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STIELTJES DIFFERENTIAL-BOUNDARY OPERATORS III, MULTIVALUED OPERATORS-LINEAR RELATIONS

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This article deals with a multivalued differential-boundary operator on a nondense domain regarding it as a linear relation. The adjoint relation is derived. It is shown that these dual relations have the same form as exhibited in earlier papers where the operators involved were uniquely defined on dense domains. Self-adjoint relations are considered on the Hilbert space $\mathscr{L}_n^2[0,1]$. The connection with self-adjoint operators defined on subspaces of $\mathscr{L}_n^2[0,1]$ is made.

I. Introduction. This article is a continuation of [8] and [9]. The notation is the same. We review it briefly. X is the Banach space $\mathscr{L}_n^p[0, 1]$, $1 \leqq p < \infty$, consisting of all n-dimensional vectors

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

under the norm

$$\| y \| = \left[\int_0^1 \left[\sum_{i=1}^n |y_i|^2\right]^{p/2} dt\right]^{1/p} .$$

X^* is the dual space $\mathscr{L}_n^q[0, 1]$, $1/p + 1/q = 1$.

A and B are $m \times n$ matrices, $m \leqq 2n$, satisfying rank $(A\colon B) = m$. C and D are $(2n - m) \times n$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$ is given by $\begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix}$, where $\tilde{A}$ and $\tilde{B}$ are $m \times n$ matrices satisfying rank $(\tilde{A}\colon \tilde{B}) = m$, and $\tilde{C}$ and $\tilde{D}$ are $(2n - m \times n)$ matrices. Hence the large matrices above may be multiplied together in the usual component-like manner.

K is a regular $m \times n$ matrix valued function of bounded variation satisfying $dK(0) = 0$, $dK(1) = 0$. K_1 is a regular $r \times n$ matrix valued function of bounded variation satisfying $dK_1(0) = 0$, $dK_1(1) = 0$.

H is a regular $n \times (2m - m)$ matrix valued function of bounded variation satisfying $dH(0) = 0$, $dH(1) = 0$. H_1 is a regular $n \times s$ matrix valued function of bounded variation satisfying $dH_1(0) = 0$, $dH_1(1) = 0$. P is a continuous $n \times n$ matrix.

Now let $\mathscr{D}$ denote those elements $y \in X$ satisfying

1. For each y there is an $s \times 1$ matrix valued constant ψ such that

$$y + H[Cy(0) + Dy(1)] + H_1\psi$$

is absolutely continuous.

2. $ly = (y + H[Cy(0) + Dy(1)] + H_1\psi)' + Py$ exists a.e. and is in X.

3. $Ay(0) + \int_0^1 dK(t)y(t) + By(1) = 0$, $\quad \int_0^1 dK_1(t)y(t) = 0$.

The purpose of this article is to discuss the expression l on $\mathscr{D}$ as a linear relation L, defined by its graph

$$L = \{(y, ly): y \in \mathscr{D}\} \subset X \times X .$$

Note that ly may be multivalued. If H_1 possesses a linear combination of columns which is absolutely continuous, then ly is unique only modulo such combinations. Note also that $\mathscr{D}$ may not be dense in X. If K_1 possesses linear combinations of rows which are absolutely continuous then $\mathscr{D}$ is orthogonal to those combinations.

In seaching for the adjoint of L, we encounter the following problem even if ly is uniquely defined: If $\mathscr{D}$ is dense in X, $y \in D$, and $f \in X^*$, let $[y, f]$ denote $f(y)$. Then $\mathscr{D}^*$, the domain of L^*, in X^* is given by

$$\mathscr{D}^* = \{f: [ly, f] = [y, g] \text{ for some } g \in X^* \text{ and } y \in \mathscr{D}\} .$$

If l^* denotes the form of the adjoint, then $l^*f = g$ is uniquely defined. For if $l^*f = h$ as well, then $[y, g - h] = 0$ for all $y \in \mathscr{D}$. If $y \rightarrow y_0$, then $[y, g - h] \rightarrow [y_0, g - h] = 0$, and $g - h = 0$ in X^*

However, if $\mathscr{D}$ is not dense in X, then $l^*f = g$ is defined only modulo $\mathscr{D}^\perp$ (Kelley and Namioka [7; p. 120]). If $d^\perp \in \mathscr{D}^\perp$ and $f \in \mathscr{D}^\perp$, then

$$[ly, f] = [y, g + d^\perp] = [y, g] .$$

Hence $l^*f = g + d^\perp$ for all $d^\perp \in \mathscr{D}^\perp$. The adjoint is not unique.

