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SOME ASPECTS OF T-NILPOTENCE II: LIFTING PROPERTIES OVER T-NILPOTENT IDEALS

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It has been shown by Năstăsescu and Popescu that every nonzero (left, unital) module over a ring R has a simple submodule if and only if the Jacobson radical J of R is right T-nilpotent and every nonzero R/J-module has a simple submodule. The work presented here arose largely from an attempt to find a general framework for results like this.

In § 2 it is shown that if R has a right T-nilpotent ideal I, then a bijection from the torsion classes of R/I-modules to those of R-Modules can be obtained by associating with each $\mathscr{T} \subseteqq \operatorname{Mod}(R/I)$ the lower radical class it defines as a class of R-modules. § 3 contains applications involving the lifting of torsion properties and in § 4 it is shown that if R has a right T-nilpotent ideal I such that R/I is the direct sum of its torsion and divisible ideals, then R has this property also.

In the present paper we continue the general investigation of T-nilpotence begun in [8]. The emphasis here is on T-nilpotent ideals, whereas previously we looked at whole rings, but there are some methodological similarities between the two papers.

1. Preliminaries. Except where we state otherwise, all rings considered are associative and have identities (though some results hold more generally) and modules are always left, unital modules. An ideal I (one or two-sided) is *left T-nilpotent* if for every sequence $a_1, a_2, \cdots$ of its elements, $a_1 a_2 \cdots a_n = 0$ for some n. *Right T-nilpotence* is defined analogously. The terms are due to Bass [2], though the concepts originated with Levitzki [19]. The terminology relating to the radical theory of modules varies somewhat; we shall describe things in the following way: An ordered pair $(\mathscr{R}, \mathscr{S})$ of classes of modules over a ring will be called a *radical theory* if (i) $\mathscr{R} \cap \mathscr{S} = \{0\}$, (ii) $\mathscr{R}$ is homomorphically closed, (iii) $\mathscr{S}$ is hereditary (i.e. closed under taking submodules), (iv) every module M has a largest submodule $\mathscr{R}(M)$ from $\mathscr{R}$ and (v) $M/\mathscr{R}(M) \in \mathscr{S}$ for all M. In this situation, $\mathscr{R}$ will be called a *radical class*, $\mathscr{S}$ a *semi-simple class*. If also $\mathscr{R}$ is hereditary, it will be called a *torsion class*, while if $\mathscr{S}$ is homomorphically closed it will be called a *TTF* class. $L_R(\mathscr{M})$ is the lower radical class defined by a class $\mathscr{M}$ of R-modules. With these conventions established, we refer the reader to [4], [5] and [14] for further details. Mod(R) is the class of all modules

over a ring R. $(0: M)$ is the annihilator of a module M.

2. Torsion from iterated annihilators. Until further notice, I is a right ideal of a ring R.

For any R-module M, we define submodules $M^{(\alpha)}$ for all ordinals α, as follows:

$$M^{(0)} = 0; M^{(\alpha+1)} = \{m \in M \mid Im \subseteq M^{(\alpha)}\} ;$$
$$M^{(\beta)} = \bigcup_{\alpha<\beta} M^{(\alpha)}, \text{ if } \beta \text{ is a limit} .$$

Eventually, $M^{(\mu+1)} = M^{(\mu)}$. If $M^{(\mu)} = M$, then

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(\alpha)} \subseteq M^{(\alpha+1)} \subseteq \cdots \subseteq M^{(\mu)} = M$$

is called the *upper ascending I-series* of M. More generally, a transfinite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\mu = M$$

of submodules is called an *ascending I-series* for M if

$$IM_{\alpha+1} \subseteq M_\alpha \text{ for each } \alpha \text{ and } \bigcup_{\alpha<\beta} M_\alpha = M_\beta \text{ for all limits } \beta .$$

Finally, $\mathscr{M}_I = \{M \in \mathrm{Mod}\,(R) \mid IM = 0\}$.

THEOREM 2.1. *The following conditions are equivalent for a module M:*

(i) *The upper ascending I-series of M exists.*

(ii) *M has an ascending I-series.*

(iii) *Every nonzero homomorphic image of M has a nonzero submodule in $\mathscr{M}_I$.*

(iv) *For any $x \in M$ and $a_1, a_2, \cdots \in I$, there exists n such that $a_n a_{n-1} \cdots a_2 a_1 x = 0$.*

(v) *$M \in L_R(\mathscr{M}_I)$.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): If $N \subsetneqq M$, let

$$\beta = \min \{\alpha \mid M_\alpha \nsubseteq N\}$$

for some ascending I-series

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\mu = M .$$

Then $\beta \neq 0$ and β is not a limit, so $(M_\beta + N)/N$ is a nonzero submodule of M/N with $I(M_\beta + N)/N \subseteq (M_{\beta-1} + N)N = 0$.

