PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

INFINITE SUBRINGS OF INFINITE RINGS AND NEAR-RINGS

HOWARD E. BELL

Leffey has proved that every infinite associative ring contains an infinite commutative subring, and thereby suggested the problem of finding reasonably small classes $\mathscr{F}$ of infinite rings with the property that (*) every infinite ring contains a subring belonging to $\mathscr{F}$. Clearly, there is no minimal class $\mathscr{F}$ in the obvious sense, for in any class satisfying (*) a ring may be replaced by any proper infinite subring of itself. In §§ 1–3 we determine a class $\mathscr{F}_0$ satisfying (*) and consisting of familiar and easily-described rings; and § 4 we indicate how our results subsume and extend known finiteness results formulated in terms of subrings and zero divisors.

Section 5 identifies classes which satisfy (*) and are minimal in a certain loose sense, and § 6 extends the major result of the first three sections to distributive nearrings. The ring-theoretic results are proved in the setting of alternative rings.

In the remainder of the paper, Z stands for the ring of integers and Z_p for the ring of integers modulo p, where p is prime. The term *J-ring* refers to a ring R such that for each $x \in R$, there exists an integer $n(x) > 1$ for which $x^{n(x)} = x$. The cyclic group of order p is denoted by C_p, the infinite cyclic group by C_∞, the Prüfer p-group by $C(p^\infty)$.

Let p be a prime and $\lambda = \langle p_i \rangle$ an infinite strictly-increasing sequence of primes. Then $G(\lambda)$ will denote the direct sum of the groups C_{p_i}, $H(\lambda)$ will denote the direct sum of the fields Z_{p_i}, and $F(p, \lambda)$ will denote the field $\bigcup_{n=1}^{\infty} GF(p^{\pi_n})$, where $\pi_n = p_1 p_2 \cdots p_n$. If λ is replaced by an infinite sequence all terms of which are equal to the same prime q, then the analogous groups and rings will be denoted by $G(q)$, $H(q)$, and $F(p, q)$.

Finally, for any subset S of R, $A_L(S)$, $A_R(S)$, and $A(S)$ will denote respectively the left, right, and two-sided annihilators of S.

The major theorem of the paper is

THEOREM 1. *Let $\mathscr{F}_0$ be the class consisting of all rings of the following kinds*:

(a) *the zero ring on one of the groups* C_∞, $C(p^\infty)$, $G(q)$, or $G(\lambda)$;

(b) *rings generated by a single element and isomorphic to a subring of Z, to the ring $XZ[X]$, or to the ring $XZ_p[X]$ for some prime p*;

(c) *a ring* $H(q)$, *a ring* $H(\lambda)$, *a field* $F(p, q)$, *or a field* $F(p, \lambda)$. *Then every infinite alternative ring contains a subring belonging to* $\mathscr{I}_0$.

1. Preliminary results. We begin this section with some preliminary results on alternative rings. Denoting the associator $(xy)z - x(yz)$ by (x, y, z), we note that it is skew-symmetric and satisfies the identity [4, p. 379]

$$\text{(A)} \qquad (y, x^2, z) = (y, x, xz + zx) = x(y, x, z) + (y, x, z)x .$$

LEMMA 1. *For any alternative ring* R, *the following results hold*:

(i) $A(R)$ *is a two-sided ideal of* R;

(ii) *if* $x \in R$, $A(x)$ *is a subring of* R;

(iii) *if* $x \in R$ *and* $x^2 = 0$, *and if* $H = A_R(x)$, *then* Hx *is a zero ring*;

(iv) *if* e *is an idempotent of* R *which commutes elementwise with* R, *then* e *is in the nucleus and* R *is the direct sum of the orthogonal ideals* Re *and* $A(e)$.

Proof. (i) The proof is trivial and is omitted.

(ii) Clearly $A(x)$ is an additive subgroup. Also, if $a_1, a_2 \in A(x)$, we have $(a_1, x, a_2) = (a_1x)a_2 - a_1(xa_2) = 0$; thus $(a_1a_2)x = x(a_1a_2) = 0$.

(iii) Since H is an additive subgroup, so is Hx. Moreover, letting $h_1, h_2 \in H$ and applying (A), we get $0 = (h_1, 0, h_2) = (h_1, x^2, h_2) = (h_1, x, xh_2 + h_2x) = (h_1, x, h_2x) = (h_1x)(h_2x) - h_1(x(h_2x)) = (h_1x)(h_2x)$; therefore, Hx is a zero ring.

