PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

CLASSIFICATION OF SINGULAR INTEGRALS OVER A LOCAL FIELD

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The singular integral operators over a local field K whose kernels are multiplicative characters of the unit sphere of K are shown to be precisely those continuous operators on $\mathscr{L}_2(K)$ which commute with translation and dilation, anti-commute with an appropriately defined rotation, and whose multipliers satisfy a smoothness condition. The characterization is analogous to that of the Hilbert transform over the real numbers.

1. Classically, the Hilbert transform over $\boldsymbol{R}$ is, up to a constant multiple, the only continuous operator on $\mathscr{L}_2(\boldsymbol{R})$ which commutes with translation and (positive) dilation and anti-commutes with reflection. See [9], page 55. The Hilbert transform is a singular integral operator with kernel the only (nontrivial) multiplicative character of the unit sphere of $\boldsymbol{R}$.

Singular integrals over a local field have been developed. (See, for example, Phillips [6], Phillips-Taibleson [7], and Chao [1].) Those with kernel a multiplicative character of the unit sphere satisfy a classification similar to that of the classical Hilbert transform.

The classification theorem is in § 4. The main results are Theorems 4.1 and 4.2. Section 3 contains the necessary results regarding the character group of the unit sphere of a local field; § 2 contains other preliminary results, notation, and definitions.

2. Let Z, Z^+, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the integers, the positive integers, the rational number, the real numbers, and the complex numbers, respectively. $\boldsymbol{F}_{p^n}$ will denote the (unique) field with p^n elements. The symbols $\boldsymbol{Q}_p$ and Z_p will denote the p-adic numbers and the p-adic integers, respectively. For any set S, ξ_S will denote the characteristic function of S. The complement of S will be written S_c.

The necessary analysis on local fields is stated without proof below. Most of it may be found in Chapters I and II of Weil [11].

A local field is a nondiscrete, locally compact, zero-dimensional topological (commutative) field. These have been completely classified. Those of characteristic $p \neq 0$ can be identified as the fields of formal power series over a finite field. Those of characteristic 0 are either the p-adic numbers of finite extensions of the p-adic numbers. See [11], page 11.

Let K be a local field with λ Haar measure for $(K, +)$. The modular function for K, $|\cdot|$, is given by $|x| = \lambda(xS)/\lambda(S)$ for $0 < \lambda(S) < \infty$. Haar measure for the multiplicative group $K^\times = K \sim \{0\}$ is $\lambda/|\cdot|$.

Let R be the ring of integers of the local field K and P be the unique maximal ideal of R. Then $\operatorname{ord}(R/P) = q$, the module of K, a prime power. The ideal P has a generator π, so that $\pi R = P$. We have $|\pi| = q^{-1}$, and, in fact, any $x \in K$ with $|x| = q^{-1}$ will generate P. Those elements of modulus q^{-1} will be called primes in K.

For $n \in Z$ we define

$$P^n = \{x \in K\colon |x| \leqq q^{-n}\};\ D^n = \{x \in K\colon |x| = q^{-n}\}\ .$$

Then $P^1 = P$, $P^0 = R$, and $R \sim P = D^0$. The set $\{P^n\}_{n=0}^{\infty}$ is a neighborhood base at 0 of open and closed subgroups of $(K, +)$. The set $\{1 + P^n\}_{n=1}^{\infty}$ is a neighborhood base at 1 of open and closed subgroups for the topological group $(K^\times, \cdot)$.

We define the operators τ_δ for $\delta \neq 0$ on functions by $\tau_\delta f(x) = f(\delta x)$. Regarding the prime π as fixed, we single out a set of such operators, the dilation operators, $\mathscr{D}_j$, defined by $\mathscr{D}_j f(x) = f(\pi^j x) j \in Z$. A function f is homogeneous degree zero if $\mathscr{D}_j f = f$ for all $j \in Z$. For $x \in K$, translation operators T_x are defined on functions by $T_x f(y) = f(x + y)$.

