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FINDING A MAXIMAL SUBALGEBRA ON WHICH THE TWO ARENS PRODUCTS AGREE

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Arens has given two ways of defining a Banach algebra product on the second dual of a Banach algebra $\mathscr{A}$. In this paper we give a construction for finding a maximal subalgebra on which the two Arens products agree. Moreover, we give an example which shows that there is not necessarily a unique maximal subalgebra on which the two Arens products agree. This example is a Banach algebra whose second dual has a *nonunique* element I which is simultaneously a right identity under the first Arens product and a left identity under the second Arens product.

1. Preliminaries. The two Arens products are defined according to the following rules. Let $\mathscr{A}$ be a Banach algebra. Let $A, B \in \mathscr{A}$, $f \in \mathscr{A}^*$, $F, G \in \mathscr{A}^{**}$.

DEFINITION 1.1.
$(f*_1A)B = f(AB)$ This defines $f*_1A$ as an element of $\mathscr{A}^*$.
$(G*_1f)A = G(f*_1A)$ This defines $G*_1f$ as an element of $\mathscr{A}^*$.
$(F*_1G)f = F(G*_1f)$ This defines $F*_1G$ as an element of $\mathscr{A}^{**}$.
We will call $F*_1G$ the first or the m_1 product.

DEFINITION 1.2. $(f*_2A)B = f(BA)$; $(F*_2f)A = F(f*_2A)$; $(F*_2G)f = G(F*_2f)$.
We will call $F*_2G$ the second or the m_2 product.

PROPOSITION 1.3. *If $\mathscr{A}$ has an approximate identity, then $\mathscr{A}^{**}$ has an element I which is simultaneously a right m_1 identity and a left m_2 identity. Call such an element I a simultaneous identity.*

Proof. $\mathscr{A}^{**}$ has a right m_1 identity by [2, p. 146] the proof that it also has a left m_2 identity is similar.

EXAMPLE 1.4. A simultaneous right m_1 and left m_2 identity, unlike a two-sided identity, is not necessarily unique.

Let $X = c_0 \bigoplus_{sup} \ell^1$. Let $\{x_1, x_2, x_3, x_4, \cdots\}$ be the basis $\{d_1, e_1, d_2, e_2, \cdots\}$ where $\{d_i\}$ and $\{e_i\}$ are the canonical bases for c_0 and ℓ^1 respectively. Let $\mathscr{D}$ be the norm closure of operators in $\mathscr{B}(X)$ which have a finite matrix with respect to $\{x_i\}$. For each $f \in \mathscr{D}^*$ we can associate a matrix (f_{ij}) by defining $f_{ij} = f(E_{ij})$ when

E_{ij} is the matrix in $\mathscr{D}$ with a 1 in the ij^{th} place and 0's elsewhere. $\mathscr{D}$ has an approximate indentity, namely the operators E_n with 1's down the first n entries on the diagonal and 0's elsewhere.

Let T_n be the matrix with 1's in the $j+1$, j^{th} slots for $j = 1, 3, 5, \cdots, 2n-1$ and 0's elsewhere. Clearly $\|T\| = n$ and so by the Hahn Banach theorem there exists an $f_n \in \mathscr{D}^*$ of norm one with $f_n(T_n) = n$. Since f_n has norm one, each of its entries must have modulus $\leqq 1$. This can be seen directly or from [7, Prop. 2.6]. Hence the matrix for f_n must have $j+1$, j^{th} entries $= 1$ for $j = 1, 3, \cdots, 2n-1$.

By the weak star compactness of the unit ball of $\mathscr{D}^*$ there exists an f which is a weak star cluster point of the f_n. Note that the $j+1$, j^{th} entries of f must all be 1, because if $f_{m+1,m} \neq 1$ for some m, then the weak star neighborhood of f given by $\mathscr{N}(f; E_{m+1,m}; \varepsilon)$ would not contain infinitely many f_n for ε small. It is clear that f is not in the subspace of $\mathscr{D}^*$ spanned by those functionals whose matrices have either a finite number of rows or columns. Hence, there exists and $H \in \mathscr{D}^{**}$ such that $H(f) = 1$ and $H(g) = 0$ if the matrix for g has a finite number of rows or a finite number of columns.

Note that $I *_1 H = 0$ because for arbitrary $g \in \mathscr{D}^*$, $(I *_1 H)g = \lim_n (E_n *_1 H)g$ by the left weak star continuity of m_1. See [1]. This equals $\lim_n E_n(H *_1 g) = \lim_n (H *_1 g)E_n = \lim_n H(g *_1 E_n)$. But it is easily seen that for each n, the matrix for the functional $g *_1 E_n$ has the same first n rows as that of g and zeroes elsewhere. This can be computed directly. Hence $H(g *_1 E_n) = 0$ and so $I *_1 H = 0$. Similarly it can be seen that $(H *_2 I)g = \lim (H *_2 E_n)g = \lim E_n(H *_2 g) = \lim (H *_2 g)E_n = \lim H(g *_2 E_n)$ and that the functional $g *_2 E_n$ has as its matrix the first n columns of g and zeroes elsewhere. Thus $H *_2 I = 0$.

