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NORMS OF RANDOM MATRICES

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Under rather general conditions on the matrix entries, we obtain estimates for the probability distribution of the norm of a random matrix transformation from ℓ_n^2 to ℓ_m^q $(2 \leqq q < \infty)$. Asymptotically, the expected norm is remarkably small and this enables us to produce an interesting class of bounded linear operators from ℓ^2 to ℓ^q. As an application, we complete the characterization of (p, q)-absolutely summing operators on Hilbert space, thereby answering a question left open by several previous authors.

1. Introduction.

Many questions in the theory of ℓ^p spaces require, for their solution, the existence of finite matrices with ± 1 entries whose norms satisfy prescribed conditions. In several cases the required matrices have been given explicitly: the simplest examples stem from the orthogonality of the Walsh functions (see, for example, [10] where non-complemented subspaces of ℓ^p are constructed) or from the Rademacher functions via Khintchine's inequality (see, for example, [6] where the p-absolutely summing operators on Hilbert space are characterized). In many problems, however, the construction of suitable matrices leads to formidable combinatorial difficulties. We consider in this paper one such problem for which no constructive method is available. The appropriate matrix is obtained here probabilistically by showing that "most" matrices satisfy the prescribed norm inequalities.

Specifically, the problem we consider is that of characterizing the ideal, $\prod_{p,q}$, of (p, q)-absolutely summing operators on Hilbert space. Recall that a bounded linear operator T on ℓ^2 is (p, q)-*absolutely summing* $(1 \leqq q \leqq p \leqq \infty)$ if $(\|Tx_n\|)_{n=1}^{\infty} \in \ell^p$ whenever $(x_n)_{n=1}^{\infty}$ is a sequence of elements of ℓ^2 with the property that $(\langle x_n, y\rangle)_{n=1}^{\infty} \in \ell^q$ for each $y \in \ell^2$. This problem has received a good deal of attention in recent years and the known results are described below. We denote by $\mathfrak{S}_r(1 \leqq r < \infty)$ the *Schatten r-class* of all compact linear operators T on ℓ^2 for which $\sum_{n=1}^{\infty} |\lambda_n|^r < \infty$ where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of $(T^*T)^{1/2}$, counted according to multiplicities (and arranged in order of decreasing modulus); for convenience the class of *all* bounded linear operators on ℓ^2 is denoted by $\mathfrak{S}_\infty$.

We then have:

(a) *if* $p = q < \infty$, $\Pi_{p,q} = \mathfrak{S}_2$;

(b) *if* $p = \infty$ *or* $(1/q) - (1/p) \geqq 1/2$, $\Pi_{p,q} = \mathfrak{S}_\infty$;

(c) *if* $(1/q) - (1/p) < 1/2$ *and* $q \leqq 2$, $\Pi_{p,q} = \mathfrak{S}_r$ *with* $1/r = (1/p) - (1/q) + (1/2)$;

(d) *if* $2 < q < p < \infty$, $\mathfrak{S}_{2p/q} \subsetneqq \Pi_{p,q} \subsetneqq \mathfrak{S}_p$.

(a) is due to Pełczynski [6]; (b) to Kwapien [4]; (c) and the first part of (d) to Mitiagin [4]; and the second part of (d) to Pietsch-Triebel [8]. Special cases were discovered earlier by Grothendieck ((a) for $p = 1$) [3]; Pietsch ((a) for $1 \leqq p \leqq 2$) [7]; and Orlicz ((b) for $p = 2$, $q = 1$) [5].

The only outstanding case is thus $2 < q < p < \infty$. This case is more subtle than the others in that a new ideal, $\mathfrak{S}_{r,s}$, generated by the Lorentz sequence space $\ell^{r,s}$ is involved. $\mathfrak{S}_{r,s}(1 < r, s < \infty)$ is the set of all compact linear operators T on ℓ^2 for which $\sum_{n=1}^{\infty} n^{(s/r)-1} |\lambda_n|^s < \infty$ where the λ_n's are defined as above. In [1], it is shown that

(e) *if* $2 < q < p < \infty$, $\mathfrak{S}_{2p/q,p} \subseteqq \Pi_{p,q}$, *with equality when* q *is an even integer*;

we here remove the restriction that q be an even integer, thereby completing the description of $\Pi_{p,q}$. This is done in Section 3 of the paper by using the following result whose proof is given in Section 2.