There is a character χ of the additive group of K which is identically one on R and nontrivial on P^{-1}. Then for any $y \in K$, $\chi_y(x) = \chi(xy)$ defines a character of K. In fact, the mapping $y \rightarrow \chi_y$ is a topological isomorphism of $(K, +)$ onto its dual. We thus identify K with its dual.

The Fourier transform for K is initially defined on $\mathscr{L}_1(K)$ by

$$\mathscr{F} f(x) = \hat{f}(x) = \int_K f(y)\overline{\chi(xy)}dy\ .$$

[The integral is taken with respect to λ. Here and elsewhere the λ will be suppressed.] The transform $\mathscr{F}^{-1}$ is defined by $\mathscr{F}^{-1}f(x) = \check{f}(x) = \int_K f(y)\chi(xy)dy$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ extend uniquely to $\mathscr{L}_2$. It is easy to see that, as $\mathscr{L}_2$ operators, $\tau_\delta \mathscr{F} = |\delta|^{-1}\mathscr{F}\tau_{\delta^{-1}}$ and $\tau_\delta \mathscr{F}^{-1} = |\delta|^{-1}\mathscr{F}^{-1}\tau_{\delta^{-1}}$.

The following result will be used extensively in the sequel: Let L be a continuous linear operator from $\mathscr{L}_2(K)$ to $\mathscr{L}_2(K)$. Then a necessary and sufficient condition that L commute with translation is that there exist a function m, in $\mathscr{L}_\infty(K)$, such that $\mathscr{F}(Lf) = m\mathscr{F} f$ for all $f \in \mathscr{L}_2(K)$. See [5], pp. 92-94.

The space $\mathscr{S}$ of test functions on K and its topological dual $\mathscr{S}'$, the space of distributions, are defined as in [8]. Both are complete linear spaces. The action of a $\mu \in \mathscr{S}'$ on an $f \in \mathscr{S}$ will be denoted (μ, f).

The space $\mathscr{S}$ is contained densely in $\mathscr{L}_p$, $1 \leqq p < \infty$. The Fourier transform is thus will-defined on $\mathscr{S}$. The Fourier transform on $\mathscr{S}'$ is given by $(\hat{\mu}, f) = (\mu, \hat{f})$. Thus defined, the Fourier transform is a linear topological isomorphism on both $\mathscr{S}$ and $\mathscr{S}'$.

Functions and measures will be identified with the distributions they induce. Convolution of a distribution and a test function is defined by $\mu * f(x) = (\mu, T_x \tilde{f})$, where $\tilde{f}(x) = f(-x)$.

Let $\mu \in \mathscr{S}'$, and let σ be a (not necessarily unitary) multiplicative character of $K^\times (= K \sim \{0\})$. Then, as in [8], we say μ is homogeneous of degree σ if for all $t \in K^\times$, $\mu_t = \sigma(t)\mu$, where μ_t is that distribution defined by $(\mu_t, \phi) = (\mu, |t|^{-1}\tau_{t^{-1}}\phi)$.

We take M to be $M^\times \cup \{0\}$, where $M^\times$ is the group of roots of unity in K of order prime to p. Then $M^\times$ is the unique cyclic group of order $q - 1$ ([11], p. 16). Let g be a generator of $M^\times$. Then each $0 \neq x \in K$ may be written uniquely as $x = \pi^j g^k (1 + p_x)$, where $k, j \in Z$, $0 \leqq k \leqq q - 2$, $p_x \in P$. A multiplicative character of $K^\times$ is given by its values at π, g, and on $1 + P$.

Let ω be a multiplicative character of $K^\times$. There is some $n \in Z$ such that ω is trivial on $1 + P^n$. If ω is trivial on $1 + P^n$ but not on $1 + P^{n-1}$, $n \geqq 1$, we say ω is ramified of degree n. If ω is trivial on D^0, we say ω is unramified. Given a character ω of $1 + P$, ω is the restriction of a character of $K^\times$, say ω'. The ramification degree of ω' depends only on ω, and we define the ramification degree of ω to be that of ω'.