(iv) Taking $x = e$ in (A) and using the skew-symmetry of the associator yields $(e, y, z) = e(e, y, z) + (e, y, z)e = 2e(e, y, z)$. Multiplying through by e then gives $e(e, y, z) = (e, y, z) = 0$. The result of (iv) now follows trivially.

LEMMA 2. (See [5].) *Let* R *be an infinite alternative ring containing no infinite zero ring. Then for each nilpotent element* $x \in R$, $A(x)$ *is infinite.*

Proof. Let G denote any infinite additive subgroup of R, and define the additive subgroup homomorphism $\phi: G \to xG$ by $y \mapsto xy$. Application of the first isomorphism theorem shows that either xG is infinite or $\{y \in G \mid xy = 0\} = G \cap A_R(x)$ is infinite; similarly, one of Gx and $G \cap A_L(x)$ must be infinite. Using these results, we proceed by induction on the index of nilpotence of x.

Suppose first that $x^2 = 0$. Since either xR or $A_R(x)$ must be infinite, $A_R(x)$ is infinite in any event. By (iii) of Lemma 1, $(A_R(x))x$ is a zero ring, hence finite; therefore $A_R(x) \cap A_L(x) = A(x)$ must be infinite.

Now assume the result for nilpotent elements of index less than k, $k \geqq 3$; and suppose $x^k = 0$. Since $(x^2)^{k-1} = 0$, $A(x^2)$ is infinite and hence either (i) $xA(x^2)$is infinite or (ii) $A(x^2) \cap A_R(x)$ is infinite. In the event that (i) holds, then one of $(xA(x^2))x$ and $xA(x^2) \cap A_L(x)$ is infinite; and since both are contained in $A(x)$, we are done. If (ii) holds then either $(A(x^2) \cap A_R(x))x$ is infinite or $A(x^2) \cap A_R(x) \cap A_L(x)$ is infinite, and again we are finished because both are contained in A(x).

Finally, we present for the sake of completeness some easy results on periodic (alternative) rings R—that is, rings with the property that for each $x \in R$ there exist distinct positive integers n, m for which $x^n = x^m$.

LEMMA 3. *Let R be a periodic alternative ring. Then* (i) *if R is not nil, R has a non-zero idempotent*; (ii) *if R has no non-zero nilpotent elements, R is a J-ring.*

Proof. (i) If $x^n = x^m$ for $n > m$, then $x^{j+k(n-m)} = x^j$ for each positive integer k and each $j \geqq m$. Thus $x^{m(n-m)}$ is idempotent.

(ii) Let $x^n = x^m$, $n > m > 1$. Then $x^{m-2}x(x - x^{n-m+1}) = 0 = x^{m-2}x^{n-m+1}(x - x^{n-m+1}) = x^{m-2}(x - x^{n-m+1})^2$. The obvious induction shows that $x - x^{n-m+1}$ is nilpotent, hence 0.

2. Initial reduction of the problem.

PROPOSITION 4. *Every infinite alternative ring contains an infinite subring of one of the following kinds*:

(a) *a nil ring;*

(b) *a ring generated by a single element;*

(c) *a J-ring.*

Proof. Let R be any infinite ring containing no infinite subring of type (a) or (b); note that every infinite subring of R has the same property. Since every element of R generates a finite subring, R must be periodic.

Suppose for the time being that for every set $S_N = \{0 = x_1, x_2, \cdots, x_N\}$ of distinct elements of R such that

$$x_i x_j = 0 \text{ for all } i, j = 1, \cdots, N \tag{4.1}$$

and

$$R_N = A(S_N) \text{ is infinite ,} \tag{4.2}$$

it is possible to find $y \in R_N \backslash S_N$ for which $y^2 = 0$. By (ii) of Lemma 1 and our earlier observations on R, R_N is an infinite ring with no infinite zero subring; thus by Lemma 2, the annihilator of y in R_N, namely $R_N \cap A(y)$, must be infinite, and $S_{N+1} = S_N \cup \{y\}$ is a set of $N + 1$ distinct elements satisfying (4.1) and (4.2). Thus, beginning with $S_1 = \{0\}$ we can construct inductively an infinite sequence of pairwise-orthogonal elements squaring to zero and therefore an infinite zero ring—a contradiction.

