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FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

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Let T be an orientation preserving homeomorphism defined on a subset of the plane which interchanges two points, P and Q. Let Γ be a simple curve joining P and Q and let Ω be a simply connected set contained in the domain and range of T such that $\Gamma \subset \Omega$, $T(\Gamma) \subset \Omega$, $T^{-1}(\Gamma) \subset \Omega$. Then T has a fixed point in Ω. A corollary concerning fixed points of homeomorphisms on S^2 follows.

The proof would be trivial if T were necessarily an element of a flow on the plane, however an example given in this paper shows that this need not be the case.

If, in particular, T is defined in the whole plane or if its domain and range are the same halfplane, then the existence (but no constructive information about the location) of a fixed point could also be derived from classical results of Brouwer [2] (see for instance Proposition $0(a \Rightarrow b)$ and Proposition 1.1 of S. A. Andrea [1]).

1. The theorem. We use $\mathscr{R}$ to denote the real numbers and $\mathscr{R}^2$ for the coordinate plane. A curve is a continuous function whose domain is a compact interval of $\mathscr{R}$ and whose range is a subset of $\mathscr{R}^2$. If $[a, b]$ is an interval within the domain of the curve Φ, we use $\Phi[a, b]$ as a shorthand for $\{\Phi(t): t \in [a, b]\}$; $\Phi(a, b)$ and $\Phi[a, b)$ have analogous meanings. The terms close surve, simple curve, and simple closed curve have the standard meanings.

For the following lemmas we fix two simple closed curves, Φ_1 and $\Phi_2: [0, 3] \rightarrow \mathscr{R}^2$. The first, Φ_1, is the triangle defined by

$$\Phi_1(t) = \begin{cases} (2t - 1, 2t) & \text{for} \quad 0 \leqq t \leqq 1 \\ (1, 4 - 2t) & \text{for} \quad 1 \leqq t \leqq 2 \\ (5 - 2t, 0) & \text{for} \quad 2 \leqq t \leqq 3 \end{cases}$$

Referring to Figure 1, $\Phi_1[0, 1]$ is the segment MH, $\Phi_1[1, 2]$ is the segment HK, and Φ_1 $[2, 3]$ is the segment KM.

The second curve, Φ_2, is defined so that the following conditions are satisfied:

(I) $\Phi_2[0, 1]$ *is the segment from* $L = (\lambda, 0)$ (with $\lambda > 0$) to $M = (-1, 0)$; one has therefore: $\Phi_2(\rho) = 0 = (0, 0)$ for a suitable ρ with $0 < \rho < 1$.

(II) $\Phi_2[0, 3]$ has winding number -1 about each of its interior points (just as Φ_1 has); and therefore $\Phi_2[1, 3]$ has winding number $-1/2$ about the origin (just as $\Phi_1[0, 2]$ has).

(III) $\Phi_2(1, 2]$ is disjoint from $\Phi_1[1, 3]$. In Figure 1, $\Phi_2[1, 2]$ is represented by the curve MN, which except for M is disjoint from HK and KM.

(IV) $\Phi_2[2, 3)$ is disjoint from $\Phi_1[2, 3]$. In Figure 1, $\Phi_2[2, 3]$ is represented by the curve NL, which except perhaps for L, is disjoint from KM.

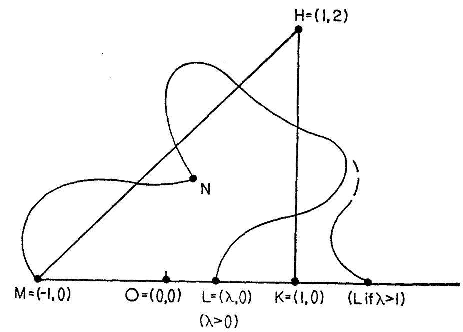

FIGURE 1

LEMMA 1. *With Φ_1 and Φ_2 as above, the closed curve $\Psi(t) = \Phi_1(t) - \Phi_2(t)$, $0 \leqq t \leqq 3$, has winding number -1 about the origin.*

Proof. It is clear that $\Psi[0, 3]$ is a closed curve with $\Psi(t) \neq (0, 0)$ for all t, so that the winding number of Ψ about the origin is defined. The idea of the following proof is to deform Ψ without touching the origin, into a curve which obviously has winding number -1 about the origin.