We define the local field gamma function on ramified characters of $K^\times$ by

$$\Gamma(\omega) = \text{p.v.} \int_K \frac{\chi(x)\omega(x)dx}{|x|} ,$$

where

$$\text{p.v.} \int_K f(x)dx = \lim_{n \to \infty} \int_{P^{-1} \cap (P^{+n})^c} f(x)dx .$$

See [8] for details and further definition of Γ.

3. Lemma 3.1. *Let K be a local field of characteristic $p \neq 0$ with module $q = p^f$. Let $\{\alpha_1, \cdots, \alpha_f\}$ be a basis for F_q over F_p. Then given $x \in P$ and $N \in Z^+$,*

(a) *there are unique integers $a_{i,j}$, n_j, ν_j, with $0 \leqq a_{i,j} < p$, $(n_j, p) = 1$ for $1 \leqq i \leqq f$, $1 \leqq j \leqq N$, such that $1 + x = \prod_{j=1}^{N} \prod_{i=1}^{f} (1 + \alpha_i \pi^{n_j})^{a_{i,j} p^{\nu_j}} (p^{N+1})$, and*

(b) $1 + x \in (1 + P^N) \sim (1 + P^{N+1})$ *if and only if* $a_{i,j} = 0$ *for* $1 \leqq i \leqq f, 1 \leqq j \leqq N$ *and at least one of the* $a_{i,N} \neq 0, 1 \leqq i \leqq f$.

Proof. The proof is similar to that of Proposition 10, page 34 of [11], and is omitted.

Given $N \in Z^+$ we establish the following notation to be used in the following lemma and theorem. For each $j, 1 \leqq j \leqq N$, write $j = n_j P^{\nu_j}$, where $(n_j, p) = 1$; define m_j as the smallest integer such that $m_j \geqq \log_p((N+1)/n_j)$ then define β_j as a primitive $p^{m_j\text{th}}$ root of 1 in $\boldsymbol{C}$.

LEMMA 3.2. *With the above notation,* $m_N = \nu_N + 1$.

Proof. The proof is a direct computation and is omitted.

THEOREM 3.1. *Let* K *be a local field of characteristic* $p \neq 0$ *and* ω *a character of* $1 + P \subset K$ *ramifield degree* $N + 1$. *Then for* $x \in 1 + P$, ω *is given by*

$$\omega(x) = \prod_{j=1}^{N} \prod_{i=1}^{f} \beta_j^{k_{i,j} a_{i,j} p^{\nu_j}} ,$$

where

$$(*) \qquad x \equiv \prod_{j=1}^{N} \prod_{i=1}^{f} (1 + \alpha_i \pi^{n_j})^{a_{i,j} p^{\nu_j}} (P^{N+1})$$

for some unique $k_{i,j}$, $0 \leqq k_{i,j} < p^{m_j}$ *with at least one of* $k_{i,N}$, $1 \leqq i \leqq f$, *relatively prime to* p.

Proof. Since ω is constant on cosets of P^{N+1} it suffices to consider $x \bmod P^{N+1}$. For any $x \in 1 + P$, the numbers $a_{i,j}$, n_j, ν_j are determined as in Lemma 3.1 so that (*) holds. Clearly ω will be completely determined by its values on $\{1 + \alpha_i \pi^{n_j}\}$, $1 \leqq i \leqq f, 1 \leqq j \leqq N$, and the range of ω is contained in the p^{th} power roots of unity.

The definition of m_j as the smallest integer greater than or equal to $\log_p (N+1)/n_j$ makes m_j the smallest integer such that

$$(1 + \alpha_i \pi^{n_j})^{p^{m_i}} \in 1 + P^{N+1} .$$

Thus $(\omega(1 + \alpha_i \pi^{n_j}))^{p^{m_j}} = 1$, and $\omega(1 + \alpha_i \pi^{n_j}) = \beta_j^{k_{i,j}}$ for some unique $k_{i,j}$, $0 \leqq k_{i,j} < p^m$. Thus ω has the form required. The remainder of the theorem follows easily from the fact that β_N is a $p^{m_N\text{th}}$ root of unity and ω must be nontrivial on P^N.