Thus, R contains some set S_N satisfying (4.1) and (4.2) such that every element y of R_N squaring to zero already belongs to S_N; replacing R by R_N, we assume henceforth that R has the property that $S = \{x \in R \mid x^2 = 0\}$ is finite and equal to $A(R)$.

By (i) of Lemma 3, R contains a nonzero idempotent e. Now for each $x \in R$, and every nonzero idempotent e of R, $ex - exe$ and $xe - exe$ are elements of R squaring to zero, hence are in S and are annihilated by e. Thus $xe - exe = ex - exe = 0$; and by (iv) of Lemma 1, e is central in R and $R = eR \oplus A(e)$. Since $S \subseteqq A(e)$, eR can contain no nonzero elements squaring to zero, hence no nonzero nilpotent elements; thus eR is a J-ring. We may assume that eR is finite for all nonzero idempotents e, for otherwise we are done. A straightforward induction yields an infinite sequence of pairwise orthogonal nonzero idempotents e_i such that for each m, $R = e_1R \oplus \cdots \oplus e_mR \oplus T_m$, where $T_m = \bigcap_{i=1}^{m} A(e_i)$. The restricted direct sum $\Sigma \oplus e_iR$ of the J-rings e_iR is therefore an infinite J-ring contained in R.

PROPOSITION 5. *Every infinite alternative nil ring contains an infinite zero ring.*

Proof. Assume the result is false. Then by the second and third paragraphs of the proof of Proposition 4, every counterexample must contain as a subring a counterexample R with the property that

(P) $S = \{x \in R \mid x^2 = 0\}$ is finite and is equal to $A(R)$.

We first show that R must have bounded index of nilpotence. Denote the number of elements of S by N, and suppose that $x^{2k} = 0$ for $k \geqq N + 1$; note that $x^k, \cdots, x^{2k-1}$ all square to zero. Since $k > N$ these elements cannot be distinct, and there exist positive integers j_1 and j_2 such that $j_1 < j_2 \leqq 2k - 1$ and $x^{j_1} = x^{j_2 + j(j_2 - j_1)}$ for all positive integers j. Thus $x^{j_1} = 0$ and it follows that $x^{2N} = 0$ for all $x \in R$.

We assume now that R has degree of nilpotence K, minimal for the family of counterexamples with property (P). Clearly $\bar{R} = R/A(R)$

is infinite; and since $(x^{K-1})^2 = 0$ for all $x \in R$, $\bar{R}$ must have index of nilpotence at most $K - 1$. If R were a counterexample to Proposition 5, then it would contain a counterexample with property (P), thereby contradicting the minimality of K; thus, $\bar{R}$ has an infinite zero subring $\bar{T} = T/A(R)$. Clearly T is an infinite subring of R such that $(xy)z = x(yz) = 0$ for all $x, y, z \in T$; in particular, $x^3 = 0$ for all $x \in T$. Since T contains only a finite number of elements squaring to zero, each necessarily of finite additive order, there must exist a positive integer n such that $y^2 = 0$ implies $ny = 0$. Thus, $nx^2 = 0$ for all $x \in T$, so that nT has each of its elements squaring to zero, hence is finite. Therefore $\tilde{T} = \{x \in T \mid nx = 0\}$ must be infinite.

Replacing R by $\tilde{T}$, we now have a counterexample R such that $S = \{x \in R \mid x^2 = 0\} = A(R)$ is finite, $A(R) \supseteqq R^2$, and $nR = 0$ for some positive integer n. For any finite or infinite sequence $\langle x_i \rangle$ of elements of R, denote by W_i the subring generated by $S \cup \{x_1, \cdots, x_i\}$; and note that each W_i must be finite. Using Lemma 2 and (ii) of Lemma 1 we can obtain a sequence $\langle x_i \rangle$ of elements of R such that

(5.1) $$S = W_0 \subset W_1 \subset \cdots \subset W_i$$

for each i, the inclusions all being strict ;

(5.2) $$x_i x_j = 0 \text{ for all } i \neq j \;;$$

(5.3) $$\text{for each } k,\ T_k = A(W_k) \text{ is infinite .}$$

Specifically we begin by choosing any $x_1 \notin S$ and proceed inductively — once $x_1, \cdots, x_k$ have been defined, the finiteness of W_k permits the choice of $x_{k+1} \in T_k \backslash W_k$; and Lemma 2 applied to T_k guarantees that $T_k \cap A(x_{k+1}) = T_{k+1}$ is infinite.

