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MORE SUM THEOREMS FOR TOPOLOGICAL SPACES

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By a sum theorem for topological spaces is meant a theorem of the following type: If $\mathscr{F}$ is a cover of a space X, each element of which possesses a property $\mathscr{P}$, then X also possesses the property $\mathscr{P}$. Different types of sum theorems for various classes of topological spaces have been obtained from time to time by various authors. Perhaps the simplest known sum theorem is the locally finite sum theorem which states the following:

If $\{F_\alpha: \alpha \in \Lambda\}$ be a locally finite closed covering of a space X such that each F_α possesses a property $\mathscr{P}$, then X possesses $\mathscr{P}$.

The locally finite sum theorem is shown to hold for a large number of important topological properties.

R. E. Hodel [7] obtained three sum theorems which were applicable to all those properties for which the locally finite sum theorem is true and which are closed hereditary (A property is said to be closed hereditary if when possessed by the space, it is also shared by every closed subspace.) Two of these three theorems of Hodel were improved by the authors in [15]. As applications of these theorems, the authors showed that these theorems not only offered new results for many important properties but also improved results of H. Tamano [20, Theorem 2], Y. Katuta [8, Theorem 5], S. Hanai and A. Okuyama [5, Theorem 3] and A. H. Stone [19, Theorem 3]. In the present paper, some more sum theorems are presented. As an application of Theorem 1, a result of A. H. Stone [19, Theorem 2(ii)] is improved. Theorem 2 is a slight improvement of Theorem 3 obtained by the authors in [15]. Applications of Theorem 3 add substantially to the list of properties for which the locally finite sum theorem is true. In Theorem 4, a general technique for proving the locally finite sum theorem for topological properties has been developed. As a consequence, the locally finite sum theorem has been shown to hold for many more properties. Also, it provides a simple and neat proof of the locally finite sum theorem for many properties for which it is known to hold already.

Suppose, for the first two theorems, that $\mathscr{P}$ is a property for which the locally finite sum theorem holds and which is closed hereditary.

THEOREM 1. *If X is a regular space and $\{V_\alpha: \alpha \in \Lambda\}$ is a locally* *finite open covering of X such that each V_α possesses the property $\mathscr{P}$ and $fr\, V_\alpha$* (frontier of V_α) *is Lindelöf for each $\alpha \in \Lambda$, then X possesses the property $\mathscr{P}$.*

Proof. In view of Theorem 2 in [15], it is sufficient to prove that each cl V_α possesses the property $\mathscr{P}$. Let $\alpha \in \Lambda$ be fixed. Since $fr\, V_\alpha$ is Lindelöf, there exists a countable subfamily $\{V_{\alpha_i}: i = 1, 2, \cdots\}$ such that $fr\, V_\alpha \subset \bigcup_{i=1}^{\infty} V_{\alpha_i}$. Let $V_1 = fr\, V_\alpha \sim \bigcup_{i=2}^{\infty} V_{\alpha_i}$. Then V_1 is a closed subset $fr\, V_\alpha$.

V_1 and $fr\, V_{\alpha_1}$ are disjoint closed Lindelöf subsets of X. By a simple modification of Tychonoff's construction for separating a pair of disjoint closed Lindelöf subsets of a regular space, it is easy to obtain an open set G_1 such that $V_1 \subset G_1 \subset \mathrm{cl}\, G_1 \subset V_{\alpha_1}$.