From Proposition 9 of Chapter II, § 3 of [11], we have:

PROPOSITION. *Let K be a d-dimensional extension of $\boldsymbol{Q}_p$. Then there is an integer $m \geqq 0$ such that $1 + P$, as a multiplicative group is isomorphic to the additive group $Z_p^d \times F_{p^m}$, where m is the largest integer such that K contains a primitive $p^{m\text{th}}$ root of unity. For proof see* [11].

Let $\{u_i\}_{i=1}^d$ be those elements of $1 + P$ which map to the vectors with 1 in the i^{th} coordinate and zeros elsewhere by the isomorphism in the proposition. Let u_{d+1} be a primitive p^{th} power root of unity in K of maximal order, say p^m. Then any $x \in 1 + P$ is given uniquely by $x = \prod_{i=1}^{d+1} u_i^{a_i}$, where $a_i \in Z_p$, $1 \leqq i \leqq d$ and $a_{d+1} \in Z$, $0 \leqq a_{d+1} < p^m$.

LEMMA 3.3. *Let K be a d-dimensional extension of $\boldsymbol{Q}_p$. Then given nonnegative integers k_i, $1 \leqq i \leqq d$, each $x \in 1 + P \subset K$ has a representation as*

$$x = u_{d+1}^{a_{d+1}} \prod_{i=1}^d u_i^{n_i} u_i^{b_i}\ , \quad \textit{where}$$

$b_i \in Z_p$ with $|b_i|_{Z_p} < p^{-k_i}$ and n_i is a nonnegative integer. If n_i is picked to be as small as possible, this representation is unique.

Proof. The proof is direct from the above proposition and the density of Z^+ in Z_p.

Given $N \in Z^+$, define, for $1 \leqq i \leqq d + 1$, $\mathscr{L}_1$ to be the smallest integer such that $u_i^{p^{l_i}} \in 1 + P^{N+1}$ and β_i to be a fixed primitive $p^{l_i\text{th}}$ root of $1 \in \boldsymbol{C}$. With this notation we have the following:

THEOREM 3.2. *Let K be a local field of characteristic 0 and ω a character of $1 + P \subset K$ ramified of degree $N + 1$. Then for $x \in 1 + P$, ω is given for some unique k_i, $0 \leqq k_i < p^{l_i}$, $1 \leqq i \leqq d + 1$, by*

$$\omega(x) = \prod_{i=1}^d \beta_i^{k_i n_i} \beta_{d+1}^{k_{d+1} a_{d+1}} \ \textit{for}\ x = \prod_{i=1}^d u_i^{n_i} u_i^{b_i} u_{d+1}^{a_{d+1}}\ ,$$

where for $1 \leqq i \leqq d$, $b_i \in Z_p$ with $|b_i|_{Z_p} < p^{-l_i}$ and $n_i \in Z^+$.

Proof. The density of Z in Z_p shows that an (additive) character of Z_p is determined by its value at 1. Thus a (multiplicative) character $1 + P$ will be determined by its values at the u_i, $1 \leqq i \leqq d + 1$. Here $\omega(u_i)^{p^{l_i}} = 1$ since $u_i^{p^{l_i}} \subset 1 + P^{N+1}$ and ω is ramifield of degree $N + 1$. Thus $\omega(u_i) = \beta_i^{k_i}$ for some (unique) k_i, $0 \leqq k_i < p^{l_i}$.

This characterization of the character group of K depends on the p^{th} roots of unity in K. Since K is a finite dimensional extension of $\boldsymbol{Q}_p$, we look for a relationship between the degree d of K over $\boldsymbol{Q}_p$ and the existence of p^{th} roots of unity in K.