Let

$$\Xi(u, v) = \Phi_1(u) - \Phi_2(v) .$$

From conditions (I) (II) (III) (IV) it follows rather easily that $\Xi^{-1}(0)$ is a subset of the hatched area in Figure 2. For our purpose it is enough to prove that the origin is never in the range of $\Xi(u, v)$ restricted to the dotted region in Figure 2, i.e. the region bounded by the segment AG and the piecewise linear curve $\Sigma = ABCDEFG$ where $A = (0, 0)$, $B = (0, \rho)$, $C = (2, \rho)$, $D = (2, 1)$, $E = (5/2, 1)$, $F = (5/2, 3)$, $G = (3, 3)$. In details: for $0 \leqq u \leqq 1$, $0 \leqq v \leqq \rho$, $\Phi_1(u)$ is a point of the segment MH wnile $\Phi_2(v)$ is on the segment LO, hence $\Phi_1(u) \neq \Phi_2(v)$; for $0 < u \leqq 2$, $\rho \leqq v \leqq 1$, $\Phi_1(u)$ is on one of the segmehts MH,

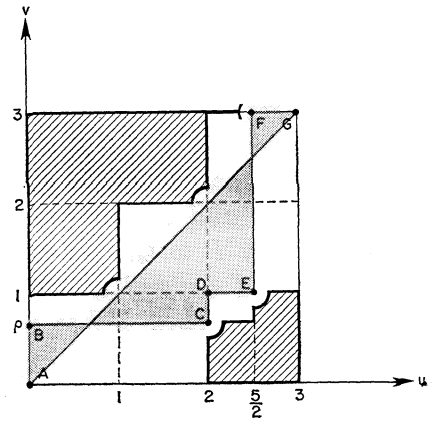

FIGURE 2

HK but not at M (since $u > 0$) while $\Phi_2(v)$ is on the segment OM, hence again $\Phi_1(u) \neq \Phi_2(v)$; for $1 \leqq u \leqq 5/2$, $1 \leqq v \leqq 2$, $\Phi_1(u)$ is on one of the segments HK, KO while $\Phi_2(v)$ is a point of the curve MN which by condition (III) cannot intersect HK, KM except at M, but M is not on HK, KO, hence again $\Phi_1(u) \neq \Phi_2(v)$; for $2 \leqq u \leqq 3$, $2 \leqq v < 3$ the same conclusion follows from condition (IV); finally for $5/2 \leqq u \leqq 3$, $v = 3$, $\Phi_1(u)$ is on the segment OM while $\Phi_2(v) = L$, hence $\Phi_1(u) \neq \Phi_2(v)$ everywhere in the dotted region.

The diagonal $\Delta(t) = (t, t)$, $0 \leqq t \leqq 3$, is obviously homotopic to $\Sigma = ABCDEFG$ staying within the dotted region in which, as just proved, $\Xi(u, v)$ is never the origin. Hence

$$\Psi(t) = \Phi_1(t) - \Phi_2(t) = (\Xi \circ \Delta)(t)$$

is homotopic to $(\Xi \circ \Sigma)(t)$ never hitting the origin, and therefore the winding numbers about the origin are the same.

It only remains to check that $\Xi \circ \Sigma$ has winding number -1 about the origin, which follows just by adding the winding numbers of $\Xi \circ AB$, $\Xi \circ BC$, $\Xi \circ CD$, $\Xi \circ DE$, $\Xi \circ EF$, $\Xi \circ FG$ which turn out to be respectively 0, $-1/2$ (because of condition (II)), 0, 0, $-1/2$ (because of condition (II)), 0.

LEMMA 2. *Let Φ_1 and Φ_2 be defined as above. Let T be a homeomorphism defined on Φ_1 as well as in its interior and such that $T(\Phi_1(t)) = \Phi_2(t)$, $0 \leqq t \leqq 3$. Then T has a fixed point which is contained in the intersection of the interiors of Φ_1 and Φ_2.*

Proof. Assume T has no fixed point in the intersection of the interiors of the simple closed curves Φ_1, Φ_2. Then

$$H_s(t) = s\Phi_1(t) - T(s\Phi_1(t))$$

is a homotopy from the constant $-T(0)$ map to $\Phi_1(t) - \Phi_2(t)$ which never hits the origin. This contradicts Lemma 1.

