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ON THE LATTICE OF NORMAL SUBGROUPS OF A DIRECT PRODUCT

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Suzuki has determined that if G is a direct product $G = \Pi_{i=1}^{k} G_i$ of groups $G_i \neq 1$, then the lattice $L(G)$ of subgroups of G is the direct product of the lattices $L(G_i)$ if and only if the order of any element in G_i is finite and relatively prime to the order of any element in $G_j (i \neq j)$. An exercise in Zassenhaus' *The Theory of Groups* asks the reader to prove an analogous result for the lattice of normal subgroups. In §1, we derive this result for the case of the direct product of two groups. (The generalization to the direct product of any finite number of groups is straightforward.) In §2, we use results obtained in §1 to study in detail the normal subgroup lattice of the direct product of finitely many symmetric groups.

1. The lattice of normal subgroups. If G_1 and G_2 are groups, we denote elements of the direct product $G_1 \times G_2$ by ordered pairs (a, b), $a \in G_1$, $b \in G_2$. If A and B are subgroups of a group G, we define $[A, B] = \langle aba^{-1}b^{-1} \mid a \in A, b \in B \rangle$, and note that if $A \lhd G$, then $[A, B] \lhd A$. We let ρ_1 and ρ_2 denote the first and second projection maps on $G_1 \times G_2$, and finally, we denote by $o(g)$ the order of the element g.

If N is a subgroup of $G_1 \times G_2$, we put $N_1 = \rho_1(N)$ and $N_2 = \rho_2(N)$. Thus N_i is a subgroup of G_i, called the *i*th projection of N. Furthermore, if $N \lhd G_1 \times G_2$, then $N_i \lhd G_i$.

LEMMA 1. *If $N \lhd G_1 \times G_2$, then $N \supseteq [G_1, N_1] \times [G_2, N_2]$.*

Proof. Let $a \in N_1$. Then there exists $y \in N_2$ such that $(a, y) \in N$. Thus $(a^{-1}, y^{-1}) \in N$, and since $N \lhd G_1 \times G_2$, $(g, 1)(a, y)(g^{-1}, 1) = (gag^{-1}, y) \in N$. It follows that $(gag^{-1}, y)(a^{-1}, y^{-1}) = (gag^{-1}a^{-1}, 1) \in N$, so $N \supseteq [G_1, N_1] \times \{1\}$. Similarly, $N \supseteq \{1\} \times [G_2, N_2]$, completing the proof.

The following lemma, whose proof is immediate, will be used in the discussion that follows.

LEMMA 2. *Let G be a group, $H \lhd G$. Then any subgroup L of G such that $[G, H] \subseteq L \subseteq H$ is normal in G.*

Since $[G_1 \times G_2, A \times B] = [G_1, A] \times [G_2, B]$ whenever $A \subseteq G_1$, $B \subseteq G_2$, if $N \lhd G_1 \times G_2$ with projections N_1 and N_2, then any subgroup of $G_1 \times G_2$ lying between $[G_1, N_1] \times [G_2, N_2]$ and $N_1 \times N_2$ is normal in

$G_1 \times G_2$. Moreover, as $[N_i, N_i] \subseteq [G_i, N_i]$, we see that $C_i = N_i/[G_i, N_i]$ is abelian, as is $C_1 \times C_2 = (N_1 \times N_2)/[G_1, N_1] \times [G_2, N_2]$. There is thus a 1–1 correspondence ϕ between subgroups of $C_1 \times C_2$ and subgroups of $G_1 \times G_2$ lying between $[G_1, N_1] \times [G_2, N_2]$ and $N_1 \times N_2$.

DEFINITION. A normal subgroup S of $G_1 \times G_2$ is called $G_1 - G_2$ *decomposable* if $S = S_1 \times S_2$, $S_1 \lhd G_1$, $S_2 \lhd G_2$.

It is easy to see that a subgroup H of $C_1 \times C_2$ is $C_1 - C_2$ decomposable if and only if $\phi(H)$ is $G_1 - G_2$ decomposable. Furthermore, if $B \subseteq C_1 \times C_2$, then the ith projection of B is C_i if and only if the ith projection of $\phi(B)$ is N_i.

Assuming we can determine the subgroup lattice structure of arbitrary abelian groups, we now have a systematic way of describing the normal subgroups of $G_1 \times G_2$ in terms of those of G_1 and G_2. Namely, choose $S_1 \lhd G_1$, $S_2 \lhd G_2$ and consider the subgroups M of the abelian group $S_1/[G_1, S_1] \times S_2/[G_2, S_2]$ with the property that $M_i = S_i/[G_i, S_i]$. To each such M, there corresponds a normal subgroup N of $G_1 \times G_2$ with $[G_1, S_1] \times [G_2, S_2] \subseteq N \subseteq S_1 \times S_2$ and $N_i = S_i$. As S_1 and S_2 run through the normal subgroups of G_1 and G_2 respectively, we obtain each normal subgroup of $G_1 \times G_2$ exactly once.