THEOREM 3.3. *Let K be a local field of characteristic* 0. *If K is the p-adic field $\boldsymbol{Q}_p$ for some prime $p \neq 2$, then K has no nontrivial p^{th} roots of unity. If K is an extension of $\boldsymbol{Q}_p$, $p \neq 2$, let the degree of ramification* (see [11]) *of K over $\boldsymbol{Q}_p$ be e; then,*

(a) *K has no p^{th} roots of* 1 *if* $(p-1)$ *does not divide e,*

(b) *K may or may not have p^{th} roots of* 1 *if $p-1$ divides e.*

Proof. For the proof of (a) see [2]. Part (b) follows from [2] and the fact that the extension of $\boldsymbol{Q}_p$ by a root of $x^{p-1}-p$ is fully ramified of degree $p-1$ and has no p^{th} roots of unity.

LEMMA 4.1. *Let ω be a homogeneous degree zero multiplicative character of K^x, ramified of degree $k>0$. Then ω is a kernel for a singular integral operator. The multiplier m for the singular integral operator T with kernel ω satisfies*

$$m(x) = \omega(-1)\Gamma(\omega)\omega^{-1}(x) .$$

Proof. The operator T is defined for $f \in \mathscr{L}_p$, $1 \leqq p < \infty$ by

$$Tf(x) = \lim_{k\to\infty} \int_{(P^k)^c} \frac{\omega(y)}{|y|}(f(x-y)dy .$$

Theorem 3.1 of [7] gives sufficient conditions on the kernel ω for the limit to exist (in $\mathscr{L}_p$). That ω satisfies those conditions is easily verified. Then from [7] we know T is bounded on $\mathscr{L}_p$, $1 < p < \infty$ and weak type (1, 1).

The remainder of the lemma is done by Chao [1] for the case ω ramified of degree 1. The same proof establishes the result stated here.

Note. Chao [1] uses Theorem 4 of [8] to establish the conclusion of Lemma 4.1 for the case ω ramified of degree 1. However, he fails to compensate for the fact that he defines the Fourier transform as herein, i.e., $\mathscr{F}f(y) = \int f(x)\overline{\chi(xy)}dx$, while in [8] it is defined as $\int f(x)\chi(xy)dx$. Thus the result of [1] which corresponds to the conclusion of Lemma 4.4 above does not contain the necessary factor of $\omega(-1)$.

With notation as in Theorem 3.1, we define rotation operators $S_{i,j}$ for functions on a p-series field as follows:

$$S_{1,0}f(x) = f(gx) ,$$

where g is a fixed primitive $(q-1)^{st}$ root of unity in K; and

$$S_{i,j}f(x) = f((1 + \alpha_i \pi^{n_j})x)$$

for $1 \leqq i \leqq f$, $j \geqq 1$.

Given N we determine β_j, $1 \leqq j \leqq n$ as in Theorem 3.1, and let β_0 be a $(q-1)^{st}$ root of unity in $\boldsymbol{C}$. Also as in that theorem, note that given N the choice of integers $k_{i,j}$, $0 \leqq k_{i,j} < p^{m_j}$, $1 \leqq i \leqq f$, $1 \leqq j \leqq N$ determines a character of $1 + P$. If we also pick a $k_{1,0}$, $0 \leqq k_{1,0} < q - 1$, and set $\omega(g) = \beta^{k_{1\,0}}$, then the set $\{k_{i,j}\}$ determines character of D^0. That character will be called the character determined by $\{k_{i,j}\}$. As it may be used as a kernel for a singular integral operator, that operator will be identified as the one determined by $\{k_{i,j}\}$.

We can how state

THEOREM 4.1. *Let K be a p-series field and L a continuous linear operator from $\mathscr{L}_2(K)$ to $\mathscr{L}_2(K)$ which satisfies*

(a) *$\mathscr{L}$ commutes with translation and dilation,*

(b) *there is some $N \geqq 0$ such that the multiplier corresponding to L is constant on cosets of P^{N+1},*

(c) *L anti-commutes with the rotations $S_{i,j}$, $1 \leqq i \leqq f$, $1 \leqq j \leqq N$, and $S_{1,0}$ in the sense that*

$$LS_{i,j} = \beta_j^{-k_{i,j}} S_{i,j} L ,$$

for some $k_{i,j}$. Then L is a constant multiple of the singular integral operator determined by $\{k_{i,j}\}$.

