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METRIZATION OF SPACES WITH COUNTABLE LARGE BASIS DIMENSION

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With the following results, we generalize known metrization theorems for spaces with large basis dimension 0 i.e., non-archimedean spaces) to the higher dimensions: *Theorem.* **If X is a normal Σ-space with countable large basis dimension, then X is metrizable.** *Theorem.* **If X is a normal $w\Delta$-space with countable large basis dimension, then X is metrizable.**

I. Introduction.

A collection Γ of subsets of a set X is said to have *rank* 1 if whenever g_1 and g_2 are in Γ with $g_1 \cap g_2 \neq \emptyset$ then $g_1 \subset g_2$ or $g_2 \subset g_1$. According to P. J. Nyikos [13], a topological space X has *large basis dimension* $\leqq n$ (denoted $\operatorname{Bad} X \leqq n$) if X has a basis which is the union of $n+1$ rank 1 collections of open sets. X has *countable large basis dimension* ($\operatorname{Bad} X \leqq \aleph_0$) if X has a basis which is the union of a countable number of rank 1 collections such that each point of X has a basis belonging to one of the collections (a property which is automatically true in the finite case). $\operatorname{Bad} X$ coincides with $\operatorname{Ind} X$ and $\dim X$ for metric spaces.

Spaces having large basis dimension 0 are usually called *non-archimedean* spaces. Theorems of Nyikos [11] and A. V. Archangelskii [3] show that a non-archimedean space is metrizable if and only if it is a Σ-space or a $w\Delta$-space. In this paper we show that these results are valid, under mild assumptions, for the higher dimensions. Our results also improve a result of G. Gruenhage [6], who showed that compact spaces having finite large basis dimension are metrizable.

II. Main results.

According to Nyikos [11], a *tree of open sets* is a collection Γ of open sets such that if $g \in \Gamma$, then the set $\{g' \in \Gamma \mid g' \supset g\}$ is well-ordered by reverse inclusion; that is, $g \leqq g'$ if and only if $g \supset g'$. Nyikos shows that the rank 1 collections for spaces with $\operatorname{Bad} X \leqq \aleph_0$ can be considered as rank 1 trees of open sets. The following fact will be used in our proofs:

LEMMA 1. *Let T be a rank 1 tree of open subsets of a regular space X which contains a basis at each point of a subset X' of X. Then if $\mathscr{U}$ is a cover of X' by open subsets of X, there exists a subset T' of T such that*

(i) *T' is a cover of X';*

(ii) *the elements of T' are pairwise disjoint;*

(iii) *$t \in T'$ implies that either t is degenerate or $\bar{t}$ is a proper subset of some member of $\mathscr{U}$.*

Proof. Put t in T' if and only if (a) either t is degenerate or there is a member U of $\mathscr{U}$ such that $\bar{t}$ is a proper subset of U and (b) there is no predecessor of t in T whose closure is a proper subset of some element of $\mathscr{U}$. Since T contains a basis at each point of X' and since the predecessors of a given $t \in T$ are well-ordered, it is easy to see that T' covers X'. Further, since T is a tree, the members of T' are mutually exclusive.

Nyikos calls a space *basically screenable* if it has a basis which is the union of countably many rank 1 trees of open sets. Every space X with Bad $X \leqq \aleph_0$ is basically screenable. Basically screenable spaces are, of course, screenable; that is, every open cover has a σ-pairwise disjoint open refinement. While the following result is known, for the sake of completeness, we include its easy proof:

LEMMA 2. *A screenable countably compact space X is compact* |2|.

Proof. Let $\mathscr{U}$ be an open cover of X and let $\mathscr{V} = \cup\{\mathscr{V}_n | n = 1, 2, \cdots\}$ be an open refinement of U covering X such that, for each i, the members of $\mathscr{V}_i$ are mutually exclusive. The set $\{V_n = \bigcup \mathscr{V}_n | n = 1, 2, \cdots\}$ is a countable open cover of X; hence, there exists a finite subcover $\{V_{n_1}, V_{n_2}, \cdots, V_{n_k}\}$. Then $\mathscr{V}_{n_1} \cup \mathscr{V}_{n_2} \cup \cdots \cup \mathscr{V}_{n_k}$ is a point-finite refinement of $\mathscr{U}$. Thus, X is metacompact and it is well-known that a metacompact countably compact space is compact.

According to C. R. Borges [4], a space X is a *wΔ-space* if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that whenever $x \in X$ and $x_n \in \mathrm{St}(x, \mathscr{G}_n)$ for each n, then $\{x_1, x_2, \cdots\}$ has a cluster point.

THEOREM 1. *If X is a regular wΔ-space with countable large basis dimension, then X has a point countable basis.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a sequence of open covers of X satisfying the properties given in the definition of a $w\Delta$-space. Let $\mathscr{B}_1, \mathscr{B}_2, \cdots$ and $X_1, X_2, \cdots$ be sequences such that $X = \bigcup\{X_i | i = 1, 2, \cdots\}$ and, for each i, $\mathscr{B}_i$ is a rank 1 tree of open sets containing a basis at each point of X_i.