THEOREM 1. *Let* $A = (a_{ij})$ *be an* $m \times n$ *matrix whose entries are independent, mean-zero random variables with* $|a_{ij}| \leqq 1$ *for all* i, j. *For* $2 \leqq q < \infty$, *there is a constant* K, *depending only on* q, *such that*

$$E(\|A\|_{2,q}) \leqq K \max(m^{1/q}, n^{1/2})$$

where $\|A\|_{2,q}$ *denotes the operator norm of* $A\colon \ell_n^2 \to \ell_m^q$.

It should be noted that the estimate of Theorem 1 is best possible (up to the choice of K) in the sense that every $m \times n$ matrix $A' = (a'_{ij})$ with $a'_{ij} = \pm 1$ has $\|A'\|_{2,q} \geqq \max(m^{1/q}, n^{1/2})$.

2. Random matrices.

Theorem 1 is proved below using the following lemmas. Our techniques were suggested by methods used in deriving limit theorems for large deviations of sums of random variables ([2] and [9]). We begin with a standard result whose proof is included for completeness.

LEMMA 1. *Let* $(X_j)_{j=1}^n$ *be independent, mean-zero random variables with* $|X_j| \leqq 1$ *for all* j; *then for any* $\lambda > 0$ *and real* $b_1, \cdots, b_n$,

$$P\left(\left|\sum_{j=1}^n b_j X_j\right| \geqq \lambda\right) \leqq 2\exp\left(-\lambda^2/4 \sum_{j=1}^n b_j^2\right).$$

Proof. Using the elementary inequality, $\exp(x) - x \leqq \exp(x^2)$, it follows that for all real μ,

$$E[\exp(\mu X_j)] = E[\exp(\mu X_j) - \mu X_j] \leqq E[\exp(\mu^2 X_j^2)] \leqq \exp(\mu^2) .$$

The independence of $(X_j)_{j=1}^n$ thus yields

$$E\left[\exp\left(\mu \sum_{j=1}^n b_j X_j\right)\right] \leqq \exp\left(\mu^2 \sum_{j=1}^n b_j^2\right) .$$

Applying Chebyshev's inequality, we obtain, for $\mu > 0$,

$$P\left(\sum_{j=1}^n b_j X_j \geqq \lambda\right) \leqq \exp\left(\mu^2 \sum_{j=1}^n b_j^2 - \lambda\mu\right) .$$

Taking $\mu = \lambda/2 \sum_{j=1}^n b_j^2$ gives the desired result.

The next lemma is really the technical key to our results (at least for $q \neq 2$).

LEMMA 2. *For each $q \geqq 2$, there is a constant C, depending only on q, so that if $(X_j)_{j=1}^n$ satisfy the hypotheses of Lemma 1,*

$$E\left[\exp\left(\mu\left|\sum_{j=1}^n b_j X_j\right|^q\right)\right] \leqq 1 + C\mu\left(\sum_{j=1}^n b_j^2\right)^{q/2}$$

for $0 \leqq \mu \leqq (\sum_{j=1}^n b_j^2)^{-q/2} n^{1-q/2}/8$.

Proof. Without loss of generality we may take $\sum_{j=1}^n b_j^2 = 1$.

$$\begin{aligned} E\left[\exp\left(\mu\left|\sum_{j=1}^n b_j X_j\right|^q\right)\right] &= \int_0^\infty \exp(\mu\lambda^q) dP\left(\left|\sum_{j=1}^n b_j X_j\right| \leqq \lambda\right) \\ &= 1 + \int_0^\infty q\mu\lambda^{q-1} \exp(\mu\lambda^q) P\left(\left|\sum_{j=1}^n b_j X_j\right| > \lambda\right) d\lambda \end{aligned}$$

via integration by parts. Since $|\sum_{j=1}^n b_j X_j| \leqq (\sum_{j=1}^n b_j^2)^{1/2} n^{1/2}$, we have $P(|\sum_{j=1}^n b_j X_j| > n^{1/2}) = 0$. We apply the estimate from Lemma 1 to obtain

$$\begin{aligned} &E\left[\exp\left(\mu\left|\sum_{j=1}^n b_j x_j\right|^q\right)\right] \\ &\quad \leqq 1 + 2q\mu \int_0^{n^{1/2}} \lambda^{q-1} \exp(\mu\lambda^q - \lambda^2/4) d\lambda \\ &\quad \leqq 1 + 2q\mu \int_0^\infty \lambda^{q-1} \exp(-\lambda^2/8) d\lambda \end{aligned}$$

since $0 \leqq \mu \leqq n^{1-q/2}/8$. This gives the desired result with

$$C = 2q \int_0^\infty \lambda^{q-1} \exp(-\lambda^2/8) d\mu = 8^{q/2} q \Gamma(q/2) .$$