Since $x_i^2 \in S$ for each i, the finiteness of S implies the existence of $s \in S$ for which $x_i^2 = s$ for n distinct x_i. Letting z be the sum of these x_i, we have the result that $z^2 = ns = 0$ but $z \notin S$. This contradiction completes the proof of Proposition 5.

PROPOSITION 6. *Every infinite alternative ring contains an infinite subring which is both associative and commutative.*

Proof. Since one-generator subrings and zero rings are obviously associative and commutative, we need only establish the same for alternative J-rings. These are commutative by a theorem of Smiley [10]; the associativity follows from the general result that a commutative alternative ring with no nonzero nilpotent elements is associative [7, Lemma 3].

3. Proof of Theorem 1.

The proof of Theorem 1 is completed

by three lemmas, which further refine the classes (a), (b) and (c) of Proposition 4 (in that order). In view of Proposition 6, we may assume that our rings are associative.

LEMMA 7. *Every infinite zero ring contains a zero ring on one of the following groups*: (i) C_∞; (ii) $G(\lambda)$ *for some strictly-increasing sequence* λ *of primes*; (iii) $G(q)$ *for some prime* q; (iv) $C(p^\infty)$ *for some prime* p.

Proof. Of course, we wish to prove that every infinite abelian group contains one of the indicated groups as a subgroup.

Suppose then that G is any infinite abelian group. If G contains an element of infinite order, it contains an infinite cyclic subgroup; hence we may suppose that G is periodic, in which case $G = \Sigma \bigoplus G_p$, where the G_p are the p-primary components for all primes p. If there are infinitely many nontrivial G_p, then G has a subgroup of type (ii); thus we consider the case of only finitely many nontrivial G_p and assume without loss that G is a *countable p-group* for some prime p. Let H be the subgroup of G consisting of elements of order p.

If G has no elements of infinite height, then G has a subgroup of type (iii) by Theorem 11.3 of [3]; if H is infinite, then we can replace G by H and apply the same argument. Thus, we suppose that H is finite and that G contains an element x_0 of infinite height such that $px_0 = 0$. There exists a sequence x_i of elements of G for which $p^i x_i = x_0$, $i = 1, 2, \cdots$; and the set $\{x_1 - p^{i-1}x_i \mid i = 2, 3, \cdots\}$ is a subset of H. There is, therefore, a smallest integer $M \geqq 2$ for which $p^{M-1}x_M$ is equal to $p^{i-1}x_i$ for infinitely many i; and it follows that $x_1' = p^{M-1}x_M$ is of infinite height and $px_1' = x_0$. Proceeding inductively, we get a sequence $x_0, x_1, x_2, \cdots$ where $px_0 = 0$ and $px_i = x_{i-1}$, $i = 1, 2, \cdots$; hence G must contain $C(p^\infty)$ as a subgroup.

LEMMA 8. *Let R be an infinite ring which is generated by a single element, and suppose R contains no infinite zero ring. Then R must contain $XZ[X]$, or $XZ_p[X]$ for some prime p, or a subring of Z.*

Proof. Suppose initially that R is generated by an element a of infinite additive order. Clearly, if a is not algebraic over the integers, $R \cong XZ[X]$. Consider now the case where a is algebraic over the integers, and let a satisfy

$$n_1a^{k_1} + n_2a^{k_2} + \cdots + n_sa^{k_s} = 0 , \tag{8.1}$$

where $k_1 < k_2 < \cdots < k_s$, $n_1 \neq 0$, and k_1 is the smallest positive integer occurring as the lowest power of a in any such relation. If $k_1 > 1$, then $ab = 0$, where $b = n_1a^{k_1-1} + \cdots + n_sa^{k_s-1}$; and since the annihilator of a is the annihilator of R, which under our assumptions is finite, either $b = 0$ or $jb = 0$ for some positive integer j. In either case the minimality of k_1 is contradicted; therefore $k_1 = 1$ and

$$n_1a = ap(a) \, , \tag{8.2}$$

where $p(X) \in Z[X]$ has zero constant term. Letting $b = p(a)$ in (8.2), we see that b has infinite order and $b^2 = n_1b$. Thus, the subring of R generated by b is isomorphic to the subring of Z generated by n_1.

