FACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

MISCELLANY ON BIEBERBACH GROUP ALGEBRAS

DANIEL R. FARKAS

One of the longstanding problems in the the theory of infinite group algebras is the zero divisor conjecture: does the group algebra of a torsion free group have zero divisors?

Results presented here grew out of an attempt to settle the conjecture for abelian-by-finite groups. Since the problem is not solved it seems valuable to collect in one paper most of the information about this case.

The conjecture has been verified for some limited classes of groups ([9], [8], [5]).

A finitely generated group Γ is called a *Bieberbach group* if it has a torsion free, self-centralizing, abelian subgroup of finite index. It is easy to see that this is equivalent to the existence of a short exact sequence

$$1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

where Δ is the finite conjugate subgroup of Γ, assumed to be torsion free, and G is a finite group. We will often refer to G as the top of Γ. It is worth observing that Δ is a finitely generated, torsion free abelian group containing all abelian subgroups of Γ of finite index; consequently G acts faithfully by conjugation Δ. ([9] is a good source for elementary properties of the finite conjugate subgroup.)

If F is an arbitrary field then the group algebra $F[\Gamma]$ can be partially described by exploiting the theory of central simple algebras. Adopt the notation $R = F[\Delta]$, let K be the field of fractions of R, and let C denote the center of $F[\Gamma]$. By inverting the nonzero elements of C we get the tower of rings

$$\begin{array}{c} C^{-1}F[\Gamma] \\ \cup \\ C^{-1}R \\ \cup \\ C^{-1}C \end{array}$$

It turns out that $C^{-1}R = K$ and K is a Galois extension of $C^{-1}C$ whose group is G (acting by conjugation). Moreover, $C^{-1}F[\Gamma]$ is a prime ring with a basis over K consisting of a transversal to the cosets Δ in Γ. It is now clear that $C^{-1}F[\Gamma]$ is a cross-product "in" $H^2(G, K^*)$. (The reader who wishes more details should consult Thm 6.5 in [9]). To prove that $F[\Gamma]$ has no zero divisors is equivalent to proving that $C^{-1}F[\Gamma]$ is a division algebra.

1. A cohomology result. In tnis section we prove.

THEOREM 1. *There is a natural injection* $H^2(G, \Delta) \subsetneqq H^2(G, K^*)$. (The notation here is continued from the introduction. $(\ )^*$ denotes the units of the ring in question.)

This theorem was proved for cyclic G in [4]; a similar result is contained, implicitly in unpublished work of Procesi and Schacher. All three observed that as an immediate consequence of this theorem, the zero divisor conjecture is true when G is cyclic of prime order. (If $|G| = p$ then all nonzero elements of $H^2(G, K^*)$ represent cross-products of degree p and $C^{-1}F[\Gamma]$ has dimension p^2 over its center.) The following lemma is a well-known analogue of the polynomial result.

LEMMA 1. $R = F[\Delta]$ *is a unique factorization domain and* $R^* = F^* \times \Delta$.

LEMMA 2. *Suppose G acts on a set X. If $\hat{X}$ is the free abelian group on X then* $H^1(G, \hat{X}) = 0$.

Proof. Partition $X = \bigcup_I X_i$ as a union of orbits. Then $\hat{X} = \sum_I \hat{X}_i$ and so $H^1(G, \hat{X}) = \sum_I H^1(G, \hat{X}_i)$.

(The cohomology "distributes" because G is finite. Equivalently, one may reduce to the case of finitely many orbits and a finite sum by considering only those generators in X which appear in the image of a crossed-homomorphism.) We need only show that $H^1(G, \hat{X}_i) = 0$ where X_i is an orbit. In that case $\hat{X}_i \cong \boldsymbol{Z}[G/H]$ as modules, where H is the stabilizer of the orbit. By Shapiro's lemma ([6]) $H^1(G, \boldsymbol{Z}[G/H]) \cong H^1(H, \boldsymbol{Z})$. (The explict map is the group ring trace of the restriction of a crossed-homomorphism.) $\boldsymbol{Z}$ has the trivial H-action, so $H^1(H, \boldsymbol{Z}) = \operatorname{Hom}(H, \boldsymbol{Z})$. Since H is finite, $\operatorname{Hom}(H, \boldsymbol{Z})$.