It is, of course, not always easy to determine all subgroups of a given abelian group. For finite groups, however, Suzuki's result shows that it suffices to consider the case of abelian p-groups.

EXAMPLE. Let $G_1 = S_3$ (symmetric group on 3 letters)
$G_2 = \mathbf{Z}_2$ (cyclic group of order 2)
We calculate the following:

S_1	S_2	$[G_1, S_1]$	$[G_2, S_2]$	C_1	C_2	ν
1	1	1	1	1	1	1
1	$\mathbf{Z}_2$	1	1	1	$\mathbf{Z}_2$	1
$\mathbf{Z}_3$	1	$\mathbf{Z}_3$	1	1	1	1
$\mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	1	1	$\mathbf{Z}_2$	1
S_3	1	$\mathbf{Z}_3$	1	$\mathbf{Z}_2$	1	1
S_3	$\mathbf{Z}_2$	$\mathbf{Z}_3$	1	$\mathbf{Z}_2$	$\mathbf{Z}_2$	2

Here $C_i = S_i/[G_i, S_i]$, and ν denotes the number of subgroups $M \subseteq C_1 \times C_2$ with $M_i = C_i$.

From this, we see that $S_3 \times \mathbf{Z}_2$ has seven normal subgroups, all of which are $S_3 - \mathbf{Z}_2$ decomposable, except for one of order six.

We now determine the condition for every normal subgroup of $G_1 \times G_2$ to be $G_1 - G_2$ decomposable. Recall that a group G is called perfect if $G = G'$. We will say that G is *super-perfect* if $[G, H] = H$ for all $H \lhd G$.

THEOREM 1. *Let G_1 and G_2 be groups. Then every normal subgroup of $G_1 \times G_2$ is $G_1 - G_2$ decomposable if and only if either* (i) *at least one of G_1 and G_2 is super-perfect, or* (ii) *for all $S_1 \lhd G_1$, $S_2 \lhd G_2$, the elements of $S_1/[G_1, S_1]$ have orders relatively prime to those of $S_2/[G_2, S_2]$.* (In particular, these orders must be finite.)

Proof. ($\Leftarrow$) Suppose $N \lhd G_1 \times G_2$ is not $G_1 - G_2$ decomposable. Then the subgroup $\phi(N)$ of $N_1/[G_1, N_1] \times N_2/[G_2, N_2] = C_1 \times C_2$ is not $C_1 - C_2$ decomposable. If G_i is super-perfect, then $C_i = \mathbf{1}$, a contradiction. Otherwise (ii) holds, and we have a contradiction to Suzuki's result **[2]**. ($\Rightarrow$) Let $S_i \lhd G_i$. By hypothesis, every normal subgroup $N \subseteq G_1 \times G_2$ with $[G_1, S_1] \times [G_2, S_2] \subseteq N \subseteq S_1 \times S_2$ is $G_1 - G_2$ decomposable, and therefore all subgroups of $S_1/[G_1, S_1] \times S_2/[G_2, S_2] = C_1 \times C_2$ are $C_1 - C_2$ decomposable. If C_1 has elements of infinite order, then G_2 must be super-perfect. For if not, there is a normal subgroup H of G_2 such that $D = H/[G_2, H] \neq \mathbf{1}$. By Suzuki's result, $C_1 \times D$ contains a subgroup which is not $C_1 - D$ decomposable, a contradiction. Similiarly, if G_1 is not super-perfect, then C_2 must be a torsion group.

Finally, if neither G_1 nor G_2 is super-perfect, then the order of any element in C_1 must be relatively prime to the order of any element in C_2, for if not, by Suzuki's result, there would be a subgroup of $C_1 \times C_2$ which is not $C_1 - C_2$ decomposable.

COROLLARY 1. *Every normal subgroup of $G \times G$ is $G - G$ decomposable if and only if G is super-perfect.*

COROLLARY 2. *If G_1 and G_2 are torsion groups, and the order of any element in G_1 is relatively prime to the order of any element in G_2, then every normal subgroup of $G_1 \times G_2$ is $G_1 - G_2$ decomposable.*

DEFINITION. A torsion group G is called *quasi-nilpotent* if for every prime p with $p = o(a)$ for some $a \in G$, there exists $H \lhd G$ such that $H/[G, H]$ has an element of order p.

It is easy to see that every nilpotent torsion group is quasi-nilpotent. The first example of a quasi-nilpotent group which is not nilpotent is the group $S_3 \times \mathbf{Z}_3$, of order 18. The first indecomposable example is SL(2, 3) of order 24.