We turn now to the case where the generator a has finite order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, the p_i being distinct primes. Since R is the direct sum of its p_i-primary components and since each of these is generated by a single element, we may assume that $n = p^{\alpha}$ for some prime p. If $\alpha = 1$, in which case R may be regarded as an algebra over Z_p, then $R \cong XZ_p[X]$; otherwise the generator would be algebraic over Z_p and hence R would be finite.

Suppose, then, that $\alpha > 1$. Since pR is nil, it must be finite by Proposition 5; therefore $\tilde{R} = \{x \in R \mid px = 0\}$ is infinite. If a denotes the generator of R, then for appropriate positive integers n, m we have $pa^n = pa^m = pa^{m+k(n-m)}$ for all integers $k \geqq 1$; and it follows that $b = \sum_{j=1}^{p^{\alpha-1}} a^{m+j(n-m)}$ is an element of $\tilde{R}$. Moreover, $b \neq 0$ since a would otherwise generate a finite ring. Clearly b cannot be algebraic over Z_p—that too would imply R is finite; hence b generates a subring isomorphic to $XZ_p[X]$.

LEMMA 9. *Let R be an infinite J-ring. Then R must contain a subring of one of the following forms: $H(\lambda)$ for some strictly-increasing sequence λ of primes; $H(q)$ for some prime q; a field $F(p, \lambda)$ for some strictly-increasing sequence of primes; a field $F(p, q)$.*

Proof. Let R be any infinite J-ring. Since the additive group of a J-ring is a torsion group, $R = \Sigma \oplus R_p$, where R_p are the p-primary components of R^+. Clearly each nontrivial R_p contains a non-zero idempotent of additive order p, hence R contains a ring $H(\lambda)$ if there are infinitely may nontrivial R_p. Otherwise we may assume that the additive group of R is a p-group; moreover, since R has no nonzero nilpotent elements, the additive order of each nonzero element is square-free and we have $pR = 0$. If $x \in R$ satisfies $x^{n+1} = x$, then $x^n = e$ is an idempotent such that $ex = xe = x$; thus if R has a unique nonzero idempotent, it is an identity element and R is a field. On the other hand, if e is a nonzero idempotent which is not an identity element, we get a nontrivial decomposition $R = Re \oplus A(e)$ with one of the summands infinite. Thus, by continuing direct sum decompositions as long as possible, we either get a ring $H(p)$ or an infinite field. The proof is completed by showing that every infinite J-field contains a subfield of type $F(p, \lambda)$ or $F(p, q)$.

Accordingly, let F be an infinite J-field and note that every finitely-generated subring is a finite field. Thus, F contains a subfield $\tilde{F}$ which is the union of a strictly ascending tower $Z_p = F_0 \subset F_1 \subset F_2 \cdots$ of finite fields; and we may assume that the tower has been so refined that there are no subfields properly contained between any two of its members. It follows that for each $i = 1, 2, \cdots$, $[F_i : F_{i-1}]$ is a prime p_i. Using the basic facts about finite fields, it is easy to construct a field $F(p, \lambda)$ if there are infinitely many different p_i and a field $F(p, q)$ otherwise. This completes the proof of Lemma 9 and hence of Theorem 1.

4. Some consequences of Theorem 1.

Theorem 1 leads directly to the following two extensions of Szele's result [11] that an associative ring must be finite if it has both a.c.c. and d.c.c. on subrings.

THEOREM 2. *If R is an alternative ring satisfying both ascending chain condition and descending chain condition on commutative associative subrings, then R is finite.*

Proof. If there were a counterexample the class $\mathscr{J}_0$ of Theorem 1 would include rings having both a.c.c. and d.c.c. on subrings; but it does not.

THEOREM 3. *Let R be an alternative ring having only a finite number of zero subrings and a finite number of subrings which are integral domains. Then R is finite.*

Proof. The rings of type (a) in Theorem 1 all contain infinitely many zero subrings; those of types (b) and (c) all contain infinitely many integral domains.

REMARKS 1. Of course we could have obtained Theorem 2 by invoking our Proposition 6 and Szele's proof for the associative case. The proof of Theorem 1, however, is conceptually more elementary than Szele's proof.