(ii) $\Rightarrow$ (iv): With the same notation, for $x \in M$ and $a_1, a_2, \cdots \in I$

let

$$\gamma = \min\{\alpha \mid M_\alpha \text{ contains some } a_n a_{n-1} \cdots a_2 a_1 x\}\ ,$$

and let $a_m a_{m-1} \cdots a_2 a_1 x$ be in M_γ. Then γ is not a limit, and if $\gamma \neq 0$, then $a_{m+1}(a_m a_{m-1} \cdots a_2 a_1 x) \in IM_\gamma \subseteqq M_{\gamma-1}$, contrary to the assumed minimality of γ. Hence $\gamma = 0$, so $a_m a_{m-1} \cdots a_2 a_1 x = 0$.

(iv) $\Rightarrow$ (iii): Let $0 \neq x \in M$. If $Ix \neq 0$, let $a_1 x \neq 0$, where $a_1 \in I$. If $Ia_1 x \neq 0$, let $a_2 a_1 x \neq 0$, where $a_2 \in I$. Proceeding thus, we obtain an element $b = a_n a_{n-1} \cdots a_2 a_1 x \neq 0$ where $a_1, a_2, \cdots, a_n \in I$ and $Ib = 0$. Thus M has a submodule which is annihilated by I. Since property (iv) is preserved in homomorphic images, the result follows.

(iii) $\Rightarrow$ (i): Let $M^{(\mu)} = M^{(\mu+1)}$. Then $[M/M^{(\mu)}]^{(1)} = M^{(\mu+1)}/M^{(\mu)} = 0$, so $M/M^{(\mu)} = 0$.

The equivalence of (iii) and (v) is well-known.

The proof just given is patterned after some proofs in [8]. Kashu [15], working with modules over a ring R not necessarily having an identity, has shown that $L_R(\mathscr{Z})$, where $\mathscr{Z}$ is the class of modules with trivial multiplication, consists precisely of those modules M such that for every sequence $a_1, a_2, \cdots$ in R and for every $x \in M$, $a_n a_{n-1} \cdots a_2 a_1 x = 0$ eventually. Kellett [16], in the same setting, has shown that $L_R(\mathscr{Z})$ is the class of modules with an upper ascending R-series.

Since $L_R(\mathscr{M}_I) = \mathrm{Mod}\,(R)$ if and only if $R \in L_R(\mathscr{M}_I)$ we have the following consequence of Theorem 2.1.

COROLLARY 2.2. *$L_R(\mathscr{M}_I) = \mathrm{Mod}\,(R)$ if and only if I is right T-nilpotent.*

PROPOSITION 2.3. *$L_R(\mathscr{M}_I)$ is a torsion class. Moreover, if $\mathscr{R}$ is a torsion subclass of $L_R(\mathscr{M}_I)$, then $\mathscr{R} = L_R(\mathscr{R} \cap \mathscr{M}_I)$.*

Proof. Since $\mathscr{M}_I$ is hereditary, the first assertion is clear. If $\mathscr{R}$ is a torsion subclass of $L_R(\mathscr{M}_I)$, then clearly $L_R(\mathscr{R} \cap \mathscr{M}_I) \subseteqq \mathscr{R}$. On the other hand, if $M \in \mathscr{R} \subseteqq L_R(\mathscr{M}_I)$, then every nonzero homomorphic image of M has a non zero submodule $N \in \mathscr{M}_I$. But any such N is in $\mathscr{R}$, as the latter is a torsion class. Thus each nonzero homomorphic image of M has a nonzero submodule in $\mathscr{R} \cap \mathscr{M}_I$, so $M \in L_R(\mathscr{R} \cap \mathscr{M}_I)$.

For the remainder of this section, I is a *two-sided* ideal.

The proof of the following result is straightforward.

PROPOSITION 2.4. *With the notation of Proposition 2.3, $\mathscr{R} \cap \mathscr{M}_I$ is a torsion class in* $\mathrm{Mod}\,(R/I)$.

THEOREM 2.5. *The correspondence* $\mathscr{T} \mapsto L_R(\mathscr{T})$ *defines a bijection from the set of torsion classes in* Mod (R/I) *to the set of torsion subclasses of* $L_R(\mathscr{M}_I)$ *in* Mod (R).