We now state a partial converse to Corollary 2:

COROLLARY 3. *If* G_1 and G_2 *are quasi-nilpotent groups such that every normal subgroup of* $G_1 \times G_2$ *is* $G_1 - G_2$ *decomposable, then the order of any element in* G_1 *is relatively prime to the order of any element in* G_2.

Theorem 1 can be easily generalized to yield the following:

THEOREM. 2. *Let* $G = \Pi_{i=1}^k G_i$. *Then every normal subgroup* N *of* G *is a direct product* $N = \Pi_{i=1}^k N_i$ *of normal subgroups* N_i *of* G_i *if and only if either* (i) *at most one of the* G_i *is not super-perfect, or* (ii) *whenever* $H_i \lhd G_i$ *and* $H_j \lhd G_j (i \neq j)$, *the order of any element in* $H_i/[G_i, H_i]$ *is relatively prime to the order of any element in* $H_j/[G_j, H_j]$. *(In particular, these orders must be finite.)*

The proof is identical in nature to that of Theorem 1, and will therefore be omitted.

Instead of studying the lattice of normal subgroups, one can look at other systems of subgroups which form a lattice. For example, one could ask when every characteristic (resp. fully invariant) subgroup of $\Pi_{i=1}^k G_i$ is a direct product of characteristic (resp. fully invariant) subgroups of the individual G_i. These problems appear to be substantially more difficult than the one treated in this section.

2. Direct products of symmetric groups. We begin with two definitions. If G is a group and H is any subgroup containing G', then H is called a *CC-subgroup* of G. All CC-subgroups are therefore normal. Secondly, if $G = \Pi_{i=1}^k G_i$ and ρ_i is the projection on the ith factor, then an automorphism ϕ of G is called *rigid* if $\phi(\rho_i(G)) = \rho_i(G)$ for all i. The group of rigid automorphisms of G is thus isomorphic to $\Pi_{i=1}^k \operatorname{Aut}(G_i)$.

Now let $(S_n)^k$ be the direct product of k copies of the symmetric group S_n, where $n > 4$. (The results of this section are not in general true for $n \leqq 4$, although analogous results may be obtained by treating each case separately.)

We wish to determine all normal subgroups of $(S_n)^k$. For $k = 1$, there are exactly three: $\mathbf{1}$, A_n, and S_n. Suppose the normal subgroups of $(S_n)^r$ for $r < k$ have been determined. Then if $N \lhd (S_n)^k$, we may assume that N is not contained in a product of fewer than k copies of S_n. By the simplicity of A_n, $N \supseteq (A_n)^k$ and so N is a CC-subgroup.

Now $(S_n)^k/(A_n)^k$ is an elementary abelian 2-group, which may be considered as the vector space P (over $\mathbf{Z}_2$) of subsets of the set $K = \{1, 2, \cdots, k\}$, where addition is defined by symmetric difference. There is therefore a $1-1$ correspondence σ between CC-

subgroups of $(S_n)^k$ and subspaces of P. We proceed to show that this correspondence can be made canonical.

If N is a CC-subgroup of $(S_n)^k$, let $\sigma(N)$ be the subspace of P spanned by the set of all $U \subseteq K$ such that $\Pi_{i \in U} x_i \in A_n$ for all $(x_1, \cdots, x_k) \in N$. Conversely, if S is a subspace of P, let N be that CC-subgroup of $(S_n)^k$ consisting of all elements $(x_1, x_2, \cdots, x_k)$ such that $\Pi_{i \in R} x_i \in A_n$ for all $R \in S$. It is easily verified that $\sigma(N) = S$, and that σ is a Galois correspondence, i.e., it is $1-1$ and reverses inclusion.

For example, $(S_n)^2/(A_n)^2$ is isomorphic to the Klein group $\mathbf{Z}_2 \times \mathbf{Z}_2$, which has 5 subgroups (subspaces). There are therefore 5 CC-subgroups of $(S_n)^2$ viz., $(A_n)^2$, $A_n \times S_n$, $S_n \times A_n$, $\{(x_1, x_2) \mid x_1 x_2 \in A_n\}$, and $(S_n)^2$. If we add to these the subgroups $\mathbf{1} \times \mathbf{1}, \mathbf{1} \times A_n, A_n \times \mathbf{1}, \mathbf{1} \times S_n$, and $S_n \times \mathbf{1}$, we find that there is a total of 10 normal subgroups in $(S_n)^2$.

Now $e_1, e_2, \cdots, e_k$ where $e_i = \{i\}$ form a basis of P. Under σ, e_i corresponds to the subgroup of all $(x_1, x_2, \cdots, x_k) \in (S_n)^k$ such that x_i is even. We define a *coordinate plane* to be a subspace of P spanned by some collection of the e_i, and call it *proper* if it has dimension $< k$. It is not hard to see that a CC-subgroup N of $(S_n)^k$ is a nontrivial direct product of two normal subgroups of $(S_n)^k$ if and only if $\sigma(N)$ is the direct sum of two subspaces of P, contained respectively in complementary proper coordinate planes.