From the fact that $I *_1 H = 0$ it follows that $G *_1 H = 0$ for all $G \in \mathscr{D}^{**}$ since $G *_1 H = (G *_1 I) *_1 H = G *_1 (I *_1 H)$. Similarly, $H *_2 I = 0$ implies $H *_2 G = 0$ for all $G \in \mathscr{D}^{**}$. It is easy to see that $H + I$ is a simultaneous right m_1 and left m_2 identity.

2. The main result. Let $\mathscr{A}$ be a Banach algebra and suppose the two Arens products agree on $\mathscr{B}$ where $\mathscr{A} \subset \mathscr{B} \subset \mathscr{A}^{**}$. Then by Zorn's lemma, it follows that there exists an algebra $\mathscr{M}$ with $\mathscr{B} \subset \mathscr{M} \subset \mathscr{A}^{**}$ such that the two Arens products agree on $\mathscr{M}$ and $\mathscr{M}$ is maximal with respect to this property.

EXAMPLE 2.1. Let $\mathscr{D}$ be the same Banach algebra as in Example 1.4. Then there is not a unique maximal subalgebra of $\mathscr{D}^{**}$ on which the Arens products agree. Note that the Arens products agree on the algebra generated by $[\mathscr{D}, I]$. Also they agree on the algebra generated by $[\mathscr{D}, H]$, since they agree if one factor is in $\mathscr{D}$, and also $H *_1 H = H *_2 H = 0$. However the Arens products cannot agree on any algebra containing I and H, since $I *_1 (I + H) = I$ but $I *_2 (I + H) = I + H$.

DEFINITION 2.2. Let $\mathscr{A}$ be a Banach algebra and E_α an approximate identity with weak star limit I in $\mathscr{A}^{**}$. Then E_α is called *projecting* if for each $F \in \mathscr{A}^{**}$, $E_\alpha *_1 F *_1 E_\beta$ is in $\mathscr{A}$ for E_α and E_β sufficiently far out.

THEOREM 2.3. *Let E_α be a projecting weak identity for $\mathscr{A}$ and let $I *_1 (F *_2 I) = F *_2 I$ for all $F \in \mathscr{A}^{**}$. Then*

(1) $m_1 = m_2$ on $\mathscr{N}$ where $\mathscr{N} = \{F *_2 I : F \in \mathscr{A}^{**}\}$

(2) $\mathscr{N}$ is an algebra which is maximal with respect to the property that $m_1 = m_2$.

Proof. One of the difficulties is the fact that mixed Arens products like $(F *_1 G) *_2 H$ are not necessarily associative. In this proof all limits will be in the weak star topology on $\mathscr{A}^{**}$. We will make frequent use of the fact that the two Arens products agree if one of the factors is in $\mathscr{A}$. Also we will make very careful use of the *left* weak star continuity of m_1 and the *right* weak star continuity of m_2. Furthermore note that by the hypothesis on I, it follows that $I *_1 V = V$ for any $V \in \mathscr{N}$.

Given $S = F *_2 I$ and $T = G *_2 I$ we must show that $S *_2 T = S *_1 T$. Note that $S *_2 T = I *_1 (S *_2 T)$ since $S *_2 T$ is in $\mathscr{N}$ and equals

$$(\lim_\alpha E_\alpha) *_1 (S *_2 T) = \lim_\alpha [E_\alpha *_1 (S *_2 T)] .$$

Note also that

$$S *_1 T = (I *_1 S) *_1 T = \lim (E_\alpha *_1 S) *_1 T = \lim [E_\alpha *_1 (S *_1 T)] .$$

Hence it is sufficient to show that $E_\beta *_1 (S *_2 T) = E_\beta *_1 (S *_1 T)$ for all E_β far enough out.