For each $i < \omega_0$ and $\alpha < \omega_1$, we construct a collection $\mathscr{B}(i, \alpha)$ as follows: let $\mathscr{B}(i, 1)$ be a collection of mutually exclusive members of $\mathscr{B}_i$ that refines $\mathscr{G}_1$ and covers X_i.

Suppose $\mathscr{B}(i, \beta)$ has been defined for $\beta < \alpha$. If α is not a limit ordinal, applying Lemma 1, let $\mathscr{B}(i, \alpha)$ be a collection of mutually exclusive members of $\mathscr{B}_i$ such that

(i) if $j < \omega_0$, then $\mathscr{B}(i, j)$ refines $\mathscr{G}_j$;

(ii) $\mathscr{B}(i, \alpha)$ covers $(\cup \mathscr{B}(i, \alpha - 1)) \cap X_i$;

and (iii) $g \in \mathscr{B}(i, \alpha)$ implies $\bar{g}$ is a proper subset of some member of $\mathscr{B}(i, \alpha - 1)$, or g is degenerate. If α is a limit ordinal, for each $x \in X_i$, let $B(\alpha, x) = \operatorname{Int}(\bigcap_{\beta<\alpha} \{g \in \mathscr{B}(i, \beta) | x \in g\})$. Note that if x and y are in X_i, then either $B(\alpha, x) = B(\alpha, y)$ or $B(\alpha, x) \cap B(\alpha, y) = \varnothing$. Let $\mathscr{B}(i, \alpha) = \{B(\alpha, x) | x \in X_i\}$.

Let $\mathscr{B}_i^* = \bigcup_{\alpha<\omega_1} \mathscr{B}(i, \alpha)$. We will show that $\mathscr{B}_i^*$ is a point countable collection forming a basis for X_i in X.

We will say that g is a *chain in* $\mathscr{B}_i^*$ if g is a function from an initial segment of ω_1 into $\mathscr{B}_i^*$ so that (1) $g(\alpha) \in \mathscr{B}(i, \alpha)$ and (2) if $\beta < \alpha$, then $g(\beta) \supset g(\alpha)$. Note that by our construction, if $\beta < \alpha$, then $g(\beta) \supset \overline{g(\alpha)}$. Furthermore, if $x \in X_i$, then there is exactly one maximal chain, say g, such that $g(\alpha)$ contains x for every α in the domain of g.

Claim 1. The domain of each maximal chain in $\mathscr{B}_i^*$ is countable (and so, $\mathscr{B}_i^*$ is point countable in X).

Proof of Claim 1. Suppose the contrary; i.e., there is a chain, say g, of length ω_1.

Note that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact. To prove this, we will only show that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is countably compact; that $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact will then follow from Lemma 2. To this end, let N denote a countable subset of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$. There is an α so that $g(\alpha)$ does not meet N. In particular then, no point of $\overline{g(\alpha + 1)}$ is a limit point of N. Because of property (i), it must be the case that N has a limit point in $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$; and so, $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ is compact. But, $\{\overline{g(\omega_0 + 1)} - \overline{g(\alpha)} | \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0 + 1)} - \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$ with no finite subcover, which is a contradiction from which Claim 1 follows.

Claim 2: $\mathscr{B}_1^*$ is a basis for X_i in X; in particular, if $x \in X_i$ and g is the maximal chain in $\mathscr{B}_i^*$ centered at x, then $\{g(\alpha) | \alpha$ is in the domain of $g\}$ is a local basis for x in X.

Proof of Claim 2. Suppose otherwise. Then there is a point x of X_i so that the maximal chain, g, centered at x does not yield a basis at x in X; i.e., $\{g(\alpha) | \alpha \in \text{domain of } g\}$ is not a local basis for x in X. Since the domain of g is countable, there is a first $\alpha_0 < \omega_1$ not in the domain g. There is a member B of $\mathscr{B}_i$ so that if $\alpha < \alpha_0$, then $g(\alpha)$ is not a subset of B but this means that B is a subset of each $g(\alpha)$. Then x is in the interior of $\bigcap_{\alpha<\alpha_0} g(\alpha)$. Thus, by our construction of $\mathscr{B}(i, \alpha_0)$, there is a member of $\mathscr{B}(i, \alpha_0)$ that contains x. This contradicts the maximality of g and it follows that $\{g(\alpha) \mid \alpha$ is in the domain of $g\}$ is a local basis for x in X.

We now have that $\bigcup_{i<\omega_0} \mathscr{B}_i^*$ is a point countable basis for X.