We now recall the following well-known facts:

(i) For $n > 4, n \neq 6$, $\operatorname{Aut} S_n \cong \operatorname{Aut} A_n \cong S_n$. Moreover, $\operatorname{Aut} S_6 \cong \operatorname{Aut} A_6$ and $[\operatorname{Aut} S_6 : \operatorname{Inn} S_6] = 2$.

(ii) (Mathewson [1]) For $n > 4$, $\operatorname{Aut}(S_n)^k \cong \operatorname{Aut}(A_n)^k \cong (\operatorname{Aut} S_n)^k x_s S_k$.

In words, every automorphism of $S_n (n > 4, n \neq 6)$ is inner, while every automorphism of $(S_n)^k (n > 4, n \neq 6)$ is the product of an inner automorphism and an automorphism which permutes the k factors. For $n > 4$, the automorphism group of A_n (resp. $(A_n)^k$) is the same as that of S_n (resp. $(S_n)^k$).

THEOREM 3. *Let N be a CC-subgroup of $(S_n)^k$. Then every automorphism of N is induced by an automorphism of $(S_n)^k$.*

Proof. Let $\theta \in \operatorname{Aut} N$. By the above, the action of θ on $(A_n)^k$ is that of the product of a rigid automorphism of $(A_n)^k$ and an automorphism which permutes the k factors of $(A_n)^k$. Multiplying θ by a rigid automorphism of $(S_n)^k$ (itself an automorphism of N), we may assume that the action of θ on $(A_n)^k$ is simply a permutation π of the k factors.

Let $x \in N$ with $\theta(x) = y$ and $e \in (A_n)^k$ with $\theta(e) = e^\pi$. Since $x^{-1}ex \in (A_n)^k$, we have $\theta(x^{-1}ex) = x^{-\pi}e^\pi x^\pi$. But also $\theta(x^{-1}ex) = y^{-1}e^\pi y$. As e is arbitrary in $(A_n)^k, x^\pi y^{-1}$ is in the centralizer of $(A_n)^k$, which is trivial, so $y = x^\pi$. Thus every automorphism of N is the product of a rigid automorphism of $(S_n)^k$ and an automorphism which permutes the k factors. The theorem follows.

In general, not all automorphisms of $(S_n)^k$ actually restrict to automorphisms of a given CC- subgroup N, since not all permutations of the k factors leave N invariant. If $S = \sigma(N)$ is the corresponding subspace of P, let Γ denote the group of permutation matrices (with respect to the basis $e_1, e_2, \cdots, e_k$) which leave S invariant. Then $\operatorname{Aut} N \cong (\operatorname{Aut} S_n)^k\ x_s\ \Gamma$.

THEOREM 4. *The characteristic subgroups of* $(S_n)^k$ *are:* $\mathbf{1}, (S_n)^k, (A_n)^k, T_1,$ *and* T_2, *where* $T_1 = \{(x_1, x_2, \cdots, x_k) \mid \Pi_{i=1}^k x_i \in A_n\}$, *and* $T_2 = \{(x_1, x_2, \cdots, x_k) \mid \Pi_{i<j} x_i x_j \in A_n$ *for all* $i, j\}$. *(Note that* $T_1 = T_2$ *in case* $k = 2$.*)*

Proof. It is clear that except for $\mathbf{1}$, any characteristic subgroup of $(S_n)^k$ contains $(A_n)^k$, for otherwise it would be contained in a direct product of fewer than k copies of S_n, and hence not be characteristic. In terms of P, we must show that the only subspaces S invariant under all permutations of the coordinates are $\varnothing$, P, the 1-dimensional subspace V_1 spanned by $e_1 + e_2 + \cdots + e_k$, and the $(k-1)$-dimensional subspace V_2 spanned by all $e_i + e_j$.

That these are all invariant is immediate. Assume now that S is invariant, and suppose that $a_1e_1 + \cdots + a_re_r + \cdots + a_se_s + \cdots + a_ke_k \in S$, where $a_r + a_s \neq 0$ for some choice of r and s (otherwise $S = \varnothing$ or $S = V_1$). By invariance, $a_1e_1 + \cdots + a_re_s + \cdots + a_se_r + \cdots + a_ke_k \in S$, and adding gives $(a_r + a_s)e_r + (a_r + a_s)e_s \in S$, so that $e_r + e_s \in S$. Again by invariance, we conclude that $e_i + e_j \in S$ for all i, j, so either $S = P$ or $S = V_2$.

The question of determining exactly which groups can arise as the group Γ of "admissible" permutation matrices for a given CC-subgroup N will be dealt with in a future paper.

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