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STRONGLY UNICOHERENT CONTINUA

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In this journal the author introduced the concept of strongly unicoherent continua and proved that such a continuum is aposyndetic at a point p if and only if it is connected im kleinem at p. In the present paper we obtain some new results on strongly unicoherent continua. The main theorem states that the strongly unicoherent continuum M contains a unique irreducible subcontinuum between two points p and q provided M is both aposyndetic and semi-locally-connected at p.

Throughout this paper a *continuum* is a compact connected metric space and M will denote a continuum. The continuum M is *unicoherent* if whenever $M = A \cup B$, with A and B subcontinua of M, $A \cap B$ is connected. M is *hereditarily unicoherent* if every subcontinuum of M is unicoherent. A continuum M is said to be *irreducible between a pair of points* p and q of M provided no proper subcontinuum of M contains both p and q. M is said to be *irreducible* if there exists two points so that M is irreducible between the pair of points.

If N is a subset of M, the interior of N in M will be denoted by int N and the closure of N in M by $\bar{N}$.

For other terms not defined herein see [3] and [5].

DEFINITION 1. A unicoherent continuum M is *strongly unicoherent* provided that for any pair of proper subcontinua H and K such that $M = H \cup K$, each of H and K is unicoherent.

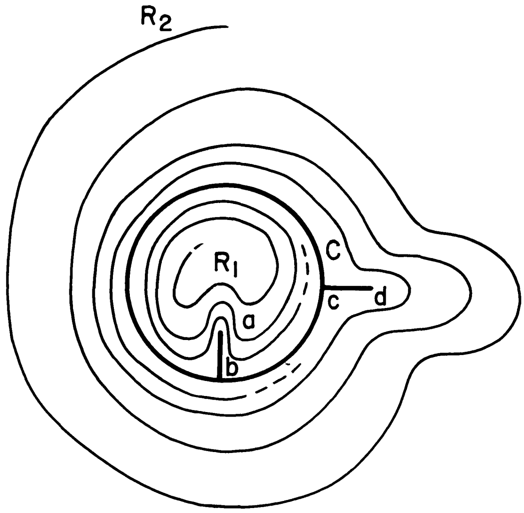
It is easily seen that this notion is a stronger form of unicoherence, but is somewhat weaker than hereditarily unicoherence. For example, a continuum which consists of a ray limiting on a circle is strongly unicoherent but fails to be hereditarily unicoherent.

In [4] Miller proved the following theorem.

THEOREM 1. *If the atriodic unicoherent continuum M is the sum of two proper subcontinua H and K, then H and K are unicoherent, and if N is a nonunicoherent subcontinuum of M intersecting H, it is a subset of H.*

Thus, atriodic unicoherent continua are examples of continua which are strongly unicoherent. A reasonable conjecture is that the last portion of the theorem holds for strongly unicoherent continua, but this is not the case as shown by the following example.

EXAMPLE 1.



The strongly unicoherent continuum M consist of a circle C, two arcs $[a,b]$ and $[c,d]$ which intersect C at the point b and point c respectively, a ray R_1 which limits on $C \cup [a,b]$, and a ray R_2 which limits on $C \cup [c,d]$.

Let $H = \bar{R}_1$ and $K = \bar{R}_2 \cup [a,b]$. Then H and K are proper subcontinua of M and $M = H \cup K$. Let

$$N = C \cup [a,b] \cup [c,d].$$

N is non-unicoherent, $N \cap H \neq \varnothing$, but $N \not\subset H$.

Note that while $M = H \cup K$ and K is unicoherent it does not follow that K is strongly unicoherent as shown by the example above.

COROLLARY 1. *If M is atriodic, then every unicoherent subcontinuum of M is strongly unicoherent.*

Next we investigate the relationship of a strongly unicoherent continuum M and its nonunicoherent subcontinua.

DEFINITION 2. A subcontinuum N of M is said to be a *continuum of condensation* if each point of N is a limit point of $M - N$.

The next theorem is an immediate consequence of definitions 1 and 2.

THEOREM 2. *If N is a nonunicoherent subcontinuum of a strongly unicoherent continuum M then either* (i) *N separates M or,* (ii) *N is a continuum of condensation.*

COROLLARY 2. *If N is a nonunicoherent subcontinuum of a strongly unicoherent continuum M then each subcontinuum of N either* (i) *separates M or* (ii) *is a continuum of condensation.*

Question. Is every nonunicoherent subcontinuum of a strongly unicoherent continuum M a continuum of condensation?

