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A CHARACTERIZATION OF PRÜFER DOMAINS IN TERMS OF POLYNOMIALS

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Assume that D is an integral domain with identity and with quotient field K. Each element of K is the root of a polynomial f in $D[X]$ such that the coefficients of f generate D if and only if the integral closure of D is a Prüfer domain.

All rings considered in this paper are assumed to be commutative and to contain an identity element. By an *overring* of a ring R, we mean a subring of the total quotient ring of R containing R. The symbol X in the notation $R[X]$ denotes an indeterminate over R.

In the study of integral domains, Prüfer domains arise in many different contexts. See, for example, [**1**; Exer. 12, p. 93] or [**2**; Chap. IV] for some of the multitudinous characterizations of Prüfer domains. Among such characterizations there are at least two in terms of polynomials: (1) The domain D is a Prüfer domain if and only if $A_fA_g = A_{fg}$ for all f, $g \in D[X]$, where A_h denotes the ideal of D generated by the coefficients of the polynomial $h \in D[X]$ (A_h is called the *content* of h) [**3**], [**10**], [**2**; p. 347]. (2) D is a Prüfer domain if and only if D is integrally closed and for each prime ideal P of D, the only prime ideals of $D[X]$ contained in $P[X]$ are those of the form $P_1[X]$, where P_1 is a prime ideal of D contained in P [**2**; p. 241]. In Theorem 2 we provide another characterization of Prüfer domains in terms of polynomials: D is a Prüfer domain if and only if D is integrally closed and each element of the quotient field K of D is a root of a polynomial $f \in D[X]$ such that $A_f = D$. Then in Theorem 5 we obtain an extension of this result to the case where D need not be integrally closed.

Our interest in domains D such that each element of K is a root of a polynomial $f \in D[X]$ with $A_f = D$ stemmed from the fact that this property is common to both Δ-domains—that is, integral domains whose set of overrings is closed under addition [**4**]—and to integral domains having property (n) for some $n > 1$—that is, integral domains D with the property that $(x, y)^n = (x^n, y^n)$ for all x, $y \in D$ [**9**]. Thus, if D is a Δ-domain with quotient field K and if $t \in K$, then since $D[t^2] + D[t^3]$ is an overring of D, $t^5 = t^2t^3 \in D[t^2] + D[t^3]$, whence it is evident that t is the root of a polynomial in $D[X]$ in which the coefficient of X^5 is a unit. If D has property (n) for some $n > 1$ and if $t = a/b \in K$, where a, $b \in D$ and $b \neq 0$, then from the equality $(a, b)^n = (a^n, b^n)$ it follows that $a^{n-1}b = d_1a^n + d_2b^n$ for some d_1, $d_2 \in D$; dividing both sides of this equation by b^n yields $d_1X^n - X^{n-1} + d_2$ as a polynomial satisfied by t.

We show that the condition described in the preceding paragraph is equivalent to the condition that each element of the quotient field of D satisfies a polynomial with a unit coefficient.

THEOREM *Let $f = \Sigma_{i=0}^n f_i X^i$ be an element of $R[X]$. Then $A_f = (f_0, f_1, \cdots, f_n)$ is the set of coefficients of elements of the principal ideal of $R[X]$ generated by f.*

Proof. Denote by E the set of coefficients of elements of (f); E is an ideal of R and the inclusion $A_f \supseteq E$ is clear. Conversely, if $t = \Sigma_0^n r_i f_i$ is an element of A_f, then $(\Sigma_{i=0}^n r_i X^{n-i})f$ is an element of (f) and the coefficient of X^n in this polynomial is t. Hence $t \in E$ and the equality $E = A_f$ holds, as asserted.

A modification of the proof of Theorem 1 shows that the result generalizes to polynomials in an arbitrary set of indeterminates, and this observation, in turn, yields a further generalization of Theorem 1.

COROLLARY 1. *Let $\{f_\alpha\}$ be a subset of the polynomial ring $R[\{X_\lambda\}]$, and for each α,* let *A_{f_α} be the ideal of R generated by the coefficients of f_α. Then $\Sigma_\alpha A_{f_\alpha}$ is the set of coefficients of the ideal of $R[\{X_\lambda\}]$ generated by $\{f_\alpha\}$.*

The equivalence of the two conditions mentioned in the paragraph immediately preceding Theorem 1 also follows at once from this result. If S is a unitary extension ring of R, we say that R *has property* (P) *with respect to* S or that S *is a P-extension of* R if each element of S satisfies a polynomial in $R[X]$ one of whose coefficients is a unit of R, or, equivalently, whose coefficients generate the unit ideal of R. The next result is not unexpected.