Suppose now that for each $i = 1, 2, \cdots, n-1$, we have defined the sets V_i and G_i such that $V_i \subset G_i \subset \mathrm{cl}\, G_i \subset V_{\alpha_i}$. If $V_n = fr\, V_\alpha \sim [(\bigcup_{k=1}^{n-1} G_k) \cup (\bigcup_{k=n+1}^{\infty} V_{\alpha_k})]$, then $V_n \subset V_{\alpha_n}$. Again, since X is regular, there exists an open set G_n such that $V_n \subset G_n \subset \mathrm{cl}\, G_n \cup V_{\alpha_n}$. Thus by induction there exists a countable family $\mathfrak{G} = \{G_n: n = 1, 2, \cdots\}$ of open sets which covers $fr\, V_\alpha$ and is such that $\{\mathrm{cl}\, G_n: n = 1, 2, \cdots\}$ is locally finite. To prove that $\mathfrak{G}$ covers $fr\, V_\alpha$, let $x \in fr\, V_\alpha$ and let n be the largest integer such that $x \in V_{\alpha_n}$. Then $x \in \bigcup_{k=1}^{n} V_{\alpha_k}$ and $x \notin \bigcup_{k=n+1}^{\infty} V_{\alpha_k}$. If $x \notin \bigcup_{k=1}^{n-1} G_k$, then $x \in fr\, V_{\alpha_k} \sim [\bigcup_{k=1}^{n-1} G_{\alpha_k} \cup (\bigcup_{k=n+1}^{\infty} V_{\alpha_k})] = V_n \subset G_n$. Also each cl G_n, being a closed subset of V_{α_n}, possesses the property $\mathscr{P}$. If $F_0 = \mathrm{cl}\, V_\alpha \sim \bigcup_{k=1}^{\infty} G_k$, then F_0 is a closed subset of $\mathrm{cl}\, V_\alpha$ and hence of X. Since $fr\, V_\alpha \subset \bigcup_{k=1}^{\infty} G_k$, we have, $F_0 \subset \mathrm{cl}\, V_\alpha \sim fr\, V_\alpha = V_\alpha$. It follows that F_0 possesses the property $\mathscr{P}$. Thus $\{\mathrm{cl}\, G_n: n = 1, 2, \cdots\} \cup F_0$ is a locally finite closed covering of $\mathrm{cl}\, V_\alpha$, each member of which possesses the property $\mathscr{P}$. It follows that $\mathrm{cl}\, V_\alpha$ possesses $\mathscr{P}$. This completes the proof of the theorem.

DEFINITION 1. [Y. Katuta, 8]. A family $\{A_\alpha: \alpha \in \Lambda\}$ is said to be order locally finite if there is a linear ordering '$<$' of the index set Λ such that for each $\alpha \in \Lambda$, the family $\{A_\beta: \beta < \alpha\}$ is locally finite at each point of A_α.

Every σ-locally finite family is order locally finite but not conversely.

DEFINITION 2. [Aull, 1]. A subset A of a space X is said to be α-paracompact if every open (in X) covering of A has a locally finite (at points of X) open (in X) refinement.

THEOREM 2. *Let X be a regular space and let $\mathscr{V}$ be an order locally finite open covering of X such that each $V \in \mathscr{V}$ possesses $\mathscr{P}$*

and $fr\,V$ is α-paracompact for each $V \in \mathscr{V}$. Then X possesses $\mathscr{P}$.

Proof. Let $V \in \mathscr{V}$. Since $fr\,V$ is α-paracompact and X is regular, there exists a locally finite open (*in* X) covering $\{U_\alpha : \alpha \in \Lambda\}$ of $fr\,V$ such that each $\mathrm{cl}\,U_\alpha$ is contained in some member of $\mathscr{V}$. For each $\alpha \in \Lambda$, let $W_\alpha = \mathrm{cl}\,U_\alpha \cap \mathrm{cl}\,V$ and let $W_0 = \mathrm{cl}\,V \sim \bigcup_{\alpha \in \Lambda} U_\alpha$. It follows that W_0 and each W_α possesses the property $\mathscr{P}$. Thus, $\{W_\alpha : \alpha \in \Lambda\} \cup W_0$ is a locally finite closed covering of $\mathrm{cl}\,V$ every member of which possesses the property $\mathscr{P}$. Hence X possesses $\mathscr{P}$ in view of Theorem 2 in [14].

COROLLARY 1 [Singal and Arya, 15]. *If $\mathscr{V}$ be an order locally finite open covering of a regular space X such that each $V \in \mathscr{V}$ possesses $\mathscr{P}$ and $fr\,V$ is compact for each $V \in \mathscr{V}$, then X possesses $\mathscr{P}$.*

We shall now examine those properties of topological spaces to which Theorems 1 and 2 are applicable.