2. In the hypotheses of Theorem 3, finiteness cannot be replaced

by a.c.c. and d.c.c., as can be seen by considering the ring $H(\lambda)$ for a strictly-increasing sequence λ.

The following theorem, which presents a new finiteness criterion for rings, does not seem to be a direct corollary of Theorem 1 but uses some of the machinery from its proof.

THEOREM 4. *Let R be an alternative ring having nonzero divisors of zero. If $A(x)$ is finite for each nonzero* (*two-sided*) *zero divisor x, then R is finite.*

Proof. Suppose that R is an infinite ring with nonzero divisors of zero. If R has nonzero nilpotent elements, then by Lemma 2, R has a nonzero element x for which $A(x)$ is infinite; thus, assume that R has no nonzero nilpotent elements, in which case $ab = 0$ if and only if $ba = 0$, so that there is no distinction between right and left annihilators. If for some nonzero pair a, b we have $ab = 0$ and a generating an infinite subring, then $A(b)$ is infinite; if there exists no such pair, for each nonzero zero divisor a, we have $a^m = a^n$ for distinct positive integers n, m and some power of a is a nonzero idempotent, necessarily central. In the latter case, the decomposition $R = Re \oplus A(e)$ is nontrivial with at least one of the summands infinite, so we again have a nonzero x with $A(x)$ infinite.

An immediate consequence of Theorem 4 is the following theorem which extends Theorem 3 of [1].

THEOREM 5. *If R is an alternative ring with nonzero divisors of zero and has* a.c.c. *and* d.c.c. *on subrings consisting of two-sided zero divisors of R, then R is finite.*

Our final application of Theorem 1 deals with the question of when an infinite ring contains infinitely many infinite subrings.

THEOREM 6. *If R is an infinite alternative ring containing no zero ring on a Prüfer p-group and no field $F(p, q)$, then R has infinitely many infinite* (commutative associative) *subrings. In particular, if R contains no infinite subring whose subrings are totally ordered by inclusion, R must have infinitely many infinite subrings.*

Proof. The first assertion is obtained by noting that all the members of the class $\mathscr{I}_0$ with the exception of zero rings on groups $C(p^\infty)$ and fields $F(p, q)$ contain infinite decreasing sequences of infinite rings. The second assertion is immediate from the fact that the zero rings on the groups $C(p^\infty)$ and the fields $F(p, q)$ are precisely the infinite rings whose subrings are totally ordered by inclusion (see [9]).

5. **Minimality considerations.** In this section, we deal with a notion of minimality for classes satisfying (*).

If $\mathscr{I}$ is any such class of rings, it must obviously contain all infinite rings having no proper infinite subrings—specifically, all fields $F(p, q)$ and the zero ring on each group $C(p^\infty)$; and it must contain all infinite rings which are isomorphic to each of their proper infinite subrings—i.e., the zero ring on C_∞, the zero ring on each group $G(p)$, and all rings $H(p)$. It must include at least one decreasing sequence of subrings of Z, at least one subring of $XZ[X]$, and at least one subring of $XZ_p[X]$ for each prime p. Finally, it must include infinitely many rings of the form $H(\lambda)$, infinitely many zero rings on groups $G(\lambda)$, and infinitely many fields $F(p, \lambda)$; this fact follows at once from the observation that infinite subrings of rings of these types are of the same type.

Such a class $\mathscr{I}$ need not contain more than one decreasing sequence of rings n_iZ provided that the one sequence has the property that each nonzero integer n divides some n_i; and since every subring of $XZ[X]$ or $XZ_p[X]$ contains a subring isomorphic to the entire ring, it will be sufficient for $\mathscr{I}$ to contain any one subring of $XZ]X]$ and any one subring of each ring $XZ_p[X]$.

It is not clear exactly which classes of rings $H(\lambda)$, $F(p, \lambda)$, and zero rings on $G(\lambda)$ must be included in $\mathscr{I}$, but we can say something. Let λ_0 denote the sequence of all primes of Z in their natural order, and let J denote any strictly-increasing sequence of positive integers. Denote by λ_J the subsequence of λ_0 obtained by choosing those terms indexed by J. Then $\mathscr{I}$ need contain no rings $H(\lambda_J)$ where J has bounded gaps; similar considerations apply to fields $F(p, \lambda)$ and zero rings on $G(\lambda)$.