THEOREM 2. *Let D be an integrally closed domain with quotient field K. Then D is a Prüfer domain if and only if K is a P-extension of D.*

Proof. If D is a Prüfer domain, then D has property (n) for each positive integer n [**5**; Theorem 2.5 (e)], [**2**; Theorem 24.3], and hence, as already shown, D has property (P) with respect to K. Conversely, suppose that K is a P-extension of D. Let M be a maximal ideal of D and let t be an element of K. Then t is a root of a polynomial f in $D[X]$ such that $A_f = D$, and hence $f \not\in M[X]$. It then follows from [**11**; p. 19] that t or t^{-1} is in D_M. Consequently, D_M is a valuation ring and D is a Prüfer domain, as asserted.

To obtain a characterization of domains D for which K is a P-extension of D, we introduce some useful notation. Let R be a ring, let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal ideals of R, and let N be the set of elements f in $R[X]$ such that $A_f = R$; W. Krull **[7]** observed that N is a regular multiplicative system in $R[X]$ and he considered properties of the ring $R[X]_N$, which M. Nagata in **[8**; p. 17**]** denotes by $R(X)$. It is clear that $N = R[X] - \cup_\lambda M_\lambda[X]$, and in Chapter 33 of **[2]** it is shown that if an ideal E of $R[X]$ is contained in $\cup_\lambda M_\lambda[X]$, then E is contained in one of the ideals $M_\lambda[X]$. Consequently, $\{M_\lambda[X]\}$ is the set of prime ideals of $R[X]$ maximal with respect to not meeting N and $\{M_\lambda R(X)\}$ is the set of maximal ideals of $R(X)$. With these facts recorded, we state and prove our next theorem.

THEOREM 3. *Let T be a unitary extension ring of the ring R and let S be the integral closure of R in T.*

(a) *The ring $S(X)$ is integral over $R(X)$.*

(b) *If $T[X]$ is integrally closed, then $S(X)$ is the integral closure of $R(X)$ in $T(X)$.*

Proof. (a): Let $\{M_\alpha\}_{\alpha \in A}$ and $\{M'_\beta\}_{\beta \in B}$ be the sets of maximal ideals of R and S, respectively. If $N = R[X] - \bigcup_\alpha M_\alpha[X]$ and $N' = S[X] - \bigcup_\beta M'_\beta[X]$, then $R(X) = R[X]_N$ and $S(X) = S[X]_{N'}$. The ring $S[X]_N$ is integral over $R[X]_N$ and we prove (a) by showing that N' is the saturation of the multiplicative system N in $S[X]$. Let N^* be the saturation of N in $S[X]$; since $N \subseteq N'$ and since N' is saturated, it follows that $N^* \subseteq N'$. The multiplicative system N^* is characterized as the complement in $S[X]$ of the set $\mathcal{P}$ of prime ideals of $S[X]$ maximal with respect to not meeting N; hence, to prove that N' is contained in N^*, we prove that $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$. Thus, let $P' \in \mathcal{P}$ and let $P' \cap R[X] = P$. Since $P' \cap N = \emptyset$, P also fails to meet N—that is, $P \subseteq \bigcup_{\alpha \in A} M_\alpha[X]$; as we remarked earlier, this inclusion implies that $P \subseteq M_\alpha[X]$ for some $\alpha \in A$. Since $S[X]$ is integral over $R[X]$, there is a prime ideal Q' of $S[X]$ such that Q' contains P' and $Q' \cap R[X] = M_\alpha[X]$. Hence $(Q' \cap S) \cap R = (Q' \cap R[X]) \cap R = M_\alpha[X] \cap R = M_\alpha$, a maximal ideal of R; from the integrality of S over R we infer that $Q' \cap S$ is a maximal ideal of S, that is, $Q' \cap S = M'_\beta$ for some $\beta \in B$. It follows that $M'_\beta[X] \subseteq Q'$ and in fact, $Q' = M'_\beta[X]$ since $S[X]$ is integral over $R[X]$ and since $Q' \cap R[X] = M'_\beta[X] \cap R[X] = M_\alpha[X]$. We therefore obtain the inclusion $P' \subseteq M'_\beta[X]$. Since $M'_\beta[X]$ misses N and since P' is maximal with respect to missing N, it follows that $P' = M'_\beta[X]$ and $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$. This completes the proof of (a).