We now assume that $q \geqq 2$ is fixed and that $\{a_{ij}\}$ are independent random variables satisfying the hypotheses of Lemma 1. Given real x_j we obtain estimates (independent of $\{x_j\}$) on the probability distribution for the random variable

$$Y = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} x_j a_{ij} \right|^q \Big/ \left(\sum_{j=1}^{n} x_j^2 \right)^{q/2} .$$

LEMMA 3. *For any real λ and positive integers m, n,*

$$P(Y \geqq Cm + 8\lambda n^{q/2}) \leqq \exp(-\lambda n) ,$$

where C is the constant appearing in Lemma 2.

Proof. For real μ, we set $K(\mu) = \log E[\exp(\mu Y)]$ so that $E[\exp(\mu Y - K(\mu))] = 1$. It follows from Chebyshev's inequality that for any real ν,

$$P(\mu Y \geqq K(\mu) + \nu) \leqq \exp(-\nu) .$$

On the other hand, we have from Lemma 2 that for $0 \leqq \mu \leqq n^{1-q/2}/8$,

$$\begin{aligned} E\Big[\exp(\mu Y)\Big] &= \prod_{i=1}^{m} \left[E\left[\exp\left(\mu \left| \sum_{j=1}^{n} b_j a_{ij} \right|^q \right) \right] \right] \\ &\leqq \prod_{i=1}^{m} (1 + C\mu) \leqq \exp(mC\mu) , \end{aligned}$$

where $b_j = x_j / (\sum_{j=1}^{n} x_j^2)^{1/2}$ so that $K(\mu) \leqq mC\mu$. Setting $\nu = \lambda n$ and $\mu = n^{1-q/2}/8$, we obtain the desired result.

We now consider the random $m \times n$ matrix A with entries a_{ij} and denote the norm of $A: \ell_n^2 \longrightarrow \ell_m^q$ by

$$\|A\|_{2,q} = \sup_{\sum_{j=1}^{n} x_j^2 = 1} \left(\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^q \right)^{1/q} = \sup_{\|x\|_2 = 1} (\|Ax\|_q) .$$

LEMMA 4. *There exist constants c_1, c_2 (depending at most on q) such that for all $\lambda > 0$,*

$$P(\|A\|_{2,q} \geqq c_1(m + \lambda n^{q/2})^{1/q}) \leqq \exp(-(\lambda - c_2)n) .$$

Proof. Let $0 < \varepsilon < 1$ be fixed; by an ε-net for the unit sphere $S = \{x: \|x\|_2 = 1\}$ in $\boldsymbol{R}^n$, we mean a finite subset, N, of S satisfying

$$\sup_{x \in S} \min_{y \in N} \|x - y\|_2 < \varepsilon .$$

Any matrix A effectively attains its norm on such a set N: more precisely, we have

$$\|A\|_{2,q} = \sup_{x\in S} \|Ax\|_q \leqq \max_{y\in N} \|Ay\|_q + \sup_{x\in S} \min_{y\in N} \|A(x-y)\|_q ,$$

so that

$$\|A\|_{2,q} \leqq \frac{1}{1-\varepsilon} \max_{y\in N} \|Ay\|_q .$$

For such an N, we have, putting $c_1 = (\max(C, 8))^{1/q}/(1-\varepsilon)$,

$$\begin{aligned} P(\|A\|_{2,q} &\geqq c_1(m + \lambda n^{q/2})^{1/q}) \\ &\leqq P\Big(\max_{y\in N} \|Ay\|_q \geqq (Cm + 8\lambda n^{q/2})^{1/q}\Big) \\ &\leqq \sum_{y\in N} \exp(-\lambda n) = |N| \exp(-\lambda n) \end{aligned}$$

by Lemma 3, where $|N|$ denotes the number of elements in N. Moreover, using elementary geometrical arguments, it is straightforward to show that N may be chosen with $|N| \leqq \exp(c_2 n)$ with c_2 a constant depending only on ε; this completes the proof.

We can now proceed to prove Theorem 1.