Proof. By Propositions 2.3 and 2.4, the correspondence is surjective. If $\mathscr{T}, \mathscr{U}$ are torsion classes in Mod (R/I) such that $L_R(\mathscr{T}) = L_R(\mathscr{U})$, then every nonzero R/I-module in $\mathscr{U} \subseteq L_R(\mathscr{T})$ has a nonzero R-(and thus R/I-) submodule in $\mathscr{T}$, whence $\mathscr{U} \subseteq \mathscr{T}$, since $\mathscr{U}$ is homomorphically closed. Injectivity of the correspondence follows.

Using Proposition 2.2, we obtain.

COROLLARY 2.6. *If I is right T-nilpotent, the correspondence* $\mathscr{T} \mapsto L_R(\mathscr{T})$ *defines a bijection from the torsion classes in* Mod (R/I) *to those in* Mod (R).

3. Lifting some torsion properties. We now apply the results of the previous section to obtain some connections between the kinds of torsion classes a ring R and a factor ring R/I can have when I is right T-nilpotent.

Suppose we have canonically associated with each ring R a class $\mathscr{X}_R$ of R-modules. Let

$$X = X(R) = \bigcap \{(0: M) \mid M \in \mathscr{X}_R\} .$$

Suppose further that $\mathscr{X}_{R/X} = \mathscr{X}_R$ for all R.

THEOREM 3.1. *Every nonzero R-module has a nonzero submodule in $\mathscr{X}_R$ if and only if X is right T-nilpotent and every nonzero R/X-module has a nonzero submodule in $\mathscr{X}_R$.*

Proof. If X is right T-nilpotent, then by Proposition 2.2, every nonzero R-module has a nonzero submodule in $\mathscr{M}_X =$ Mod (R/X). If in addition every nonzero R/X-module has a nonzero submodule in $\mathscr{X}_R$, the same must therefore be true of all nonzero R-modules. If, on the other hand, nonzero R-modules all have nonzero submodules in $\mathscr{X}_R$, then

$$\text{Mod}\,(R) = L_R(\mathscr{X}_R) \subseteq L_R(\mathscr{M}_X) ,$$

so by Proposition 2.2, X is right T-nilpotent. Also, if $0 \neq M \in \mathscr{M}_X =$ Mod (R/X), then M has a nonzero submodule in $\mathscr{X}_R$.

An example of a class satisfying the requirements of Theorem 3.1 is the class of simple R-modules, together with 0. A ring R is (*left*) *semi-artinian* if every nonzero R-module has a simple submodule. The following result was obtained in a different manner by Năstăsescu and Popescu [21].

COROLLARY 3.2. *A ring R is semi-artinian if and only if its Jacobson radical J is right T-nilpotent and R/J is semi-artinian.*

We call a nonzero module *homogeneous* (the terminology varies) if it is a rational extension of every nonzero submodule. (For the relevant details concerning rational extensions, see [23].) Hudry [9], [10], [11] calls a ring R (*left*) *locally homogeneous* (briefly, *L. H.*) if every nonzero R-module has a homogeneous submodule.

For any ring R, let

$H = H(R) = \bigcap \{(0: M) \mid M$ is a homogeneous R-module$\}$.

COROLLARY 3.3. *A ring R is L. H. if and only if H is right T-nilpotent and R/H is* L.H.

Examples of L.H. rings are left noetherian ([9], [10], [11]) and semi-artinian rings.

Nonzero submodules of homogeneous modules are homogeneous (see, for example, [23], p. 625) and thus a ring R is *L. H.* if and only if every nonzero R-module has a cyclic homogeneous submodule. When R is commutative, a cyclic R-module R/P is homogeneous precisely when P is prime (see, for example [23], pp. 625–626). Thus for commutative rings R, $H(R)$ is the prime radical $B(R)$ and a commutative *L. H.* ring is one for which every nonzero module has an associated prime. The latter have been studied (under the name *D-rings*) by by Nguyen-Trong-Kham [17], [18].

COROLLARY 3.4. *A commutative ring R is a D-ring if and only if $B(R)$ is T-nilpotent and $R/B(R)$ is a D-ring.*

A nonzero module is *co-irreducible* if it is an essential extension of every nonzero submodule. (Such modules are also called *uniform*.) Năstăsescu and Popescu [20], [21] call a ring R (*left*) *locally co-irreducible* (briefly, *L.C.*) if every nonzero R-module has a co-irreducible submodule.