Before proving Theorem 4.1, we consider the p-adic case. Let u_i, $1 \leqq i \leqq d + 1$ be as in Theorem 3.2, and let $u_0 = g$, the fixed $(q-1)^{st}$ root of $1 \in K$. In this case we define rotation operators as: $S_i f(x) = f(u_i x)$, $0 \leqq i \leqq d + 1$. Given N, we determine β_i, $0 \leqq i \leqq d + 1$ by: β_0 is a primitive $(q-1)^{st}$ root of $1 \in \boldsymbol{C}$; β_i, $1 \leqq i \leqq d + 1$, is a primitive $p^{l_i\text{th}}$ root of $1 \in \boldsymbol{C}$, where l_i is the smallest integer such that $u_i^{p^{l_i}} \in 1 + P^{N+1}$. Also, for each i we consider integers k_i such that $0 \leqq k_0 < q - 1$, $0 \leqq k_i < p^{l_i}$, $1 \leqq i \leqq d + 1$.

By Theorem 3.2 and the fact that $D^0 = M^\times x(1 + P)$, the set $\{k_i\}_{i=0}^{d+1}$ determines a unique character of D^0 by $\omega(u_i) = \beta_i^{k_i}$. The character ω will be called the character determined by the $\{k_i\}$. It is clearly constant on $1 + P^{N+1}$.

THEOREM 4.2. *Let K be a local field of characteristic 0, and let L be a continuous linear operator from $\mathscr{L}_2(K)$ to $\mathscr{L}_2(K)$ which* *satisfies*

(a) *L commutes with translation and dilation,*

(b) *there is some N such that the multiplier for L is constant on cosets of P^{N+1},*

(c) *L anti-commutes with the rotations S_i, $0 \leqq i \leqq d+1$ in the sense that*

$$LS_i = \beta_i^{-k_i} S_i L \ .$$

Then L is a constant multiple of the singular integral transform determined by the $\{k_i\}$. The proof of Theorems 4.1 and 4.2 will utilize the following Lemma.

LEMMA 4.2. *Let K be a local field. If characteristic $K = 0$, let L satisfy the hypothesis of Theorem 4.2. If characteristic $K = p \neq 0$, let L satisfy the hypothesis of Theorem 4.1. Then for $f \in \mathscr{J}$, L is given by convolution with a unique distribution μ, homogeneous of degree $\omega/|\cdot|$, where ω is the character of D^0 determined the $\{k_{i,j}\}$ or $\{k_i\}$ in the characteristic $p \neq 0$ and characteristic 0 case, respectively.*

Proof. Since L is a bounded linear operator from $\mathscr{L}_2$ to $\mathscr{L}_2$ which commutes with translation, by Theorem 9 of [10], it is given, on $\mathscr{J}$, by convolution with a unique distribution μ. We need only to show μ homogeneous of degree ν, where $\nu(x) = \omega(x)/|x|$, $x \neq 0$.

There is a function m in $\mathscr{L}_\infty(K)$ so that for $f \in \mathscr{L}_2(K)$, $(Lf)^\wedge = m\hat{f}$. For $f \in \mathscr{J}$, $\hat{f} \in \mathscr{J}$, thus $m\hat{f} \in \mathscr{L}_1(K)$ since $m \in \mathscr{L}_\infty(K)$. Then $Lf = (m\hat{f})^\vee$ is continuous since it is the inverse Fourier transform of an $\mathscr{L}_1$ function.