Proof of the theorem. First apply the long cohomology sequence to $1 \to R^* \to K^* \to K^*/R^* \to 1$.

$$\cdots \longrightarrow H^0(G, K^*/R^*) \longrightarrow H^1(G, R^*) \longrightarrow H^1(G, K^*) \longrightarrow \cdots$$

G is a group of automorphisms of K so by Hilbert's Theorem 90, $H^1(G, K^*) = 0$. Thus the map $H^0(G, K^*/R^*) \to H^1(G, R^*)$ is onto.

$R^* = F^* \times \Delta$ is a direct product as G-modules. Since cohomology commutes with products, the map $H^1(G, R^*) \to H^1(G, F^*)$ is onto. Consequently the composition $H^0(G, K^*/R^*) \xrightarrow{\Phi} H^1(G, F^*)$ is onto.

K^*/R^* is a free abelian group on the irreducibles (primes) of R

by the first lemma. Since G acts on R, it acts on the set of irreducibles. We may apply Lemma 2, yielding $H^1(G, K^*/R^*) = 0$.

Apply the long cohomology sequence to $1 \to F^* \to K^*/\Delta \to K^*/R^* \to 1$.

$$\cdots \longrightarrow H^0(G, K^*/R^*) \xrightarrow{\Phi} H^1(G, F^*) \longrightarrow H^1(G, K^*/\Delta) \longrightarrow H^1(G, K^*/R^*) \longrightarrow \cdots$$

If one "chases" the fact that Φ is onto, one finds $H^1(G, K^*/\Delta) \subseteq H^1(G, K^*/R^*) = 0$.

Finally, apply the long cohomology sequence to $1 \to \Delta \to K^* \to K^*/\Delta \to 1$.

$$\cdots \longrightarrow H^1(G, K^*/\Delta) \longrightarrow H^2(G, \Delta) \longrightarrow H^2(G, K^*) \longrightarrow \cdots$$

But we have just shown that $H^1(G, K^*/\Delta) = 0$.

2. Reduction to Sylow subgroups. The zero divisor conjecture can be simplified somewhat by showing that the problem need only be solved when tops are p-groups. In proving this we will use the identification of $H^2(G, \Delta)$ inside $H^2(G, K^*)$ given by Theorem 1, although this is not strictly necessary.

Now an element of $H^2(G, \Delta)$ represents a group extension as well as a crossproduct of dimension $|G|^2$ over its center (see e.g. [7], Chapter 4). Recall that a simple algebra is a full matrix ring over a division algebra and that the degree of the simple algebra is the square-root of the dimension of this division algebra over its center. Thus for Bieberbach groups we may state the zero divisor conjecture as follows:

If $f \in H^2(G, \Delta)$ represents a torsion-free extension then its degree is $|G|$.

If H is a subgroup of G then we will write the restriction map as

$$\mathrm{res}_{G\to H}\colon H^n(G, \cdot) \longrightarrow H^n(H, \cdot)$$

We can now state

THEOREM 2. *Assume $f \in H^2(G, \Delta)$ represents a torsion free extension. If the degree of* $\mathrm{res}_{G\to P} f$ *is $|P|$ for each Sylow subgroup, P, of G then the degree of f is $|G|$.*

As a consequence of this theorem, the zero divisor conjecture is affirmed for $F[\Gamma]$ once it is known for each $F[\pi]$ where π is the inverse image in Γ of a Sylow subgroup of G.

We will need two preliminary lemmas, the first of which is trivial and the the second of which is well known.

LEMMA 3. *If $f \in H^2(G, \Delta)$ represents a torsion free extension then $\mathrm{res}_{G\to H} f$ represents a torsion free extension for every non-identity subgroup, H, of G.*

LEMMA 4. $\mathrm{res}_{G\to H}$: *$H^2(G, K^*) \to H^2(H, K^*)$ is induced from the map of simple algebras with center K^G* (fixed points under G) *to simple algebras with center K^H given by $A \mapsto A \otimes_{K^G} K^H$.*

Proof. See [1] (Chapter V §7) or [10].