Proof of Theorem 1. We let $\mu_1 = c_1^q(m + c_2 n^{q/2})$; then by Jensen's inequality,

$$\begin{aligned} (E[\|A\|_{2,q}])^q &\leqq E[(\|A\|_{2,q})^q] \\ &= \int_0^\infty P[(\|A\|_{2,q})^q \geqq \mu] d\mu \\ &\leqq \mu_1 + \int_{\mu_1}^\infty P[\|A\|_{2,q} \geqq \mu^{1/q}] d\mu \\ &\leqq \mu_1 + \int_{\mu_1}^\infty \exp\{-n^{1-q/2}(-m + \mu/c_1^q) + c_2 n\} d\mu \\ &= c_1^q(m + c_2 n^{q/2} + n^{(q/2)-1}) , \end{aligned}$$

from which the desired result immediately follows.

Before concluding this section we apply Lemma 4 in a somewhat different manner to obtain the following result.

THEOREM 2. *For* $2 \leqq q < \infty$, *there is a constant* K', *depending only on* q, *so that if* (for each $m, n = 1, 2, \cdots$) $A_{m,n}$ *is an* $m \times n$ *random matrix satisfying the hypotheses of Theorem* 1, *then with probability one,*

$$\limsup_{\max(m,n)\to\infty} \|A_{m,n}\|_{2,q}/\max(m^{1/q}, n^{1/2}) \leqq K' .$$

Proof. Using the estimate of Lemma 4 with $\lambda = c_2 + \max(1, m/n^{q/2})$ gives

$$P(\|A_{m,n}\|_{2,q}/\max(m^{1/q}, n^{1/2}) \geqq K') \leqq \exp(-\max(n, mn^{1-q/2}))$$

with $K' = c_1(2 + c_2)^{1/q}$. By the Borel-Cantelli lemma,

$$P(\|A_{m,n}\|_{2,q}/\max(m^{1/q}, n^{1/2}) \geqq K' \text{ for infinitely many } m, n) = 0$$

if $\sum_{m,n} P(\|A\|_{2,q}/\max(m^{1/q}, n^{1/2}) \geqq K') < \infty$. Hence it suffices to show that

$$\sum_{m,n=1}^{\infty} \exp(-\max(n, mn^{1-q/2})) < \infty ,$$

which is easily verified.

REMARK. Defining for an $m \times n$ matrix A,

$$L_{2,q}(A) = \inf_{\|x\|_2=1} \|Ax\|_q ,$$

we have of course that $0 \leqq L_{2,q}(A) \leqq \|A\|_{2,q}$ with $L_{2,q}(A) = 0$ for $m < n$. Estimates similar to those obtained above can be used to derive asymptotic lower bounds for $L_{2,q}(A)$ (at least with m much larger than n). It can be shown, for example, that if $\inf_{i,j} E[(a_{ij})^2] > 0$, then for $2 < q < \infty$ and any $\delta > 0$, there exists a constant $d > 0$ so that with probability one,

$$\liminf_{\substack{m\to\infty \\ m \geqq \delta n^{q/2}}} L_{2,q}(A_{m,n})/m^{1/q} \geqq d .$$

3. Absolutely summing operators. In this section we complete the description of the (p, q)-absolutely summing operators on Hilbert space, thereby answering a question left open in [4], [8] and [1].

THEOREM 3. *If $2 < q < p < \infty$, then $\Pi_{p,q} = \mathfrak{S}_{2p/q,p}$.*

Proof. The inclusion $\Pi_{p,q} \supseteqq \mathfrak{S}_{2p/q,p}$ has already been established in Theorem I of [1]. For the converse, we apply Theorem 1 (or Lemma 4 or Theorem 2), choosing the matrix entries (a_{ij}) independently with $P(a_{ij} = +1) = P(a_{ij} = -1) = 1/2$. It follows that there exists, for each positive integer n, at least one matrix of order $[n^{q/2}] \times n$ with all ± 1 entries satisfying $\|A\|_{2,q} \leqq Kn^{1/2}$, where K is a constant depending only on q. This generalizes Proposition 2 of [1], and the argument used to prove Theorem II of that paper shows that $\Pi_{p,q} \subseteqq \mathfrak{S}_{2p/q,p}$.

Added in proof. For further applications of these results the reader should consult the forthcoming paper "On uncomplemented subspaces L^p, $1 < p < 2$, being prepared jointly with L. E. Dor and W. B. Johnson.

For extensions to matrix transformations of ℓ^p into ℓ^q, $1 \leqq p$, $q \leqq \infty$, consult "Hadamard multipliers," being prepared by G. Bennett.

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