REMARK 1. We list below those properties of topological spaces for which the locally finite sum theorem holds:

Regularity [12], normality, [9], collectionwise normality [9], complete normality [9], perfect normality [9], monotone normality [6], metrizability [12], symmetrizability [3], paracompactness [15], pointwise paracompactness [7], subparacompactness [16], $\mathfrak{M}$-paracompactness and normality [14], $\mathfrak{M}$-subparacompactness [17], stratifiability [2], the property of being a normal M-space [10], the property of being a σ-space [13], the property of being an aleph space [18], the property of being a locally Lindelöf space [22], the property of being a space of countable type [21], and the property of being a Σ-space [11].

REMARK 2. The following properties of topological spaces are hereditary:

Regularity, perfect normality, complete normality, metrizability, stratifiability, the property of being a σ-space, the property of being an aleph space.

REMARK 3. The following properties of topological spaces are closed hereditary:

Normality, collectionwise normality, monotone normality, paracompactness, pointwise paracompactness, subparacompactness, $\mathfrak{M}$-paracompactness, $\mathfrak{M}$-subparacompactness, the property of being a locally Lindelöf space, the property of being a normal M-space, the property of being a space of countable type.

It follows that Theorems 1 and 2 hold for all properties mentioned in Remarks 2 and 3 above. When applied to metrizability, Theorem 1 improves a result of A. H. Stone [19, Theorem 2(ii)]. Many new results are obtained when Theorems 1 and 2 are applied to other properties of topological spaces listed in Remarks 2 and 3 above.

THEOREM 3. *Let $\mathscr{P}$ be a property for which the locally finite sum theorem holds. Then the locally finite sum theorem also holds for the property hereditarily $\mathscr{P}$.*

Proof. Let $\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite closed covering of a space X such that each F_α possesses the property $\mathscr{P}$ hereditarily. Let A be any subset of X. Then $\{A \cap F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed (*in* A) covering of A. Since each F_α possesses the property $\mathscr{P}$ hereditarily, each $A \cap F_\alpha$ possesses $\mathscr{P}$. Hence, in view of the hypothesis, X possesses $\mathscr{P}$.

In view of Theorem 2 of R. E. Hodel [7], Theorem 2 of the authors [15] and Theorems 1 and 2 obtained above, we have the following important corollaries to Theorem 3 above for any property $\mathscr{P}$ for which the locally finite sum theorem holds. In the following corollaries 2–5, $\mathscr{P}$ is a property for which LFST holds.

COROLLARY 2. *If $\mathscr{V}$ be a σ-locally finite elementary covering of a space X such that each $V \in \mathscr{V}$ possesses the property $\mathscr{P}$ hereditarily, then X possesses $\mathscr{P}$ hereditarily.*

COROLLARY 3. *If $\mathscr{V}$ be an order locally finite open covering of a space X such that for each $V \in \mathscr{V}$, $\operatorname{cl} V$ possesses the property $\mathscr{P}$ hereditarily, then X possesses $\mathscr{P}$ hereditarily.*

COROLLARY 4. *If $\mathscr{V}$ be a locally finite open covering of a regular space X such that each $V \in \mathscr{V}$ possesses $\mathscr{P}$ hereditarily and $\operatorname{fr} V$ is Lindelöf for each $V \in \mathscr{V}$, then X possesses $\mathscr{P}$ hereditarily.*

COROLLARY 5. *If $\mathscr{V}$ be an order locally finite open covering of a regular space X such that each $V \in \mathscr{V}$ possesses the property $\mathscr{P}$ hereditarily and $\operatorname{fr} V$ is α-paracompact for each $V \in \mathscr{V}$, then X possesses $\mathscr{P}$ hereditarily.*

REMARK 4. In view of Theorem 3 above, it follows that the locally finite sum theorem holds for the property hereditarily $\mathscr{P}$, where $\mathscr{P}$ stands for any property mentioned in Remark 1. Also, the above Corollaries 2 to 5 hold for any property $\mathscr{P}$ mentioned in Remark 1.