Proof of the theorem. Let G be an arbitrary finite group and let P be a Sylow p-subgroup of G.

If $f \in H^2(G, \Delta)$ we can uniquely write $f = \sum f_q$ where f_q is in the Sylow q-subgroup of $H^2(G, \Delta)$ for primes q dividing $|G|$. Since res is a homomorphism, $\mathrm{res}_{G\to P} f_q$ is annihilated by a power of q. However, every element in $H^2(P, \Delta)$ is annihilated by $|P|$. Thus $\mathrm{res}_{G\to P} f = \mathrm{res}_{G\to P} f_p$.

By assumption the degree of res f_p in $H^2(P, K^*)$ is $|P|$. As an easy consequence of Lemma 4 the degree of f_p is not less than $|P|$. By [7] (pp. 120–121), the degree of f_p is a power of p dividing $|G|$. Thus the degree of f_p is exactly $|P|$. By the argument of [7] (Theorem 4.4.6), the degree of f is the product of the degrees of the f_p for $p \| G|$. The degree of f is $|G|$.

3. The second center is trivial. Theorem 2 reduces the zero divisor conjecture for Bieberbach groups to solvable Bieberbach groups. One may ask the status of torsion free nilpotent Bieberbach groups. It is known (though unrecorded) that these must be abelian. In this section we obtain a refinement of this proposition.

Let $\zeta(\cdot)$ denote the center of a group. Pardoning the abuse of notation, we have.

THEOREM 3. *If Γ is a Bieberbach group then $\zeta(\Gamma/\zeta(\Gamma)) = 1$.*

Proof. Suppose γ is in the inverse image of $\zeta(\Gamma/\zeta(\Gamma))$. This means that $\gamma g \gamma^{-1} g^{-1} \in \zeta(\Gamma)$ whenever $g \in \Gamma$.

If $a \in \Delta$ then $\gamma a \gamma^{-1} = za$ for some $z \in \zeta(\Gamma)$. Since Γ is Bieberbach we can find an integer $n \geqq 0$ so that $\gamma^n \in \Delta$. Thus $a = (\gamma^n)a(\gamma^n)^{-1} = az^n$. That is, $z^n = 1$. Since $\zeta(\Gamma) \subseteqq \Delta$ and Δ is torsion free, $z = 1$. γ centralizes the self-centralizing subgroup Δ so $\gamma \in \Delta$.

If $g \in \Gamma$ then $g\gamma g^{-1} = w\gamma$ for some $w \in \zeta(\Gamma)$ and $g^m \in \Delta$ for some $m \geqq 0$. Since $\gamma \in \Delta$, g^m and γ commute. Now the argument above shows that $g\gamma g^{-1} = \gamma \forall g \in \Gamma$. But then $\gamma \in \zeta(\Gamma)$ as desired.

COROLLARY. *Nilpotent Bieberbach groups are abelian.*

4. Arbitrary tops. Let G be an arbitrary finite group and let $1 \to N \to \Phi \to G \to 1$ be a free presentation with Φ finitely generated. Set $\Lambda = \Phi/[N, N]$. It is not difficult to show that $\Delta(\Lambda) = N/[N, N]$.

We will need the following lemma whose proof is quoted from [11].

LEMMA 5. *If G is solvable then Λ has a finite normal series whose factors are all isomorphic to $\boldsymbol{Z}$.*

Proof. There is a finite normal series $\Phi = \Phi_0 \triangleright \Phi_1 \triangleright \cdots \triangleright \Phi_k = N$ with Φ_i/Φ_{i+1} abelian and Φ_i a finitely generated free group. Consider the series $\Phi \triangleright [\Phi_0, \Phi_0] \triangleright [\Phi_1, \Phi_1] \triangleright \cdots \triangleright [\Phi_k, \Phi_k] = [N, N]$. $\Phi/[\Phi_0, \Phi_0]$ is certainly a finitely generated torsion free abelian group. Since Φ_i/Φ_{i+1} is abelian, $\Phi_{i+1} \supseteqq [\Phi_i, \Phi_i] \supseteqq [\Phi_{i+1}, \Phi_{i+1}]$. Thus $[\Phi_i, \Phi_i]/[\Phi_{i+1}, \Phi_{i+1}]$ is a subgroup of the finitely generated torsion free group $\Phi_{i+1}/[\Phi_{i+1}, \Phi_{i+1}]$. Now refine the series in the obvious way.