This is well borne out with the adjoint actually derived in section IV. The domain of L^*, $\mathscr{D}^*$, consists of those elements $z \in X^*$ satisfying

1. for each z there is an $r \times 1$ matrix valued constant ϕ such that

$$z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi$$

is absolutely continuous.

2. $l^{+}z = -(z + K^{*}[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^{*}\phi)' + P^{*}z$ exists a.e. and is in X^{*}.

3. $\tilde{C}z(0) + \int_0^1 dH^{*}(t)z(t) + \tilde{D}z(1) = 0$, $\int_0^1 dH_1^{*}(t)z(t) = 0$.

The relation L^{*} is defined by its graph

$$L^{*} = \{(z, l^{+}z) : z \in \mathscr{D}^{*}\} \subset X^{*} \times X^{*} .$$

When $\mathscr{D}$ is not dense in X because of the absolute continuity of a linear combination of rows of K_1, then $l^{+}z$ is multivalued since ϕ is not unique. Further when ly is multivalued because of the absolute continuity of linear combination of columns of H_1, then $\mathscr{D}^{*}$ is orthogonal to those combinations and is not dense in X^{*}.

Although multivaluedness and nondensity of domains cause problems when the setting is a standard Banach space such as X, the setting of *linear* relation in $X \times X$ handles these problems quite nicely.

Further examples illustrating this phenomenon have been presented recently by Coddington [5], [6].

II. Linear relation. (See Arens [1] or Brown [12].) A linear relation T on X is a set valued mapping with domain and range in X whose graph $G(T)$ is a linear subspace of $X \times X$.

If y is in the domain of T, $D(T)$, and Ty denotes the image of y under T, then the graph of T in $X \times X$ is given by

$$G(T) = \{(y, Ty); y \in D(T)\} .$$

(It is clear that a linear operator can be identified with its graph, so that it also can be thought of as a linear relation.) It is easy to see that $T(0)$ is a subspace of the range of T, $R(T)$; that $x, y \in T(y)$ if and only if $x = y \bmod T(0)$; that if $y_T \in T(y)$, then $T(y) = y_T + T(0)$; and that

$$G(T) = \{(y, y_T + T(0)) : y \in D(T), y_T \in T(y)\} .$$

The null space of T, $N(T)$, is given by $N(T) = \{y : (y, 0) \in G(T)\}$ and is a subspace of X.

T is closed if $G(T)$ is closed. The closure of T is dermined by $\overline{G(T)}$. T is normally solvable if it is both closed and has closed range. Closure of T implies the closure of both $N(T)$ and $T(0)$.

The purpose of introducing linear relations is to be able to define an adjoint for T. Let $[y, z] = z(y)$ for $y \in X$, $z \in X^{*}$. This can be extended to $X \times X$ and $X^{*} \times X^{*}$ by setting

$$[(y_1, y_2), (z_1, z_2)] = [y_1, z_1] + [y_2, z_2]$$

when $(y_1, y_2) \in X \times X$ and $(z_1, z_2) \in X^* \times X^*$. Then T^* is identified with its graph

$$G(T^*) = \{(z_1, z_2): (z_1, z_2) \in X^* \times X^*, [y_2, z_1] - [y_1, z_2] = 0 \text{ for all } (y_1, y_2) \in G(T)\} .$$

This, of course, agrees with the standard definition when T is an operator with dense domain. T^* has a number of properties similar to adjoint operators. We refer the reader to [1] or [12] for further details. We shall use these properties implicitly throughout the remainder of the article.

III. The adjoint of L. Recall that the expression l^+z is given by

$$l^+z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + P^*z .$$

We introduce in addition the expression $l^{++}z$, given by

$$l^{++}z = -(z + K^*\phi_1 + K_1^*\phi)' + P^*z ,$$

where ϕ and ϕ_1 are appropriate vector valued constants suitably chosen so the expression within the parentheses is absolutely continuous.