THEOREM 4. *Let* $\mathscr{P}$ *be a topological property satisfying the following*:

(a) *The disjoint topological sum of spaces possessing the property* $\mathscr{P}$ *also possesses* $\mathscr{P}$.

(b) $\mathscr{P}$ *is preserved under finite-to-one, closed continuous mappings.*

Then the locally finite sum theorem holds for $\mathscr{P}$.

Proof. The proof is based on a well-known construction which is essentially due to Morita [10, p. 871]. Let $\{F_\alpha: \alpha \in \Lambda\}$ be a locally finite closed covering of X such that each F_α possesses the property $\mathscr{P}$. For each $\alpha \in \Lambda$, let K_α denote a copy of F_α and let f_α be this homeomorphism. Let X^* be the disjoint topological sum of K_α's. Let $f: X^* \to X$ be the mapping defined as follows:

For each $x \in X^*$, $f(x) = f_\alpha(x)$ if $x \in K_\alpha$. In view of the hypothesis (a), X^* possesses $\mathscr{P}$. It can be easily verified that f is a finite-to-one, closed continuous mapping. It follows, in view of hypothesis (b), that X possesses the property $\mathscr{P}$. This completes the proof of the theorem.

COROLLARY 6. *If* $\mathscr{P}$ *be a property which is preserved under disjoint topological sums and under perfect maps* (quasi-perfect maps, quotient maps or closed continuous maps), *then the locally finite sum theorem holds for* $\mathscr{P}$.

We shall now examine the preservation of different properties of topological spaces under disjoint sums and under finite-to-one closed continuous maps.

REMARK 5. The disjoint topological sum of spaces possessing the property $\mathscr{P}$ possesses $\mathscr{P}$, where $\mathscr{P}$ stands for any of the following properties:

Regularity, normality, perfect normality, collectionwise normality, complete normality, monotone normality, metrizablity, paracompactness, countable paracompactness, pointwise paracompactness, subparacompactness, stratifiability, semi-stratifiability [4], the property of being a normal M-space, local compactness, the property of being a locally Lindelöf space, the property of being a P-space (that is, every G_δ-set is open), the property of being a Hausdorff strongly paracompact and locally Lindelöf space, the property of being a σ-space, the property of being an aleph space, the property of being a Σ-space, the property of being a Čech-complete space, the property of being a kc-space, the properey of being a space of countable type, local connectedness, local pathwise connectedness.

REMARK 6. The following properties are preserved under perfect maps:

Regularity, metrizability, local compactness, paracompactness, countable paracompactness, the property of being a Hausdorff strongly paracompact and locally Lindelöf space, the property of being a space of countable type, the property of being a kc-space, the property of being a σ-space, the property of being an aleph space, the property of being a Čech-complete space, the property of being a normal M-space.

REMARK 7. The following properties are preserved under closed continuous maps:

Normality, perfect normality, collectionwise normality, complete normality, monotone normality, the property of being a P-space, local connectedness, local pathwise connectedness, pointwise paracompactness, subparacompactness, $\mathfrak{M}$-subparacompactness, paracompactness and Hausdorff property, stratifiability, semi-stratifiability, the property of being a normal T_1-σ-space.

In view of Remarks 5 to 7, it follows from Theorem 4 and Corollary 6 that the locally finite sum theorem holds for all properties mentioned in Remark 5. Thus, a simple and neat proof of the locally finite sum theorem is obtained for most of the properties mentioned in Remark 1. Also, it follows that the locally finite sum theorem holds also for the following properties of topological spaces besides the ones mentioned in Remark 1:

Local connectedness, local pathwise connectedness, local compactness, countable paracompactness, the property of being a semi-stratifiable space, the property of being a Čech complete space, the property of being a P-space, the property of being a kc-space.

Leaving aside local connectedness and local pathwise connectedness, all the above properties are closed hereditary (at least!). This means that Theorem 2 of Hodel [7], theorem 2 of the authors [15] and Theorems 1 and 2 of this paper are applicable to all these properties and thus many new results are obtained.

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