To prove (b) we recall that $S[X]$ is the integral closure of $R[X]$ in $T[X]$ **[2**, Theorem 10.7**]**, and hence $S[X]_N = S(X)$ is the integral closure of $R[X]_N = R(X)$ in $T[X]_N$. If $T[X]$ is integrally closed, then $T[X]_N$

is also integrally closed, and since $T(X)$ is an overring of $T[X]_N$, it follows that the integral closure of $R(X)$ in $T(X)$ coincides with the integral closure of $R(X)$ in $T[X]_N$. Thus $S(X)$ is the integral closure of $R(X)$ in $T(X)$, as asserted.

REMARK 1. The following result follows from the proof of part (a) of Theorem 3: Assume that S is a unitary ring extension of the ring R and that S is integral over R. Let N be a multiplicative system in R, let $\{P_\alpha\}$ be the set of prime ideals of R maximal with respect to not meeting N, and let $\{P'_\beta\}$ be the set of prime ideals of S such that $P'_\beta \cap R \in \{P_\alpha\}$. Then $S - (\cup P'_\beta)$ is the saturation of N in S (cf. [2; Proposition 11.10]). More generally, this conclusion is valid if the extension $R \subseteq S$ satisfies *going up* in the terminology of [6; p. 28].

REMARK 2. We do not know if the conclusion of (b) is valid without the hypothesis that $T[X]$ is integrally closed. As the proof of part (b) of Theorem 3 shows, sufficient conditions for $S(X)$ to be the integral closure of $R(X)$ in $T(X)$ are that $T[X]_N$ is integrally closed in $T(X)$, a quotient ring of $T[X]_N$. It is easy to give examples to show that the inclusion $T[X]_N \subseteq T(X)$ may be proper; if R is a v-domain with quotient field T, then a necessary condition that $T(X)$ should be $T[X]_N$ is that R be a Prüfer v-multiplication ring (see §33 of [2] for terminology). The condition that $T[X]$ is integrally closed is not, insofar as we know, definitive in terms of T; it implies that T is integrally closed, but the converse fails in general.

THEOREM 4. *Assume that T is a unitary extension ring of the ring R and that S is an intermediate ring integral over R. If T is a P-extension of S, then T is a P-extension of R.*

Proof. Let $t \in T$, let $Q' = \{f \in S[X] \mid f(t) = 0\}$, and let $Q = Q' \cap R[X]$. If N and N' are defined as in the proof of Theorem 3, so that $R(X) = R[X]_N$ and $S(X) = S[X]_{N'}$, then the hypothesis that T is a P-extension of S implies that $Q' \cap N' \neq \varnothing$. If we show that $Q \cap N \neq \varnothing$, then the proof of Theorem 4 will be complete. We first observe that $QR(X) = Q'(S[X])_N \cap R(X)$. That the right side contains the left side is clear, and if $f/n = d/m \in Q'(S[X])_N \cap R(X)$, where $f \in Q'$, $d \in R[X]$, and n, $m \in N$, then $fm = dn \in Q' \cap R[X] = Q$, so that $f/n = fm/nm \in QR(X)$ and $Q'(S[X])_N \cap R(X) = QR(X)$. It follows from the proof of Theorem 3 that $(S[X])_N = (S[X])_{N'}$; hence

$$QR(X) = Q'S(X) \cap R(X) = S(X) \cap R(X) = R(X),$$

which means that $Q \cap N \neq \varnothing$.

The characterization of Prüfer domains stated at the beginning of this paper is a direct consequence of the preceding results.

THEOREM 5. *Let D be an integral domain with quotient field K, and let J be the integral closure of D. Then J is a Prüfer domain if and only if K is a P-extension of D.*

Proof. Suppose that K is a P-extension of D. Then K is, *a fortiori*, a P-extension of J. We invoke Theorem 2 to conclude that J is a Prüfer domain.

If, conversely, J is a Prüfer domain, then by Theorem 2, K is a P-extension of J and hence, by Theorem 4, a P-extension of D.

There is an extension of Theorem 5 to the case where K is not the quotient field of D.

THEOREM 6. *Let D be a domain with integral closure J, and let L be an algebraic extension field of the quotient field K of D. Then J is a Prüfer domain if and only if L is a P-extension of D.*

Proof. If L is a P-extension of D, then so is K, and hence J is a Prüfer domain by Theorem 5. Conversely, if J is a Prüfer domain, and if $t \in L$, then t is a root of a nonzero polynomial $f \in J[X]$. The ideal A_f of J is finitely generated, and hence is invertible. If $A_f^{-1} = (g_0, g_1, \cdots, g_n)$, and if $g = \sum_{i=0}^{n} g_i X^i$ then $A_{fg} = A_f A_g = J$ so that $fg \in J[X]$ and $(fg)(t) = f(t)g(t) = 0$. It follows that L is a P-extension of J, and hence by Theorem 4, L is a P-extension of D.

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