A set $\mathscr{J}$ of strictly-increasing sequences of positive integers will be called *adequate* if each of its members has an unbounded set of gap lengths and if it contains at least one subsequence of every strictly-increasing sequence of positive integers. It being understood that $\mathscr{J}$ in each occurrence denotes an adequate set of sequences, we now define a class $\mathscr{I}$ satisfying (*) to be *irredundant* if it includes each of the following:

(i) all fields $F(p, q)$ and the zero ring on each group $C(p^\infty)$;

(ii) the zero ring on C_∞, the zero ring on each group $G(p)$, and the ring $H(p)$ for each prime p;

(iii) one infinite decreasing sequence $\langle n_iZ \rangle$ of subrings of Z, with the property that each positive integer n divides some n_i;

(iv) one nonzero subring of $XZ[X]$; and one nonzero subring of $XZ_p[X]$ for each prime p;

(v) the zero rings on any family of groups of the form $\{G(\lambda_J) \mid J \in \mathscr{J}\}$;

(vi) any family $\{H(\lambda_J) \mid J \in \mathscr{J}\}$;

(vii) for each prime p, one family $\{F(p, \lambda_J) \mid J \in \mathscr{J}\}$.

We can now state a characterization theorem. A formal proof is omitted, since most of the details are included in the preceding discussion.

THEOREM 7. *A class $\mathscr{J}$ of rings satisfies* (*) *if and only if it contains an irredundant subclass.*

It would, of course, be interesting to know more about adequate sets of sequences; but we are unable to give a tight description of them. It is clear, however, that they are quite large—in fact, it is easily shown that an adequate set contains uncountably many subsequences of each increasing sequence of positive integers.

6. Extensions to distributive near-rings. A near-ring R is a binary system satisfying all the (associative) ring axioms except right distributivity and commutativity of addition; R is called a distributive near-ring (dnr) if it does have right distributivity. An ideal of a *dnr* R is a normal subgroup of R^+ which is closed under left and right multiplication by elements of R; the theory of homomorphisms is the same as for rings.

A recurring consideration in the study of near-rings is the relationship between distributivity and additive commutativity. By extending our earlier results a bit, we can show that "most" infinite distributive near-rings contain infinite sub-near-rings which are additively commutative, hence rings. Clearly, not all dnr's have this property, for there exist infinite groups with no infinite abelian subgroups—we shall refer to them as *exceptional* [8, p. 35]—and the near-ring with trivial multiplication on such a group has no infinite subrings.

We shall make use of two well-known elementary results on dnr's—

(I) If R is a distributive near-ring, R^2 is a ring [2].

(II) If R is a distributive near-ring and R' is the derived group of the additive group R^+, then R' is an ideal of R and $RR' = R'R = 0$ [6].

THEOREM 8. *Let R be an infinite distributive near-ring for which the derived group of R^+ is not exceptional. Then R contains* *an infinite subring.*

Proof. In view of (I) and (II) we consider only R with both R^2 and R' finite. The finiteness of R^2 implies the existence of a positive integer n such that $nx^2 = 0$ for all $x \in R$; thus, if u is an element of R having infinite additive order, we have $(nu)^2 = 0$ and nu generates an infinite ring. Therefore, we may assume henceforth that R^+ is a periodic group. Another consequence of the finiteness of R^2 is that R is a periodic near-ring—i.e., for each $x \in R$, there are distinct positive integers n, m for which $x^n = x^m$.

Observe that R/R' is an infinite ring. If it has no infinite zero ring, then by Theorem 1 it contains an infinite subring S/R' with no nonzero nilpotent elements. Now by Lemma 3 periodic rings with no nonzero nilpotent elements are J-rings, hence are commutative by Jacobson's well-known theorem; and it follows that for all $x, y \in S$, $xy - yx \in R' \subseteqq A(R)$. In particular, if e is an idempotent of S and s an arbitrary element of S, then $(es - se)e = e(es - se) = 0$ and therefore idempotents of S are in the centre of S. Since $e \in eS \subseteqq R^2$, we easily obtain a finite set E of pairwise-orthogonal idempotents such that $S_0 = S \cap A(E)$ is infinite and contains no nonzero idempotents; and because S_0 is periodic, it must be nil. Since S_0 is infinite, we cannot have $S_0 \subseteqq R'$; thus, we have contradicted the fact that S/R' had no nonzero nilpotent elements.