Let $\gamma \in 1 + P^{N+1}$. Then $\gamma^{-1} \in 1 + P^{N+1}$, and, since m is constant on cosets of P^{N+1}, we have:

$$\begin{aligned}(L\tau_\gamma f)^\wedge(x) &= m(\tau_\gamma f)^\wedge(x) = m(x)\hat{f}(\gamma^{-1}x) \\ &= m(\gamma^{-1}x)\hat{f}(\gamma^{-1}x) = \tau_\gamma^{-1} m\hat{f}(x) \ .\end{aligned}$$

Thus

$$L\tau_\gamma f = \tau_\gamma L f \quad \text{in} \quad \mathscr{L}_2 \ .$$

Fix $t \in K^\times$. By (a) and (c) of Theorems 4.1 and 4.2 and the above equality, we have:

$$L\tau_t f = \omega^{-1}(t)\tau_t L f \quad \text{in} \quad \mathscr{L}_2$$

and

$$L\tau_t f(x) = \omega^{-1}(t)\tau_t L f(x) \quad \text{a.e.}$$

But since both $L\tau_t f$ and $\tau_t Lf$ are continuous, we have the above equality everywhere.

For $f \in \mathscr{J}$,

$$\mu * \tau_t f(0) = \omega^{-1}(t)(\mu * f)(t \cdot 0) ,$$

and

$$(\mu, \tau_t \tilde{f}) = \omega^{-1}(t)(\mu, \tilde{f}) .$$

Thus

$$\begin{aligned}(\mu_t, f) &= (\mu, |t|^{-1}\tau_{t^{-1}} f) \\ &= |t|^{-1}(\mu, \tau_{t^{-1}} f) \\ &= \frac{\omega(t)}{|t|}(\mu, f) = \nu(t)(\mu, f) .\end{aligned}$$

Since this holds for all $f \in \mathscr{J}$, $\mu_t = \nu(t)\mu$.

Now we are ready for the

Proof of Theorems 4.1 *and* 4.2. By Lemma 4.2 for $f \in \mathscr{J}$, $Lf = \mu * f$, where μ is homogeneous of degree $\omega/|\cdot|$. But by Lemma 5 of [8], the only distributions which are homogeneous of degree σ, σ multiplicative character of $K^{\times}$ such that $\sigma(x)$ is not identically $|x|^{-1}$, are constant multiples of σ. Thus $\mu = c\omega/|\cdot|$, and $Lf = (c\omega)/(|\cdot|) * f$, $f \in \mathscr{J}$. Thus, on the test functions, a dense subset of $\mathscr{L}_2$, L agrees with L', the singular integral operator defined by $L'f(x) = c\int (\omega(y))/(|y|)f(x-y)dy$. But since L and L' are continuous, $L = L'$ on $\mathscr{L}_2$.

5. **Example.** The conclusions of Theorems 4.1 and 4.2 may be obtained by direct calculation. We indicate the method in the case $q = 3$ and ω ramified of degree 1. Here $M^{\times} = \{1, -1\}$ and ω will assume only the values ± 1. [This is the "exact" analog of the Hilbert transform for the reals.]

Let H be the singular integral operator with ω as kernel. Both theorems then have the form: Theorem: Let K be local field with module $q = 3$ and L be a continuous operator on $\mathscr{L}_2(K)$ which satisfies:

(a) L commutes with translation and dilation;

(b) the multiplier, m, for L is constant on $1 + P$;

(c) L anti-commutes with the rotation τ_{-1} by $L\tau_{-1} = -\tau_{-1}L$.

Then L is a constant multiple of H.

Proof. From the relation $(Lf)\hat{} = m\hat{f}$ it follows as in the real case (see [9]) that $m(-x) = -m(x)$. Since any $x \in K^\times$ may be written $x = \pm\pi^j(1+\rho_x)$, $\rho_x \in P$, $m(x) = \pm m(1) = \omega^{-1}(x)m(1)$. The theorem then follows from Lemma 4.1.

Lemma 4.1 may also be shown directly. For the case above we may even evaluate the multiplier m_H explicitly. Taking the fundamental character χ to be that given in [1] (a variation of that given in [6]), and the form of m_H from [7], we obtain $m_H(x) = (i)/(\sqrt{3})\omega(x)$. As in [1], a similar easy calculation gives $\Gamma(\omega) = -i/\sqrt{3}$, exemplifying Lemma 4.1. In further analogy with the real case, it is apparent from the multiplier that $H^2 = -(1/3)I$.

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Received October 14, 1974 and in revised form April 12, 1975.

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