But since $E_\beta \in \mathscr{A}$, $E_\beta *_1 (S *_2 T) = E_\beta *_2 (S *_2 T)$
$= (E_\beta *_2 S) *_2 T = (E_\beta *_2 S) *_2 (I *_1 T)$
$= (E_\beta *_2 S) *_2 \lim_\alpha (E_\alpha *_2 T)$ by the left weak star continuity of m_1
$= \lim_\alpha [(E_\beta *_2 S) *_2 (E_\alpha *_2 T)]$ by the right weak star continuity of m_2
$= \lim_\alpha [((E_\beta *_2 S) *_2 E_\alpha) *_2 T]$
$= \lim_\alpha [((E_\beta *_2 S) *_2 E_\alpha) *_1 T]$ since $E_\beta * S * E_\alpha$ is in $\mathscr{A}$ for E_β and E_α far enough out
$= \lim_\alpha [(E_\beta *_2 S) *_2 E_\alpha] *_1 T = [(E_\beta *_2 S) *_2 I] *_1 T$ by weak star continuity
$= (E_\beta *_2 (S *_2 I)) *_1 T = (E_\beta *_2 S) *_1 T$ since $S \in \mathscr{N}$
$= (E_\beta *_1 S) *_1 T = E_\beta *_1 (S *_1 T)$
and this concludes the proof of part (1).

For part (2) $\mathscr{N}$ is an algebra because $(F*_2I)*_2(G*_2I) = (F*_2G)*_2I$ by the associativity of m_2, and is thus in $\mathscr{N}$. Next suppose that $F \notin \mathscr{N}$. Then $F*_2I \neq F$ and yet $F*_1I = F$ and so $\mathscr{N}$ is maximal.

3. Applications. For an infinite, Abelian group it is well known [3] that the Arens products never agree on all of $L(G)^{**}$.

COROLLARY 3.1. *If G is a compact Abelian group, then $L(G)$ satisfies the hypotheses of the above theorem.*

Proof. Let E_α be an approximate identity for $L(G)$ with weak star limit I. By [3, Thm. 2.4] $L(G)$ is a two-sided ideal in $L(G)^{**}$. So in particular E_α will be projecting. It is easily observed that if a Banach algebra $\mathscr{A}$ is commutative, then $F*_2A = A*_2F$ for all $A \in \mathscr{A}$ and $F \in \mathscr{A}^{**}$. Then

$$\begin{aligned} I*_1(F*_2I) &= \lim_\alpha \lim_\beta\, [E_\alpha *_2(F*_2E_\beta)] \\ &= \lim_\alpha \lim_\beta\, [(E_\alpha *_2 F)*_2E_\beta] = \lim_\alpha \lim_\beta\, [(F*_2E_\alpha)*_2E_\beta] \\ &= \lim_\alpha \lim_\beta\, [F*_2(E_\alpha *_2 E_\beta)] = \lim_\alpha\, [F*_2(E_\alpha *_2 I)] \\ &= \lim_\alpha\, [F*_2E_\alpha] = F*_2I\,. \end{aligned}$$

DEFINITION. A shrinking basis $\{e_j\}$ for a Banach space is called boundedly growing if there exists an $\varepsilon > 0$ and a positive integer n such that $\|x_1 + \cdots x_n\| < n - \varepsilon$ whenever the x_i's have norm 1 and are distinct block basic vectors.

COROLLARY 3.2. *If X has an unconditionally monotone, boundedly growing bases $\{e_j\}$ then $\mathscr{C}$ the algebra of compact linear operators satisfies the hypotheses of the theorem, and $\mathscr{N}$ will consist of those $F \in \mathscr{C}^{**}$ for which each of the "rows" of F are elements of X^** (as opposed to X^{**}).

Proof. The operators E_n, with ones down the first n slots of the diagonal and zeroes elsewhere, form an approximate identity for $\mathscr{C}$. For any $F \in \mathscr{C}^{**}$ and integers n, m we claim that $E_n *_1 F *_1 E_m$ is in $\mathscr{C}$. To see this first note that for $f \in \mathscr{C}^*$ $(E_n *_1 F *_1 E_m)f = E_n[(F*_1E_m)*_1f] = [(F*_1E_m)*_1f]E_n = (F*_1E_m)(f*_1E_n) = F[E_m*_1(f*_1E_n)]$. But $E_m*_1(f*_1E_n)$ which is an element of $\mathscr{C}^*$ has as its matrix, the matrix obtained from f by replacing by zeroes all rows after the n^{th} row and all columns after the m^{th} column. This can be observed directly. Thus $(E_n*F*E_m)f = \check{C}(f)$ where C is the compact operator with matrix (C_{ij}) where $C_{ij} = F(g_{ij})$ and g_{ij} has matrix with a one in the ij^{th} place and zeroes elsewhere. Hence $E_n*F*E_m = C$.

From the proof of [7, Prop. 3.3 and Cor. 4.2] it follows that if X has an unconditionally monotone, boundedly growing basis then the matrices with a finite number of rows are dense in $\mathscr{C}^*$. See the correction at the end of this paper for details. Thus $I*_1F = F$ for any $F \in \mathscr{C}^{**}$ since $(I*_1F)f = \lim (E_n*_1F)f = \lim F(f*_1E_n)$ and the matrix for $f*_1E_n$ can be obtained from that of f by replacing with zeroes all rows after then n^{th}.