THEOREM 3.1. (A Green's formula.) *Let $y, ly \in X$ and let $z, l^{++}z \in X^*$. Then*

$$\begin{aligned}\int_0^1 [z^*(ly) - (l^{++}z)^*y]dt &= [\tilde{A}z(0) + \tilde{B}z(1)]^*\Big[Ay(0) + By(1) + \int_0^1 dKy\Big] \\ &+ \Big[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z\Big]^*[Cy(0) + Dy(1)] \\ &+ \phi^*\Big[\int_0^1 dK_1y\Big] + \Big[\int_0^1 dH_1^*z\Big]^*\psi \\ &+ \Big[\phi_1 - (\tilde{A}z(0) + \tilde{B}z(1))\Big]^*\Big[\int_0^1 dKy\Big] .\end{aligned}$$

Proof. Note that since H, H_1, K, K_1 are regular, then so are y and z. Thus according to [10; Corollary 2.1] the usual integration by parts formula

$$\int_0^1 f\cdot dg + \int_0^1 df\cdot g = f\cdot g\Big|_0^1$$

holds. If to the terms z and y on the left the terms $K^*\phi_1 + K_1^*\phi$ and $H[Cy(0) + Dy(1)] + H_1\psi$ are added and elsewhere subtracted, several integration by parts results in

$$\begin{aligned}\int_0^1 &[z^*(ly) - (l^{++}z)^*y]dt \\ &= z^*y|_0^1 + \phi_1^* \int_0^1 dKy + \phi^* \int_0^1 dK_1 y \\ &\quad + \int_0^1 z^* dH[(y(0) + Dy(1)] + \int_0^1 z^* dH_1\psi \,.\end{aligned}$$

By using the formulas resulting from multiplying $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and its inverse, the term $z^*y|_0^1$ can be written in terms of end point boundary conditions in z and y. An appropriate regrouping of terms completes the proof.

We are now in a position to characterize the adjoint linear relation L^*.

THEOREM 3.2. *The domain of the adjoint relation* L^* *is* $\mathscr{D}^*$. *Further*

$$L^* = \{(z, l^+z) \colon z \in \mathscr{D}^*\} \,.$$

Proof. If $z \in \mathscr{D}$, then Green's formula shows that $(z, l^+z) \in L^*$. Hence

$$\{(z, l^+z) \colon z \in \mathscr{D}^*\} \subset L^* \,.$$

To show the reverse inclusion, let $y \in \mathscr{D} \cap C_0^1(0, 1)$, so that

$$0 = \int_0^1 dKy = -\int_0^1 K(t)y'(t)dt \,,$$

$$0 = \int_0^1 dK_1 y = -\int_0^1 K_1(t)y'(t)dt \,.$$

Thus $(y, y' + Py) \in L$. If $(z, l^*z) \in \operatorname{dom} L^*$, then

$$[y, l^*z] - [y' + Py, z] = 0 \,,$$

or

$$\int_0^1 [(l^*z)^*y - z^*(y' + Py)]dt = 0 \,.$$

If the terms involving y are integrated by parts, this is equivalent to

$$\int_0^1 \left\{z + \int_0^t [(l^*z) - P^*z]d\xi\right\}^* y'dt = 0 \,.$$

Since y vanishes at 0 and 1, y' is orthogonal constants. From comments above y' is orthogonal to $K(t)^*$. Thus for appropriate C, ϕ, ϕ_1,

$$z + \int_0^t [(l^+z) - P^*z]d\xi = C - K(t)^*\phi_1 - K_1(t)^*\phi \ .$$

Hence

$$z + K^*\phi_1 + K_1^*\phi = -\int_0^t [(l^*z) - P^*z]d\xi + C$$

is absolutely continuous, and

$$l^*z = -(z + K^*\phi_1 + K_1^*\phi)' + P^*z = l^{++}z \ .$$

Green's formula now shows for arbitrary $y \in D$

$$\begin{aligned} 0 = &\left[\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z\right]^*[Cy(0) + Dy(1)] \\ &+ \left[\int_0^1 dH_1^*z\right]^*\psi \\ &+ [\phi_1 - (\tilde{A}z(0) + \tilde{B}z(1))[^*\left[\int_0^1 dKy\right] \ . \end{aligned}$$

$Cy(0) + Dy(1)$ varies over C^{2n-m}, for it not, a linear combination of its rows would vanish, putting an extra constraint on $\mathscr{D}$. Likewise it is clear from the definition of $\mathscr{D}$ that ψ varies over C^s. Finally if a linear combination of rows of K were constant, so then would the same linear combination of components of $\int_0^1 dKy$ be 0. Its coefficient from $\phi_1 - (\tilde{A}z(0) + \tilde{B}z(1))$ would be arbitrary. But then the corresponding product within $(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1\phi)'$ would vanish. So effectively

$$\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH_1^*z = 0 \ , \qquad \int_0^1 dH_1^*z = 0 \ ,$$

and

$$\phi_1 = \tilde{A}z(0) + \tilde{B}z(1) \ .$$

Hence $\operatorname{dom} L^* = \mathscr{D},^*$ $l^*z = l^+z$, and

$$L^* = \{(z, l^+z) : z \in \mathscr{D}^*\} \ .$$

This result is identical in form with that derived in [9]. Here, however, because of greatly relaxed assumptions concerning H, H_1, K, K_1 linear relations prove to be a very convenient setting.