THEOREM. *Let T be an orientation preserving homeomorphism defined in a subset of the plane and interchanging two points P, Q. Let Γ be a simple curve joining P to Q, and Ω a simply connected set contained in the domain of T as well as in its range and such that $\Gamma \subset \Omega$, $T(\Gamma) \subset \Omega$, $T^{-1}(\Gamma) \subset \Omega$. Then T has a fixed point in Ω.*

Proof. We show that the situation of Lemma 2 must occur. We may assume that the plane is coordinatized so that: $P = (1, 0)$, $Q = (0, 0)$, and Γ is the segment PQ. We may also assume (replacing P, Q with another pair of points if necessary) that T interchanges no pair of points of Γ between P and Q. Let ϕ be the parametrization of Γ given by $\phi(t) = (1 - t, 0)$, $0 \leqq t \leqq 1$. Define:

$$t_1 = \inf \{t\colon \phi[0, t] \cap T\phi[0, t] \neq \emptyset \quad \text{or} \quad \phi[0, t] \cap T^{-1}\phi[0, t] \neq \emptyset\}$$

(the set is nonempty since it contains the number 1). It is clear that either $\phi(t_1) \in T\phi[0, t_1]$ or $\phi(t_1) \in T^{-1}\phi[0, t_1]$. First, we will assume only one of these events occurs. Later, we will consider the case when both inclusions are valid.

By replacing T by T^{-1} if necessary, we may assume the second of the two inclusions, that is $\phi(t_1) \in T^{-1}\phi[0, t_1]$, or equivalently $T\phi(t_1) \in \phi[0, t_1]$. Let $M = \phi(t_1)$, $H = T^{-1}(M)$, $L = T(M)$, and define $t_0 \leqq t_1$ to be the scalar such that $\phi(t_0) = L$. We may assume $t_0 < t_1$, otherwise $\phi(t_1)$ is a fixed point. Define $t_{-1} = \sup \{t \leqq t_0\colon T^{-1}\phi(t) \in T\phi[0, t_1]\}$, and let $N = T^{-1}\phi(t_{-1})$. Finally, choose t_* so that $T\phi(t_*) = N$ and let $K = \phi(t_*)$. The situation is summarized in Figure 3. Now the three paths

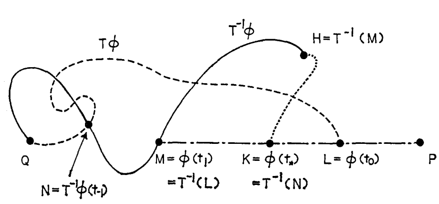

FIGURE 3

$\phi(t)$	$t_0 \leqq t \leqq t_1$	from	L	to	M
$T^{-1}\phi(t_0 - t)$	$0 \leqq t \leqq t_0 - t_{-1}$	from	M	to	N

and

$T\phi(t)$	$t_* \leqq t \leqq t_0$	from	N	to	L

form a simple closed curve which turns out to be contained in Ω (because it is composed by portions of Γ, $T(\Gamma)$, $T^{-1}(\Gamma)$) together with its interior (because Ω is simply connected). Let $\Phi_2: [0, 3] \to \mathscr{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path LM, $\Phi_2[1, 2]$ is the path MN, and $\Phi_2[2, 3]$ is the path NL. Define $\Phi_1(t) = T^{-1}\Phi_2(t)$ for $0 \leqq t \leqq 3$; notice that $\Phi_1[0, 1] = T^{-1}\phi[t_0, t_1]$ is a path from M to H, $\Phi_1[1, 2]$ is a path (not necessarily in Ω) from H to K, and $\Phi_1[2, 3] = \phi[t_*, t_1]$ is a path from K to M. Since T is a homeomorphism, Φ_1 is also a simple closed curve, and by applyidg the Schoenflies Theorem and introducing a new coordinate system on the plane (which may have the opposite orientation of the old one) we may assume that Φ_1 is identical to the triangle defined before Lemma 1 and that Φ_2 satisfies condition (I) for an appropriate choice of $\lambda > 0$. It only remains to show that Φ_2 satisfies conditions (II), (III), and (IV). Since $T\Phi_1 = \Phi_2$ and T preserves orientation, condition (II) is immediate. Condition (IV) follows from the choice of t_1. It is easily seen that the set $C_{MN} = \Phi_2(1, 2]$, which is the path from M to N, is disjoint from $\Phi_1[2, 3]$, so to verify condition (III) it suffices to show C_{MN} is disjoint from $C_{HK} = \Phi_1(1, 2] = \mathrm{T}^{-1}C_{MN}$, which is the path from H to K. To do this, observe that the path $\phi[t_{-1}, t_1]$ followed by the path C_{MN} is a simple curve, hence its image under T^{-1}, C_{NMHK} is also free of self-intersections. But the sets C_{MN} and C_{HK} are disjoint portions of the set C_{NMHK}.