The answer to the preceding question is in the affirmative if the continuum M is also irreducible.

THEOREM 3. *Suppose that M is a strongly unicoherent irreducible continuum. Then every nonunicoherent subcontinuum of M is a continuum of condensation in M.*

Proof. Let a and b be in M such that M is irreducible between a and b and suppose N is a non-unicoherent subcontinuum of M. According to Theorem 2, we may assume that N separates M. It follows that $\{a, b\} \cap N = \emptyset$ and $M - N$ has exactly two components, say A and B. Without loss of generality, assume that $a \in A$ and $b \in B$. If $\bar{A} \cap \bar{B} \neq \emptyset$ then $\bar{A} \cup \bar{B}$ is a subcontinuum of M containing $\{a, b\}$. Thus, $M = \bar{A} \cup \bar{B}$ and it follows that N is a continuum of condensation.

Suppose that $\bar{A} \cap \bar{B} = \emptyset$. Let H and K be subcontinua of N such that $N = H \cup K$ and $H \cap K$ is not connected.

Assertion. $\bar{A} \cap H \neq \emptyset \neq H \cap \bar{B}$. For suppose that this is not the case. Assume that $\bar{A} \cap H = \emptyset$. Then $\bar{A} \cap K \neq \emptyset$ and $\bar{A} \cup K \cup H$ is a proper subcontinuum of M. Since $M = (\bar{A} \cup K \cup H) \cup \bar{B}$, it follows that $\bar{A} \cup H \cup K$ is unicoherent. This implies that $(\bar{A} \cup K) \cap H = H \cap K$ is connected which is a contradiction. Thus the assertion holds.

In a similar manner, it follows that $\bar{A} \cap K \neq \emptyset \neq K \cap \bar{B}$. Since $M = \bar{A} \cup H \cup \bar{B}$, then $K - H \subset \bar{A} \cup \bar{B}$. Also since $M = \bar{A} \cup K \cup \bar{B}$, it follows that $H - K \subset \bar{A} \cup \bar{B}$.

Finally we will show that $H \cap K \subset \bar{A} \cup \bar{B}$. Let P and Q be disjoint closed sets such that $H \cap K = P \cup Q$, let $p \in P$ and C be the component of p in $\overline{H - Q}$. Then $\bar{C} \cap Q \neq \emptyset$, but note that $\bar{C} \cap (\operatorname{int} Q) = \emptyset$. Since $\bar{C} \cup K \cup \bar{A}$ is a proper subcontinuum of M and $M = \bar{B} \cup (\bar{C} \cup K \cup \bar{A})$, then $\bar{C} \cup K \cup \bar{A}$ is unicoherent. Thus $(K \cup \bar{A}) \cap \bar{C} = (K \cap \bar{C}) \cup (\bar{C} \cap \bar{A})$ is connected. Now $(K \cap \bar{C}) \cap P \neq \emptyset \neq (K \cap \bar{C}) \cap Q$ and $K \cap \bar{C} \subset P \cup Q$ which implies that $K \cap \bar{C}$ is not connected. It follows that $\bar{C} \cap \bar{A} \neq \emptyset$.

By interchanging the roles of $\bar{A}$ and $\bar{B}$ in the above argument, we have that $\bar{C} \cap \bar{B} \neq \emptyset$. Now $\bar{A} \cup \bar{C} \cup \bar{B}$ is a subcontinuum of M containing $\{a, b\}$, so $M = \bar{A} \cup \bar{C} \cup \bar{B}$. Since $(\operatorname{int} Q) \cap \bar{C} = \emptyset$, then $\operatorname{int} Q) \subset \bar{A} \cup \bar{B}$.

Suppose $q \in Q - (\text{int } Q)$ and V is an open set containing q such that $V \cap P = \varnothing$. Then $V \not\subset Q$ so there is a point $z \in V \cap (M - Q)$. Since $z \notin P \cup Q$ it follows by the first portion of this proof that $z \in \bar{A} \cup \bar{B}$. Thus q is a limit point of $\bar{A} \cup \bar{B}$ and hence $q \in \bar{A} \cup \bar{B}$. Therefore $Q \subset \bar{A} \cup \bar{B}$.

Since the preceding argument is symmetric with respect to P and Q, it follows that $P \subset \bar{A} \cup \bar{B}$.

Therefore $N = (H - K) \cup (K - H) \cup P \cup Q \subset \bar{A} \cup \bar{B}$ which implies that N is a continuum of condensation.

The following well known characterization of hereditarily unicoherent continua was given in **[4]**.