IV. Self-adjoint differential-boundary relations. In this section we restrict our attention to the Hilbert space $X = \mathscr{L}_n^2[0, 1]$ and characterize those linear relations which are self-adjoint. For convenience we replace L and L^* by M and M^*, given by

$$M = \{(y, [1/i][y + H[Cy(0) + Dy(1)] + H_1\psi]' + Qy): y \in \mathscr{D}\}$$

and

$$M^* = \{(z, [1/i][z + K^*[\tilde{A}z(0) + Bz(1)] + K_1^*z]' + Q^*z: z \in \mathscr{D}^*\}$$

where $P = iQ$.

We say that the linear relation M is self-adjoint if $M = M^*$. Hence we find

THEOREM 4.1. *The linear relation M is self-adjoint if and only if*

1. $Q = Q^*$.
2. $m = n,\ r = s$.
3. $K = [BD^* - AC^*]H^*$.
4. $AA^* = BB^*$.
5. $H[CC^* - DD^*] = 0$ a.e.
6. $K_1 = E_1H_1^*$, *where E_1 is a nonsingular $r \times r$ matrix.*

Proof. It is clear that if all these conditions are satisfied, then $M = M^*$.

Conversely if $M = M^*$, then

$$\begin{aligned}&[1/i][y + H[Cy(0) + Dy(1)] + H_1] + Qy\\ &\quad = [1/i][y + K^*[Ay(0) + By(1)] + K_1^*\phi] + Q^*y\ .\end{aligned}$$

If $y \in \mathscr{D}$ vanishes near 0 and 1, is absolutely continuous (so ψ and ϕ may be chosen 0), but is otherwise free to vary, then $Q^*y = Qy$ and $Q = Q^*$. From inspection $m = n$ and $r = s$. Otherwise either $\mathscr{D}$ or $\mathscr{D}^*$ would have more boundary constraints than the other. Further

$$Ay(0) + By(1) + \int_0^1 dKy = 0\ , \qquad \int_0^1 dK_1y = 0$$

and

$$\tilde{C}y(0) + \tilde{D}y(1) + \int_0^1 dH_1^*y = 0\ , \qquad \int_0^1 dH_1^*y = 0$$

must represent the same boundary conditions. This can only happen if $A = E\tilde{C}$, $B = E\tilde{D}$, $K = EH^*$, for some nonsingular matrix E and $K_1 = E_1H_1^*$ for some notsingular matrix E_1. The equations which result from multiplying $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} -\tilde{A} & -\tilde{C} \\ -\tilde{B} & -\tilde{D} \end{pmatrix}$$

show that $E = [BD^* - AC^*]$, as well as $AA^* = BB^*$ and $H[CC^* - DD^*] = 0$.

V. Self-adjoint operators on subspaces of $\mathscr{L}_n^2[0, 1]$. Let the columns of H_1 be suitably arranged such that the first s_1 of them form a maximal independent absolutely continuous collection. Then H_1 can be partitioned into $H_1 = (H_c: H_s)$, where H_c denotes the absolutely continuous columns, and H_s denotes those singular with respect to Lebesgue measure. In X let $\mathscr{H}_1$ denote the subspace spanned by the columns of H_c'.

Likewise, let the rows of K_1 be suitably arranged so that the first r_1 form a maximal independent absolutely continuous collection. Then K_1 can be partitioned into $K_1 = \begin{pmatrix} K_c \\ K_s \end{pmatrix}$, where K_c denotes the absolutely continuous rows, and K_s denotes those singular with respect to Lebesgue measure. In X^* let $\mathscr{K}_1^*$ denote the subspace spanned by the columns of $K_c^{*\prime}$.