We return now to the case that both $\phi(t_1) \in T\phi[0, t_1]$ and $\phi(t_1) \in T^{-1}\phi[0, t_1]$. In this case we refer to Figure 4. Let $M = \phi(t_1)$ and define t_0 and t_* so that $M = T^{-1}\phi(t_0) = T\phi(t_*)$. We have $t_* \neq t_0$, since equality would violate the assumption made in the third sentence of this proof. Replacing T by T^{-1} if necessary, we may assume $t_* < t_0$. Let $H = \phi(t_*)$, $K = \phi(t_0)$ and $N = T(K)$. The two paths

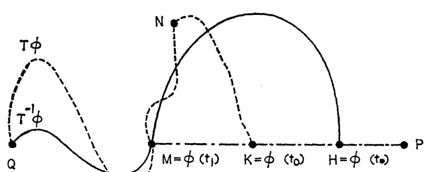

FIGURE 4

$$\phi(t) \quad t_0 \leqq t \leqq t_1 \quad \text{from} \quad K \quad \text{to} \quad M$$

and

$$T\phi(t) \quad t_* \leqq t \leqq t_1 \quad \text{from} \quad M \quad \text{through} \quad N \quad \text{to} \quad K$$

form a simple closed curve; let $\Phi_2: [0, 3] \to \mathscr{R}^2$ be a parametrization of this curve so that $\Phi_2[0, 1]$ is the path KM, $\Phi_2[1, 2]$ is the path

MN, and $\Phi_2[2, 3]$ is the path NK. Define $\Phi_1(t) = T^{-1}\Phi_2(t)$, $0 \leqq t \leqq 3$. The remainder of the proof in this case is analogous to the proof of the first case.

2. Remarks and examples. The proof of the theorem would become trivial if the hypotheses guaranteed the existence of a closed curve from P through Q to P which is transformed into itself. But this is not always true, for if $T_1: \mathscr{R}^2 \to \mathscr{R}^2$ is defined by $T_1(x, y) = (-x, -y + \sin x)$ then T_1 is an orientation-preserving homeomorphism on $\mathscr{R}^2$ which interchanges the points $P = (\pi, 0)$ and $Q = (-\pi, 0)$; however, *there exists no bounded connected set containing P and Q which is transformed into itself.* It is interesting to note that this implies that T_1 is *not* an element of any flow on $\mathscr{R}^2$.

To see that the orientation-preserving hypothesis is necessary, consider the homeomorphism $T_2: \mathscr{R}^2 \to \mathscr{R}^2$ defined by

$$T_2(x, y) = \begin{cases} (-x, y) & \text{if } |x| \geqq \dfrac{\pi}{2} \\ (-x, y + \cos x) & \text{if } |x| < \dfrac{\pi}{2} \end{cases}$$

T_2 interchanges every pair of points $\{(x, y), (-x, y)\}$ for which $|x| \geqq \pi/2$, but has no fixed point.

We conclude with a simple corollary to the Theorem (with $\Omega = \mathscr{R}^2$). Our notation follows that of [3].

COROLLARY. *Let $T: S^2 \to S^2$ be a homeomorphism such that T is of Brouwer degree 1, and T interchanges two points. Then T has two fixed points.*

Proof. Standard results in algebraic topology (see [3] page 124, Exercise 3) show that T has at least one fixed point, say U. Now $S^2 \sim U$ is homeomorphic to $\mathscr{R}^2$; let $h: S^2 \sim U \to \mathscr{R}^2$ be a homeomorphism. Then $h \circ T \circ h^{-1}$ is a homeomorphism of $\mathscr{R}^2$ which interchanges two points, and Theorem 34, page 122 of [3] shows that it also preserves orientation. Thus $h \circ T \circ h^{-1}$ has a fixed point, say W, and $V = h^{-1}(W)$ is another fixed point of T.

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