COROLLARY 3.5. *A ring R is* L.C. *if and only if*

$$C = C(R) = \bigcap \{(0: M) \mid M \text{ is a co-irreducible } R\text{-module}\}$$

is right T-nilpotent and R/C is L.C.

Let $\mathscr{X}_R$ be a nonempty class of R-modules and $\tilde{\mathscr{X}}_R$ be the class of R-modules, every nonzero homomorphic image of which has a nonzero submodule in $\mathscr{X}_R$. For the purposes of further applications of Theorem 2.5, it may be worthwhile to note that if $\mathscr{X}_R$ is not homomorphically closed, one can have $\tilde{\mathscr{X}}_R \subsetneqq L_R(\mathscr{X}_R)$. For instance, let R be the ring Z of integers and $\mathscr{X}_Z$ the class of unbounded p-groups, together with 0. Then $\tilde{\mathscr{X}}_Z$ is the class of divisible p-groups, while $L_Z(\mathscr{X}_Z)$ is the class of all p-groups. When $\mathscr{X}_R$ is the class of homogeneous modules, Hudry [12], [13] has shown that $\tilde{\mathscr{X}}_R = L_R(\mathscr{X}_R)$ for all rings R. A parallel argument establishes the corresponding equality for the co-irreducible modules.

We next consider the lifting of a property possessed by the ring of integers.

THEOREM 3.6. *Let R be a ring with a right T-nilpotent ideal I. If every nontrivial torsion class in* Mod (R/I) *is determined by simple modules, the same is true in* Mod (R).

Proof. The bijection of Corollary 2.6 clearly takes Mod (R/I) to Mod (R) and $\{0\}$ to $\{0\}$. Thus if $\mathscr{R}$ is a nontrivial torsion class in Mod (R), then $\mathscr{R} = L_R(L_{R/I}(\mathscr{C}))$ for a set $\mathscr{C}$ of simple modules. It follows that $\mathscr{R} = L_R(\mathscr{C})$.

The last result is eminently generalizable.

If a ring has the form $A \oplus B$, where A and B are ideals, the class of modules M such that $AM = 0$ is a torsion class. (The corresponding radical theory is said to be *centrally splitting* [3].)

THEOREM 3.7. *Let R be a ring a with right T-nilpotent ideal I such that $R/I = A \oplus B$* (ring direct sum). *Then*

$$L_R(\{M \in \text{Mod}\,(R) \mid IM = 0 \text{ and } AM = 0\}$$

is a TTF *class.*

Proof. The identity of A lifts to an idempotent $e \in R$, since I is nil. Since ReR is an idempotent ideal,

$$\begin{aligned}\mathscr{R} &= \{M \in \text{Mod}\,(R) \mid ReRM = 0\}\\ &= \{M \in \text{Mod}\,(R) \mid eM = 0\}\end{aligned}$$

is a TTF class [14]. Let

$$\mathscr{T} = L_R(\{M \in \text{Mod}\,(R) \mid IM = 0 \text{ and } AM = 0\})\,.$$

We show that $\mathscr{R} = \mathscr{T}$.

If $N \in \mathscr{T} \subseteqq L_R(\mathscr{M}_I)$, it has an ascending I-series

$$0 = N_0 \subseteqq N_1 \subseteqq \cdots \subseteqq N_\alpha \subseteqq N_{\alpha+1} \subseteqq \cdots \subseteqq N_\mu = N ,$$

each term of which belongs to $\mathscr{T}$. It follows from Corollary 2.6 that

$$\mathscr{T} \cap \mathscr{M}_I = \{M \in \mathrm{Mod}\,(R) \mid IM = 0 \text{ and } AM = 0\} .$$

Thus $AN_1 = 0$, so $eN_1 \subseteqq AN_1 = 0$ and $N_1 \in \mathscr{R}$. Each $N_{\alpha+1}/N_\alpha \in \mathscr{T}$, so in the same way, $e(N_{\alpha+1}/N_\alpha) = 0$, i.e. $N_{\alpha+1}/N_\alpha \in \mathscr{R}$. Clearly also $N_\beta \in \mathscr{R}$ for a limit β if $N_\alpha \in \mathscr{R}$ for every $\alpha < \beta$. Hence $\mathscr{R}$ contains all N_α, and in particular, $N \in \mathscr{R}$. This proves that $\mathscr{T} \subseteqq \mathscr{R}$.