One simple and well known consequence of this lemma is that Λ is torsion free for arbitrary G. A new one is.

THEOREM 3. *If G is an arbitrary finite group then $F[\Lambda]$ has no zero divisors.*

Proof. Let P be a Sylow p-subgroup of G and H its inverse image in Φ. Then $1 \to N \to H \to P \to 1$ is a free presentation of P and $H/[N, N]$ is the inverse image of P in Λ. The Sylow reduction Theorem 2 now implies that we might as well assume that G is a p-group.

But then G is solvable. It is well known that whenever Λ has a normal series as described in the lemma, $F[\Lambda]$ has no zero divisors. (See [9], Thm 26.7 or observe that $F[\Lambda]$ is a "twisted polynomial ring").

5. Particular tops. Arguments of the previous section show (cf. [2]) that any finite group can be the top of a torsion free Bieberbach group. That makes the short list of tops for which the zero divisor conjecture is known depressingly small.

We'll say a group is *cyclish* provided it has a cyclic subgroup of index two or less. Notice that this class of groups is closed under subgroups and homomorphic images.

If M is any group we write $\zeta^*(M) = \{m \in M \mid xmx^{-1} = m \text{ or } xmx^{-1} = m^{-1} \text{ for all } x \in M\}$. Beware that $\zeta^*(M)$ is not a subgroup of M.

However $\zeta(M) \subseteqq \zeta^*(M) \subseteqq \Delta(M)$.

If Γ is a Bieberbach group, we let $T: \Gamma \to \Delta$ denote the transfer homomorphism.

By the (Hirsch) rank of an abelian-by-finite group we mean the torsion free rank of the abelian subgroup of finite index.

LEMMA 6. *Let Γ be a Bieberbach group. Then* rank $(\Gamma/\text{Ker } T) =$ rank $(\Gamma/[\Gamma, \Gamma])$.

Proof. Let "—" be the canonical map $\Gamma \to \Gamma/[\Gamma, \Gamma]$. Since $[\Gamma, \Gamma] \subseteqq \text{Ker } T$, it suffices to show that rank $(\overline{\Gamma}/\overline{\text{Ker } T}) =$ rank $(\overline{\Gamma})$. We do this by proving that that $\overline{\text{Ker } T}$ is periodic. Set $n = |\Gamma/\Delta|$. If $g \in \text{Ker } T$ then $g^n \in \Delta$. $T(g^n)$ is the product of n conjugates of g^n by coset representatives of Δ in Γ. Thus $\overline{T(g^n)} = (\bar{g})^{n^2}$. But $T(g) = 1$.

LEMMA 7. *Let Γ be a Bieberbach group. Γ has an infinite cyclic homomnrphic image if and only if $\zeta(\Gamma) \neq 1$.*

Proof. ($\Leftarrow$) If $1 \neq z \in \zeta(\Gamma)$ then $T(z) = z^n$ where $n = |\Gamma/\Delta|$. Since $z^n \neq 1$, $T(\Gamma)$ is a finitely generated abelian group which is not finite.

($\Rightarrow$) Suppose $\zeta(\Gamma) = 1$. If $a \in \Delta$ then $T(a) \in \zeta(\Gamma)$. Since the transfer is trivial on Δ, it is trivial on Γ. In particular,

$$\text{rank}\,(\Gamma/\text{Ker } T) = 0 .$$

By Lemma 6, $|\Gamma: [\Gamma, \Gamma]| < \infty$; Γ has no infinite abelian images.