To complete the proof, we need only discuss the case where R' and R^2 are both finite and R/R' contains an infinite zero ring S/R'. For $x, y \in S$, we must have $xy \in R' \subseteqq A(R)$, so in particular $x^3 = 0$ for all $x \in S$.

By applying an inductive argument similar to that used in the proof of Proposition 5, we can show that S must contain an infinite sequence $\langle x_i \rangle$ of pairwise-orthogonal elements squaring to zero. We omit the datails, but mention that Lemma 2 holds in the context of dnr's and that the ability to choose x_{i+1} not in the subring generated by $\{x_1, \cdots, x_n\}$ depends on local finiteness of R^+, which is guaranteed by the fact that R^+ is a periodic group with finite derived group.

Consider all additive commutators of the form $[x_1, x_i] = x_1 + x_i - x_1 - x_i$, $i > 1$. Since S' is finite, we may assume that $[x_1, x_i] = [x_1, x_j]$ for all i, $j > 1$.

Defining the sequence $\langle u_i \rangle$ by $u_1 = x_1$ and $u_j = -x_2 + x_{j+1}$ for $j > 1$, we obtain a sequence of pairwise-orthogonal elements squaring to zero, such that no u_i belongs to the additive subgroup generated by the previous terms and such that all its terms commute additively with u_1 Continuing with the inductive construction this suggests, we arrive at a sequence of pairwise orthogonal elements squaring to zero and generating an infinite abelian subgroup of S^+;

therefore S contains an infinite zero ring. The proof of Theorem 8 is now complete.

COROLLARY 1. *If R is an infinite distributive near-ring, then R contains an infinite ring or an infinite near-ring with trivial multiplication.*

COROLLARY 2. *If R is an infinite distributive near-ring having ascending chain condition and descending chain condition on sub-near-rings and if R' is not exceptional, then R is finite.*

Proof. Use Theorem 2 and Theorem 8.

REMARK. Whether the hypothesis that R' is not exceptional is required in Corollary 2 is equivalent to the unsolved problem as to whether a group with acc and dcc on subgroups need be finite.

COROLLARY 3. *Let R be an infinite distributive near-ring with solvable additive group. Then R contains an infinite subring.*

Proof. Let $R^{(i)}$ denote the ith term of the derived series of R^+, $i = 1, 2, \cdots$. Since R^+ is solvable, there is a smallest positive integer m for which $R^{(m)}$ is finite. If $m = 1$, R' is not exceptional and we are finished; if $m > 1$, then $R^{(m-1)}$ is an infinite distributive near-ring whose derived group is not exceptional, and $R^{(m-1)}$ contains an infinite ring.

COROLLARY 4. *If R is an infinite distributive near-ring the additive group of which is locally finite, then R must contain an infinite subring.*

Proof. If R' is not finite, it must contain an infinite abelian subgroup by the Hall-Kulatilaka-Kargopolov theorem [8, p. 95]; therefore R' is not exceptional, and the result follows from Theorem 8.

REFERENCES

1. H. E. Bell, *Rings with finitely many subrings*, Math. Ann., **182** (1969), 314-318.
2. A. Fröhlich, *Distributively generated near-rings, I, Ideal theory*, Proc. London Math. Soc., (3) **8** (1958), 76-94.
3. L. Fuchs, *Abelian Groups*, Pergamon, Oxford, 1960.
4. M. Hall, *The Theory of Groups*, Macmillan, New York, 1959.
5. T. J. Laffey, *On commutative subrings of infinite rings*, Bull. London Math. Soc., **4** (1972), 3-5.

6. S. Ligh, *On commutativity of near-rings*, Kyungpook Math. J., **10** (1970), 105-106.
7. K. McCrimmon, *Finite power-associative division rings*, Proc. Amer. Math Soc., **17** (1966), 1173-77.
8. D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Part 1, Springer-Verlag, 1972.
9. A. Rosenfeld, *A note on two special types of rings*, Scripta Mathematica, **28** (1967), 51-54.
10. M. Smiley, *Alternative regular rings without nilpotent elements*, Bull. Amer. Math. Soc., **53** (1947) 775-778.
11. T. Szele, *On a finiteness criterion for modules*, Publ. Math. Debrecen **3** (1954), 253-256.

Received May 21, 1975.

BROCK UNIVERSITY