If $\mathscr{H}$ is a cover of the space X and if $x \in X$, then $C(x, H)$ will denote the set $\cap \{H \in \mathscr{H} \mid x \in H\}$. According to K. Nagami [9], the space X is a Σ-space if there is a sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$ of locally finite closed covers of such that if $x_0, x_1, x_2, , \cdots$ is a sequence with $x_i \in C(x_0, \mathscr{F}_i)$ for each $0 < i < \omega_0$, then $\{x_i\}$ has a cluster point. The sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$ is called a spectral Σ-sequence for X.

We will, without loss of generality, assume that each $\mathscr{F}_i$ is closed under intersections and, for each i, $\mathscr{F}_{i+1}$ refines $\mathscr{F}_i$.

LEMMA 3. *If X is a space with countable large basis dimension such that each uncountable subset of X has a limit point, then X is Lindelof.*

Proof. Since X has countable large basis dimension, X is screenable. G. Aquaro [1] has proved that every meta-Lindelof (and thus every screenable) space in which every uncountable set has a limit point is Lindelof.

THEOREM 2. *If X is a regular Σ-space with countable large basis dimension then X has a point countable basis.*

Proof. Let $\mathscr{F}_1, \mathscr{F}_2, \cdots$ be a sequence of locally finite closed coverings of X given in the definition of a Σ-space. For each n, let $\mathscr{G}_n$ be an open cover of X such that each member of $\mathscr{G}_n$ intersects only finitely many members of $\mathscr{F}_n$. Let $\mathscr{B}_1, \mathscr{B}_2, \cdots$ and $X_1, X_2, \cdots$ be sequences such that $X = \bigcup_{i<\omega_0} X_i$ and $\mathscr{B}_i$ is a rank 1 tree of open sets which contains a basis for each point of X_i.

Define $\mathscr{B}(i, \alpha)$, $i < \omega_0$, $\alpha < \omega_1$, exactly as in the proof of Theorem 1. Let $\mathscr{B}_i^* = \bigcup_{\alpha<\omega_1} \mathscr{B}(i, \alpha)$ and define chain in $\mathscr{B}_i^*$ as in the proof to Theorem 1.

Claim 1. Every chain in $\mathscr{B}_i^*$ is countable.

Proof of Claim 1. Suppose otherwise; i.e., suppose that g is a chain in $\mathscr{B}_i^*$ with length ω_1. Let $K = \bigcap_{\alpha<\omega_1} \overline{g(\alpha)}$. Every uncountable of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$ for suppose otherwise; that is, suppose that H is an uncountable subset of $\overline{g(\omega_0)} - K$ with no limit point in $\overline{g(\omega_0)} - K$.

Suppose that there is a point, h, of H such that, for each n,

$C(h, \mathscr{F}_n)$ intersects infinitely many points of H. Then there is a countable subset N of H with a limit point. Since N is countable, there is an $\alpha < \omega_1$ so that $g(\alpha)$ does not intersect N. It follows that no point of K is a limit point of N. Hence, no point of K is a limit point of N; and so, H has a limit point in $\overline{g(\omega_0)} - K$. This is a contradiction from which it follows that, for each h in H, there is an integer $n(h)$ such that $C(h, n(h))$ intersects only finitely many members of H. Thus, there is an N and an uncountable subset H^* of H so that if $h \in H^*$, then $n(h) = N$ and $\{C(h, F_N) | h \in H^*\}$ is an infinite subcollection of $\mathscr{F}_N$, each member of which intersects $g(N)$. But, $g(N)$ is in $\mathscr{B}(i, N)$ which contradicts the fact that $\mathscr{B}(i, N)$ refines $\mathscr{G}_N$. It follows that each uncountable subset of $\overline{g(\omega_0)} - K$ has a limit point in $\overline{g(\omega_0)} - K$; and so, by Lemma 3, $\overline{g(\omega_0)} - K$ is Lindelof. But $\{\overline{g(\omega_0)} - \overline{g(\alpha)} | \alpha < \omega_1\}$ is an open cover of $\overline{g(\omega_0)} - K$ with no countable subcover which is a contradiction from which Claim 1 follows.

That $\mathscr{B}_i^*$ contains a basis at each point of X_i follows exactly as in the proof of Theorem 1. Thus Theorem 2 is proved.

THEOREM 3. *If X is a normal Σ-space with countable large basis dimension, then X is metrizable.*

Proof. R. E. Hodel has proved that every Σ-space is a β-space [8], and that every β-space is countably metacompact [7]. A screenable countably metacompact space is metacompact. Nagami [10] has shown that a normal screenable metacompact space is paracompact. But a paracompact Σ-space with a point-countable base is metrizable [9].

THEOREM 4. *If X is a normal $w\Delta$-space with countable large basis dimension, then X is metrizable.*

Proof. As above, X is normal, screenable, and metacompact (since every $w\Delta$-space is a β-space), hence paracompact. But a papacompact $w\Delta$-space is an M-space, hence a Σ-space. Thus X is metrizable.

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