LEMMA 8. *Let M be an arbitrary group and let $N \lhd M$ with $|M/N| = 2$. If $\zeta(N) \neq 1$ then $\zeta^*(M) \neq 1$.*

Proof. Let $M = \langle N, b \rangle$ where $b^2 \in N$ and suppose $1 \neq \xi \in \zeta(N)$. Then $b\xi b^{-1} \in \zeta(N)$. $b(\xi(b\xi b^{-1}))b^{-1} = (b\xi b^{-1})\xi = \xi(b\xi b^{-1})$ so $\xi(b\xi b^{-1}) \in \zeta(M)$. We are done unless $\xi(b\xi b^{-1}) = 1$. In that case $b\xi b^{-1} = \xi^{-1}$ so $\xi \in \zeta^*(M)$.

THEOREM 5. *Let Γ be a Bieberbach group. Then Γ has an infinite cyclish image* (i.e. infinite cyclic or infinite dihedral) *if and only if $\zeta^*(\Gamma) \neq 1$.*

Proof. ($\Rightarrow$) If Γ has an infinite cyclish image it has a subgroup of index at most 2 with an infinite cyclic image. Apply Lemmas 7 and 8.

($\Leftarrow$) With additive notation, Δ may be regarded as a $\boldsymbol{Z}[\Gamma/\Delta]$-module. If $\zeta^*(\Gamma) \neq 1$ then $\Delta \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ has a one-dimensional $\boldsymbol{Q}[\Gamma/\Delta]$-submodule, say A. By Maschke's theorem there exists another submodule, B, so that $A \oplus B = \Delta \otimes_Z \boldsymbol{Q}$. Set $X = B \cap \Delta$. The rank of X is one less than the rank of Δ and $X \lhd \Gamma$. That is, Γ/X has rank 1. By modding out the elements of $\Delta(\Gamma/X)$ with finite order we have constructed a homomorphic image of Γ which is a Bieberbach group of rank 1. But it is easy to see that the only such groups are the infinite cyclic group ($\boldsymbol{Z}$) and the infinite dihedral group ($\boldsymbol{Z}/2\boldsymbol{Z} * \boldsymbol{Z}/2\boldsymbol{Z}$).

Theorem 5 is valuable in conjunction with a result developed in [3], [8], and especially [5]. Suppose we have a free product with amalgamation, $Y = H_1 *_N H_2$. The key result is that if $F[H_1]$ and $F[H_2]$ have no zero divisors and if $F[N]$ is a well behaved ring (e.g. noetherian) then $F[Y]$ also has no zero divisors. We are interested in the particular case $Y/N \cong \boldsymbol{Z}/2Z * \boldsymbol{Z}/2\boldsymbol{Z}$. Note that if Y is arbitrary, $Y/N \cong \boldsymbol{Z}$ and $F[N]$ has no zero divisors then $F[Y]$ has none by a "twisted polynomial ring" argument. Thus by inducting on the rank of a Bieberbach group we have.

THEOREM 6. *Suppose $\mathfrak{H}$ is a class of finite groups closed under taking subgroups and homomorphic images. If Γ always has an infinite cyclish image whenever it is torsion free and $\Gamma/\Delta \in \mathfrak{H}$ then $F[\Gamma]$ has no zero divisors.*

We now come to our main application of the previous two theorems.

THEOREM 7. *If Γ is a torsion free Bieberbach group and Γ/Δ is cyclish then $F[\Gamma]$ has no zero divisors.*

Proof. It suffices to prove that if Γ/Δ is cyclic then $\zeta(\Gamma) \neq 1$. So suppose $\Gamma = \langle \Delta, g \rangle$ where $g^k \in \Delta$. Clearly $g^k \in \zeta(\Gamma)$.

In summary, the zero divisor conjecture is true when the Sylow subgroups of the top are cyclish. These finite groups are well understood. A list of cyclish 2-groups can be found in [14] (p. 150). The solvable groups with cyclish Sylow subgroups were classified in [15] (cf. [13] p. 176). The nonsolvable groups are described in [12].

We can use these methods to settle the zero divisor conjecture for groups with low rank.

LEMMA 9. *Suppose $\Gamma/\Delta = U \oplus \boldsymbol{Z}/2\boldsymbol{Z}$. If W is the inverse image of U in Γ and $\zeta^*(W) \neq 1$ then $\zeta^*(\Gamma) \neq 1$.*

Proof. Let $b \in \Gamma$ be an inverse image of the generator of $\boldsymbol{Z}/2\boldsymbol{Z}$.