On the other hand, if $K \in \mathscr{R}$ and $IK = 0$, then $AK = 0$, so that

$$\begin{aligned} \mathscr{R} &= L_R(\{M \in \mathscr{R} \mid IM = 0\}) \\ &\subseteqq L_R(\{M \in \mathrm{Mod}\,(R) \mid IM = 0 \text{ and } AM = 0\} = \mathscr{T} . \end{aligned}$$

As a consequence, we have the following frequently-proved result [1], [6], [22].

COROLLARY 3.8. *If R is right perfect, it has finitely many torsion classes, all of which are* TTF *classes and all but* $\{0\}$ *of which are determined by simple modules.*

Proof. The Jacobson radical J of R is right T-nilpotent and any torsion class in $\mathrm{Mod}\,(R/J)$ has the form $\{M \in \mathrm{Mod}\,(R/J) \mid AM = 0\}$ for a ring direct summand A of R/J. By Corollary 2.6, the torsion classes in $\mathrm{Mod}\,(R)$ have the form

$$L_R(\{M \in \mathrm{Mod}\,(R) \mid JM = 0 \text{ and } AM = 0\}$$

and so, by Theorem 3.7, are TTF classes. The rest follows from Corollary 2.6 by an argument like that used for Theorem 3.6.

4. A splitting theorem. Throughout this section G_t and G_d denote, respectively, the maximum torsion and divisible subgroups of an abelian group G. Things such as *torsion modules* and *divisible rings* are those modules and rings whose additive groups have the stated properties.

THEOREM 4.1. *Let R be a ring with a right T-nilpotent ideal I such that R/I is the direct sum of a torsion* (necessarily bounded) *ideal and a torsion-free divisible ideal. Then R has the same property.*

Proof. Let $A = (R/I)_t$ and $B = (R/I)_d$. Our assumption implies that $R/I = A \oplus B$ (ring direct sum). Thus A, B have identities e_A, e_B respectively, and any R/I-module M has the form $M_A \oplus M_B$ where is an A-module with $BM_A = 0$ and M_B is a B-module with $AM_B = 0$. In particular, M_A is bounded and M_B is divisible. If $x \in M_B$ and $nx = 0$ for some $n > 0$, then $x = e_Bx = n(e_B/n)x = (e_B/n)nx = 0$, so M_B is torsion-free.

Let

$$\mathscr{R}_t = L_R(\{M \in \mathrm{Mod}\,(R) \mid IM = 0 \text{ and } BM = 0\}\,.$$

If N is a torsion R-module, then every nonzero homomorphic image of N has a nonzero submodule $M \in \mathscr{M}_I$ (Proposition 2.2). But M is a torsion R/I-module, so $BM = 0$. It follows that $N \in \mathscr{R}_t$ and thus $\mathscr{R}_t$ is the class of all torsion R-modules. By Theorem 3.7, $\mathscr{R}_t$ is a TTF class so $L = \prod_{i=1}^{\infty} N_i \in \mathscr{R}_t$, where each $N_i \cong N$. But L_t, as a fully invariant subgroup, is a submodule, so $\hat{L} = L/L_t \in \mathscr{R}_t$ and, if nonzero, has a nonzero submodule T such that $IT = 0$ and $BT = 0$ (so that, in particular, T is a torsion module). But $\hat{L}$ is torsion-free, so $\hat{L} = 0$; but then N must be bounded. Thus all torsion R-modules are bounded.

Let K be a nonzero torsion-free R-module. Since I is right T-nilpotent, K has a nonzero submodule P such that $IP = 0$ (Proposition 2.2). Clearly $AP = 0$, so $P = BP$ is divisible. Thus $K_d \neq 0$. But K/K_d is again a torsion-free R-module, and so, if nonzero, must, by the same argument, have a nonzero divisible submodule. Hence $K = K_d$.

Thus for any R-module V, V_t is bounded and hence a direct summand, while V/V_t is divisible. It follows that $V = V_t \oplus V_d$. In particular, $R = R_t \oplus R_d$ (module direct sum). But R_t and R_d are ideals, so the theorem is proved.

We conclude this section with a few remarks which give some indication of the range of applicability of the foregoing theorem. If a ring R has no nonzero nilpotent ideals then neither does R_t (whether it has an identity or not) so that R_t is bounded (elementary, in fact) [7], and hence a group direct summand. If in addition R/R_t is divisible, then $R = R_t \oplus R_d$ (ring direct sum). Simple rings are either torsion-free divisible or elementary p-rings; torsion-free regular rings are divisible.

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