Now l can be rewritten as follows: Let l_s be defined by

$$l_s y = (y + H[Cy(0) + Dy(1)] + H_s\psi_s)' + Py .$$

Then

$$ly = l_s y + H_c'\psi_c$$

where $\psi = \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}$. The boundary conditions determining $\mathscr{D}$ can also be more accurately written as

$$Ay(0) + By(1) + \int_0^1 dKy = 0 , \qquad \int_0^1 dK_s y = 0 , \qquad \int_0^1 K_c' y dt = 0 .$$

Similarly l^+ can be written by first defining l_s^+ by

$$l_s^+ z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_s^*\phi_s)' + P^*z .$$

Then

$$l^+ z = l_s^+ z + K_c^{*\prime}\phi_c ,$$

where $\phi = \begin{pmatrix} \phi_c \\ \phi_s \end{pmatrix}$. The boundary conditions determining $\mathscr{D}^*$ can be written as

$$\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*z = 0 , \quad \int_0^1 dH_s^* z = 0 , \quad \int_0^1 H_c^{*\prime} z dt = 0 .$$

We now face a rather odd situation. $\mathscr{D}$ is orthogonal to $\mathscr{K}_1^*$, while l is defined on D modulo $\mathscr{H}_1$. That is, for the subspace $\mathscr{K}_1^{*\perp}$, is uniquely defined only when set in $\mathscr{K}_1^{*\perp}/\mathscr{H}_1$.

Likewise $\mathscr{D}^*$ is orthogonal to $\mathscr{H}_1$, while l^+ is defined on $\mathscr{D}^*$ modulo $\mathscr{K}_1^*$. That is, for the subspace $\mathscr{H}_1^1$, l^+ is uniquely defined only when set in $\mathscr{H}_1^1/\mathscr{K}_1^*$.

This can be considerably simplified when $X = \mathscr{L}_n^2[0, 1]$, $\mathscr{H}_1 = \mathscr{K}_1^*$, and the linear relation M is self-adjoint. The spaces above are all reduced to $\mathscr{H}_1^1$ or its isomorphic copy $\mathscr{L}_n^2[0, 1]/\mathscr{H}_1$. We assume without loss of generality that the columns of H_c' are mutually orthonormal.

The restriction of M, denoted by M_1, which defines an *operator* from $\mathscr{H}_1^1$ to $\mathscr{H}_1^1$ is uniquely defined by

$$M_1 y = (1/i)l_s y + (1/i)H_c'\psi_c \,,$$

where, with $\langle \cdot \rangle$ denoting the inner product in $\mathscr{L}_n^2[0, 1]$,

$$\psi_c = -\langle l_s y, H_c' \rangle$$

Hence

$$M_1 y = (1/i)l_s y - (1/i)H_c'\langle l_s y, H_c' \rangle \,.$$

The relationships between M and M_1 can be best illustrated by the following diagram:

$$\begin{array}{llcl}
\text{Operator:} & M_1 & \xleftrightarrow[\text{Isometry}]{\text{Linear Space Homomorphism}} & M_1 \\
\text{Space:} & \mathscr{H}_1^1 & & \mathscr{L}_n^2[0, 1]/\mathscr{H}_1
\end{array}$$

$$\begin{array}{clcl}
\xleftrightarrow{\text{Linear Space Isomorphism}} & M & \xleftrightarrow{\text{Linear Relation}} & M^* \; (=M) \\
 & \mathscr{L}_n^2[0, 1] & & \mathscr{L}_n^2[0, 1]
\end{array}$$

$$\begin{array}{cl}
\xleftrightarrow{\text{Linear Space Isomorphism}} & M_1^* \; (=M_1) \\
 & \mathscr{L}_n^2[0, 1]/\mathscr{H}_1
\end{array}$$

$$\begin{array}{cl}
\xleftrightarrow[\text{Isometry}]{\text{Linear Space Homomorphism}} & M_1^* \; (=M_1) \\
 & \mathscr{H}_1^1
\end{array}$$

It is readily apparent from the diagram that:

THEOREM 6.1. *M_1 is a self-adjoint operator on the subspace $\mathscr{H}_1$ if and only if M is a self-adjoint linear on $\mathscr{L}_n^2[0, 1]$.*

We note that the description of M_1 is equivalent to that derived by Coddington [4], [5], [6] when Coddington's $n = 1$ and $H_s = 0$, H is absolutely continuous. There certainly exists an extension of the present work to higher order differential-boundary relations which will duplicate Coddington's results in full generality, although at the present time such work has not been done.

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