If $1 \neq \xi \in \zeta^*(W)$ then ξ and $b\xi b^{-1}$ are contained in the abelian group Δ. As in Lemma 8, $b(\xi(b\xi b^{-1}))b^{-1} = \xi(b\xi b^{-1})$. Let $w \in W$. The actions of w and b on Δ commute. $w\xi w^{-1} = \xi^{\varepsilon}(\varepsilon = \pm 1)$ implies $w(\xi(b\xi b^{-1}))w^{-1} = (w\xi w^{-1})b(w\xi w^{-1})b^{-1} = (\xi(b\xi b^{-1}))^{\varepsilon}$. That is, $\xi(b\xi b^{-1}) \in \zeta^*(\Gamma)$. Now finish as in Lemma 8.

THEOREM 8. *If Γ is a torsion free abelian-by-finite group of rank $\leqq 3$ then $F[\Gamma]$ has no zero divisors.*

Proof. Write r for the rank of Δ. According to the Bieberbach theorems ([13]) Γ can be realized as a discrete group of isometries of r-dimensional Euclidean space. Δ is identified with the translations in Γ and Γ/Δ is referred to as a crystallographic point group. These groups were classified for $r \leqq 3$ during the last century. They are

the cyclic groups of order 1, 2, 3, 4, or 6
the dihedral groups of order 2 '4, 6, 8, or 12
the tetrahedral group of order 12
the octahedral group of order 24
any of the groups listed above $\oplus \boldsymbol{Z}/2\boldsymbol{Z}$
(see e.g. [13])

Direct inspection shows that the Sylow subgroups of the first four types of groups are cyclish. By the Sylow reduction Theorem 2, Lemma 9, and the spirit of Theorem 7, $F[\Gamma]$ has no zero divisors.

Note added in proof. K. A. Brown has used Theorem 2 to prove the zero divisor conjecture for arbitrary Bieberbach group algebras over a field of characteristic zero. His result appears in the paper, "Zero-divisors in group." Subsequently, R. Snider and this author verified the conjecture for polycyclic group algebras in characteristic zero. That work will appear in *Journal of Algebra* under the title, "K_0 and Noetherian group rings."

REFERENCES

1. A. A. Albert, *Structure of Algebras*, American Mathematical Society, New York, 1939.
2. L. Auslander and M. Kuranishi, *On the holonomy group of locally euclidean spaces*, Annals of Math., **65** (1957), 411-415.
3. P. M. Cohen, *On the free product of associative rings. III.*, J. Algebras. **8** (1968), 376-386.
4. E. Formanek, *Matrix Techniques in Polycyclic Groups*, Ph. D. Dissertation, Rice Univ., May 1970.
5. ———, *The zero divisor question for supersolvable groups*, Bull. Austral. Math. Soc., **9** (1973), 69-71.

6. K. W. Gruenberg, *Cohomological Topics in Group Theory*, Springer-Verlag, Berlin, 19.
7. I. N. Herstein, *Noncommutative Rings*, Carus Math. Monographs, No. 15, Math. Assoc. Amer., Buffalo, 1968.
8. J. Lewin, *A note on zero divisors in group-rings*, Proc. Amer. Math. Soc., **31** (1972), 357-359.
9. D. S. Passman, *Infinite Group Rings*, Dekker, New York, 1971.
10. J.-P. Serre, *Corps Locaux*, Hermann, Paris, 1968.
11. D. M. Smirnov, *On a generalization of solvable groups and their group rings*, Mat. Sb., **67** (1965), 366-383 (Russian).
12. M. Suzuki, *On finite groups with cyclic Sylow subgroups for all odd primes*, Amer. J. Math. **77** (1955), 657-691.
13. J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York, 1967.
14. H. J. Zassenhaus, *The Theory of Groups*, 2nd ed., Chelsea, New York, 1958.
15. ———, *Über endliche Fastkörper*, Hamb. Abh., (1935), 187-220.

Received February 28, 1975 and in revised form May 7, 1975. Partially supported by NSF GP-41986.

BRANDEIS UNIVERSITY

Current Address: Virginia Polytechnic Institute and State University