the existence of Φ implies that $a'f^{-1}(b) \subseteqq \operatorname{Ker} f$ and Ψ's existence implies equality.

(2) If $\operatorname{Ker} k \cap f^{-1}(b) \neq \emptyset$ we may assume, without loss of generality that $k(a) = 1$. Since $\operatorname{Ker} k \cap \operatorname{Ker} f$ is a normal subloop of $\operatorname{Ker} f$ we have by ([B], p. 92) that the order of $\operatorname{Ker} k \cap \operatorname{Ker} f$ divides n. But $k(a'\alpha) = 1$ if and only if $k(\alpha) = 1$.

3. *Proof of Theorem* A. By 4.1 of [3] there exists a localization $L: Y \to Y_P$ such that $L_*^{-1}(\alpha) \neq \emptyset$. By 4.2 of [3] or 2.2 of [4] for any localization $\mathfrak{l}: Y \to Y_P$, $\mathfrak{l}_*^{-1}(\alpha)$ is finite. Thus we may assume $\mathfrak{l}_*^{-1}(\alpha)$ is finite and nonempty. By (1) of 2.1 the order of $\mathfrak{l}_*^{-1}(\alpha)$ is equal to the order of $\operatorname{Ker} \mathfrak{l}_*$.

We proceed by induction on the Postnikov systems for Y and Y_P. Consider the following homotopy commutative diagram:

$$\begin{array}{ccc} Y_n & \xrightarrow{\mathfrak{l}_n} & Y_{Pn} \\ \downarrow{\scriptstyle \mathfrak{p}_n} & & \downarrow{\scriptstyle \mathfrak{p}_{Pn}} \\ Y_{n-1} & \xrightarrow{\mathfrak{l}_{n-1}} & Y_{P_{n-1}} \\ \downarrow{\scriptstyle \mathfrak{k}^n} & & \downarrow{\scriptstyle \mathfrak{k}_P^n} \\ K(\pi_n(Y), n+1) & \xrightarrow{\mathfrak{l}_c} & K(\pi_n(Y_P), n+1) \end{array}$$

where $\mathfrak{k}^n$ and $\mathfrak{k}_P^n$ correspond to the the n^{th} Postnikov invariants, $\mathfrak{l}_n$, $\mathfrak{l}_{n-1}$, $\mathfrak{l}_c$ are the localization maps induced by $\mathfrak{l}: Y \to Y_P$ and $\mathfrak{p}_n$, and $\mathfrak{p}_{Pn}$

are the fibrations induced by $\mathfrak{f}^n$ and $\mathfrak{f}^n_P$ respectively. Note that all the maps in the diagram are H-maps. Let us assume that the order of Ker $\mathfrak{l}_{n-1*}$ is prime to P.

By ([5], 2.3) the commuting diagram

$$\begin{tikzcd} [X, Y_{n-1}] \arrow[r, "\mathfrak{l}_{n-1*}"] \arrow[d, "\mathfrak{f}_n"] & [X, Y_{pn-1}] \arrow[d, "\mathfrak{f}_{nP}"] \\ H^{n+1}(X; \pi_n(Y)) \arrow[r, "\mathfrak{l}_{c*}"] & H^{n+1}(X; \pi_n(Y_P)) \end{tikzcd}$$

is a diagram of nilpotent loops and homomorphisms. By 2.1, 2), the subloop H of ker $\mathfrak{l}_{n-1*}$ which lifts to $[X, Y_n]$ divides the order of ker $\mathfrak{l}_{n-1*}$ and hence is prime to P.

Let K be the subloop of H which have liftings $\beta \in [X, Y_n]$ such that $\beta \in \mathrm{Ker}\, \mathfrak{l}_{n*}$. Since ker $\mathfrak{l}_{n-1*}$ is nilpotent ([1], P. 96, 1.1), we have ([1], 93) that the order of K divides the order of H and hence is prime to P. But by ([3] 3.3 and 4.1), the set of liftings $\{\beta \in [X, Y_n] \mid \mathfrak{p}_{n*}(\beta) = \alpha, \mathfrak{l}_{n*}(\beta) = 0\}$ is in $1-1$ correspondence with a finite group of order prime to P. Thus the order of ker $\mathfrak{l}_{n*}$ is again finite of order prime to P. Since the assumption trivially holds at the first stage of the Postnikov decomposition, the result follows.

To prove Theorem B note that by [3] the finiteness of $[X, Y]$ implies that $\mathfrak{l}_*\colon [X, Y] \to [X, Y_P]$ is onto for any $\mathfrak{l}$. Thus $[X, Y_\phi]$ is finite. But $Y_\phi = \Pi K(Q, n)$, so that

$$[X, Y_\phi] = [X, \Pi K(Q, n)] = \Pi H^n(X; Q)$$

which is finite if and only if $[X, Y_\phi] = 0$.

If q is a prime and $\bar{q}$ its complimentary set of primes then by ([2], [4])

$$\begin{tikzcd} [X, Y] \arrow[r] \arrow[d] & [X, Y_{\bar{q}}] \arrow[d] \\ [X, Y_q] \arrow[r] & [X, Y_\phi] \end{tikzcd}$$

is a pullback diagram. Therefore
$\#[X, Y] = \#[X, Y_{\bar{q}}] \cdot \#[X, Y_q]$ (where $\#S$ is the order of the set S).

Since $\mathfrak{l}_*\colon [X, Y] \to [X, X_{\bar{q}}]$ is onto we see, by the proof of A, that there is an integer k such that $\#[\mathfrak{l}_*^{-1}(\alpha)] = q^k$ for all $\alpha \in [X, Y_{\bar{q}}]$.

Thus $\#[X, Y] = q^k \#[X, Y_{\bar{q}}]$ or $[X, Y_q] = q^k$. By [4], and the fact that $[X, Y_\phi] = 0$ we get $[X, Y] = \Pi[X, Y_q]$.

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