To identify $\mathscr{N}$, first note that each $F \in \mathscr{C}^{**}$ can be regarded as having "rows" which are elements of X^{***} and "columns" which are elements of X^{**}. The n^{th} "row" of F is the restriction of F to the elements of $\mathscr{C}^*$ whose matrices have zeros outside the n^{th} row; "columns" are similarly defined. (Of course, a "row" of F in this sense does not have a sequence of numbers associated with it.)

Then note that $(F*_2I)f = \lim F(f*_2E_n)$ and recall that $f*_2E_n$ has as its matrix the first n columns of f. Recall also that the hypotheses imply that the matrices with a finite number of rows are norm dense in $\mathscr{C}^*$. Thus $\lim F(f*_2E_n) = F(f)$ for all f in $\mathscr{C}^*$ if and only if each row of F is in X^*, since by hypothesis the basis for X is shrinking.

EXAMPLE 3.3. For $X = c_0$ or $X = c_0 \oplus \ell^p$ with $1 < p < \infty$ the natural basis is boundedly growing. Moreover, $\mathscr{N}$ is strictly contained between $\mathscr{B}(X)$ and $\mathscr{C}^{**}$, because it will have some elements (with "columns" in X^{**}) which won't be in $\mathscr{B}(X)$.

Correction. In [7, Props. 3.2 and 3.3] the assumption that X is reflexive was mistakenly omitted. Of course, the main Theorem 3.2 is not affected, since there X was uniformly convex. Also, in the proof of [7, Cor. 4.2] it was stated that: If X has a boundedly growing, unconditionally monotone basis then the matrices with a finite number of rows are dense in $\mathscr{C}^*$. Here is a proof of that fact: Suppose the matrices with a finite number of rows are not dense in $\mathscr{C}^*$. We will show that this implies that the basis is not boundedly growing.

First note that there exists and $f \in \mathscr{C}^*$ such that f^N does not approach 0, where f^N is the matrix formed from f by deleting the first N rows and columns. To see this observe that for $g \in \mathscr{C}^*$, if $g^N \to 0$ then $g - g^N$ approaches g. Thus for $\lambda > 0$, $\exists K$: $\| g - (g - g^K)\| < \lambda/2$. Then since each column of g can be regarded as an element of X^* and the basis for X is shrinking, there exists an M such that the matrix consisting of the first M rows of $g - g^K$ will be within $\lambda/2$ of $g - g^K$. Therefore, since the matrices with a finite number of rows are assumed to be non dense in $\mathscr{C}^*$, there must exist an f for which f^N does not approach 0. Without loss of generality [7, Prop. 2.6] we can assume that $||f^N|| \downarrow 1$.

Given ε and n, let $\delta > 0$. Pick N_1: $||f^{N_1}|| < 1 + \delta$. Since the basis is shrinking, the finite operators are dense in $\mathscr{C}$. Thus there exists an integer $N_1' > N_1$ and a finite operator T_1 of norm 1 such that T_1 is concentrated on the manifold $X_1 = [e_{N_1}, \cdots, e_{N_1'}]$ and $f^{N_1}(T_1) > 1$. Let $N_2 = N_1' + 1$. There exists an operator T_2 of norm 1, concentrated on the manifold $X_2 = [e_{N_2}, \cdots, e_{N_2'}]$ such that $f^{N_2}(T_2) > 1$. Repeating this process n times, we can find $T_1, \cdots, T_n$ such that $f^{N_k}(T_k) > 1$, and the T_k are concentrated on disjoint basic blocks. Hence

$$\begin{aligned} n &< f^{N_1}(T_1) + \cdots + f^{N_n}(T_n) = f^{N_1}(\mathrm{T}_1 + \cdots + T_n) \\ &\leqq ||f^{N_1}|| \, || T_1 + \cdots + T_n || \end{aligned}$$

thus $n/(1 + \delta) < || T_1 + \cdots + T_n ||$ and there exists an x of norm 1, wsere $x = x_1 + \cdots + x_n$ and each x_i in X_i such that $n/(1 + \delta) < || (T_1 + \cdots + T_n)x || = || T_1x_1 + \cdots + T_nx_n ||$. However, $\delta < 0$ was arbitrary. By picking δ small enough we can assure that $|| T_1x_1 + \cdots + T_nx_n ||$ is as close to n as we wish. By unconditional monotonicity, each $|| x_i || \leqq 1$. Thus each $|| T_ix_i || \leqq 1$ and since the T_ix_i are from disjoint blocks the basis won't be boundedly growing.

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