THEOREM 4 (Miller). *In order that the continuum M be hereditarily unicoherent it is necessary and sufficient that for any two points p and q of M there is only one subcontinuum of M which is irreducible between p and q.*

DEFINITION 3. A continuum M is *hereditarily unicoherent at the point p* of M provided for each $q \in M$ different from p, there is a unique subcontinuum of M which is irreducible between p and q.

Thus if M is hereditarily unicoherent at p and $q \in M - \{p\}$, then the intersection of all subcontinuum of M which contain $\{p, q\}$ is connected.

We shall show that strongly unicoherent continua are "hereditarily unicoherent at certain points", but first we prove the following lemma.

LEMMA 1. *Let p and q be points of the continuum M, I_1 and I_2 be subcontinua of M which are irreducible between p and q, and D be a subcontinuum of M containing p. If the continuum $D \cup I_1 \cup I_2$ is unicoherent, then $I_1 \cap M - D = I_2 \cap M - D$.*

Proof. Suppose that $D \cup I_1 \cup I_2$ is unicoherent. Then $(D \cup I_2) \cap I_1$ is connected and hence is a subcontinuum of I_1 containing $\{p, q\}$. Since I_1 is irreducible between p and q, then $(D \cup I_2) \cap I_1 = I_1$. Therefore $I_1 \subset D \cup I_2$ which implies that (i) $I_1 \cap (M - D) \subset I_2$.

Likewise $(D \cup I_1) \cap I_2$ is subcontinuum of I_2 containing $\{p, q\}$ so $(D \cup I_1) \cap I_2 = I_2$. Thus $I_2 \subset D \cup I_1$ and it follows that (ii) $I_2 \cap (M - D) \subset I_1$.

The conclusion follows from (i) and (ii).

THEOREM 5. *Let M be a strongly unicoherent continuum and $p \in M$. If M is both aposyndetic at p and semi-locally-connected at p, then M is hereditarily unicoherent at p.*

Proof. Suppose that M is aposyndetic and semi-locally-connected at p, $q \in M - \{p\}$, and I_1 and I_2 are subcontinua of M which are irreducible between p and q.

Since M is aposyndetic and semi-locally-connected at p, there are subcontinua H_1 and K_1 of M such that $p \in H_1 - K_1$, $q \in K_1 - H_1$, and $M = H_1 \cup K_1$ (Theorem 6 of **[2]**).

Since M is aposyndetic at p, according to Theorem 6 of **[1]**, M is also connected im kleinen at p. So there is a subcontinuum L in $M - (H_1 \cap K_1)$ such that $p \in \operatorname{int} L$. Since M is semi-locally-connected at p, there is an open set V such that $p \in V \subset (\operatorname{int} L)$ and $M - V$ has a finite number of components. Let $F_1, \cdots, F_n$ be the components of $M - V$ and without loss of generality assume that $K_1 \subset F_1$. Let $H_2 = L \cup (F_2 \cup F_3 \cup \cdots \cup F_n)$ and $K_2 = F_1$. Then H_2 and K_2 are subcontinua such that $p \in H_2 - K_2$, $q \in K_2 - H_2$, and $M = H_2 \cup K_2$.

Since M is strongly unicoherent, then $K_1 \cup I_1 \cup I_2$, $H_2 \cup I_1 \cup I_2$, and $K_2 \cup I_1 \cup I_2$ are unicoherent. So by the preceding lemma, $I_1 \cap (M - K_1) = I_2 \cap (M - K_1)$, $I_1 \cap (M - H_2) = I_2 \cap (M - H_2)$, and $I_1 \cap (M - K_2) = I_2 \cap (M - K_2)$.

Now $H_2 \cap K_2 \subset L \subset M - K_1$ so $I_1 \cap (H_2 \cap K_2) = I_2 \cap (H_2 \cap K_2)$. So it follows that

$$I_1 = [I_1 \cap (M - H_2)] \cup [I_1 \cap (H_2 \cap K_2)] \cup [I_1 \cap (M - K_2)]$$

$$= [I_2 \cap (M - H_2)] \cup [I_2 \cap (H_2 \cap K_2)] \cup [I_2 \cap (M - K_2)] = I_2.$$

Therefore M is hereditarily unicoherent at p.

COROLLARY 3. *If the strongly unicoherent M is aposyndetic at each point, then M is hereditarily unicoherent.*

Proof. By Theorem 4 of **[2]** M is semi-locally-connected at each point, so it follows from the preceding theorem that M is hereditarily unicoherent.

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