PACIFIC JOURNAL OF MATHEMATICS
Vol, 61, No. 1, 1975

THE CONSTRUCTIVE THEORY OF KT-MODULES

FRED RICHMAN

The theory of countable KT-modules is developed from a constructive point of view. Of classical interest is a characterization in terms of local properties.

1. **Introduction.** One of the high points in the theory of abelian groups is the structure theory of countable abelian p-groups, often referred to simply as *Ulm's theorem*. Attempts to extend this theory to more general classes of abelian groups culminated in the work of Hill [4] who showed that the theory carried over to the class of totally projective p-groups. The thrust here was to remove the countability restriction. From a constructive point of view this does not appear to be much of an advance, since the construction of nontrivial totally projective groups is based upon the existence of uncountable ordinals. On the other hand, Warfield's notion of a *KT-module* [5], which removes the torsion restriction, admits plenty of nontrivial constructive examples, and thus invites the formation of a constructive theory.

The constructive approach leads to an increased emphasis on local properties. A *local property* of a module is a condition on its finitely generated submodules. In the original Ulm theory the local property demanded of the group was that it be torsion, while the only global property invoked was countability. More recent treatments have centered around purely global characterizations. This was motivated by the necessity for *some* global property, coupled with a distaste for the countability restriction. From a constructive point of view all the significant examples are countable, and the finitistic approach strongly suggests looking at finitely generated submodules. Thus we are led to consider KT-modules in much the same spirit as the original Ulm theory.

As in [2] we consider the height function to be an integral part of the presentation of the module. So a "local object" is a finitely generated module with a valuation induced by the height function of the module in which it is imbedded. Our main task will be to study such objects to determine when they can be submodules of KT-modules.

We shall be dealing with modules over a discrete valuation ring R. All modules will be *discrete* in the sense that given any two elements x and y, one can decide whether $x = y$ or $x \neq y$. From a constructive point of view this precludes, for example, taking R to be the p-adic integers (unless the module is torsion), but allows R to be the rational p-adic integers or even the algebraic p-adic integers. The constructive theory of ordinal numbers developed in [2] will be adopted throughout.

2. Finitely related valuated modules. Let R be a discrete valuation ring with prime p. An R-module A is *valuated* if it is equipped with a function $h: A \to \lambda \cup \{\infty\}$, where λ is an ordinal, such that

1. $hpx > hx$
2. $h(x + y) \geqq \min(hx, hy)$
3. $hux = hx$ if u is a unit in R.

We employ the conventions that $\alpha < \infty$ and $\infty + 1 = \infty$. We say that A is *reduced* if $h^{-1}(\infty) = \{0\}$. The function h is said to be a *height function* if, in addition to satisfying properties 1 through 3, it is surjective and

4. If $hx > \alpha$, then we can find a y such that $hy \geqq \alpha$ and $py = x$.

An R-module is *finitely related* if it is a cokernel of a map between finite rank free R-modules. Although it is a classical theorem that, when R is a discrete valuation ring, every finitely generated R-module is finitely related, this is not the case constructively, so we must distinguish between these two concepts. For constructive facts about finitely related modules see [3]. Consider the category of finitely related valuated R-modules. A morphism is a homomorphism $f: A \to B$ such that for all x in A either $hfx = \infty$ or $hx \leqq hfx$ (as ordinals). We may as well assume that for each α in λ there is an x in A such that $\alpha \leqq hx \neq \infty$. This λ, which is an invariant of the valuated module A, is called the *length* of A and may be written λA. Note that $A \oplus B$ exists if and only if $\lambda A \vee \lambda B$ exists.

A *rank* 1 *h-free module* is a torsion free cyclic module such that $hpx = hx + 1$ for each $x \neq 0$. A rank 1 h-free module Re is of *type* α if $he = \alpha$. An *h-free module* of rank n is a direct sum of n rank 1 h-free modules. The number of different types of rank 1 h-free modules in such a direct sum is the *number of homogeneous components* of the h-free module. An h-free module is *homogeneous* if it has only one homogeneous component. Note that if F is h-free of length λ and $\alpha \in \lambda$, then the rank of the homogeneous part of F of type α is the dimension of the vector space

$$\frac{F(\alpha)}{F(\alpha + 1) + pF}$$

where $F(\beta) = \{x \in F: hx \geqq \beta\}$, and so is an invariant.

If B is a valuated R-module and A is a submodule of B, then A is *nice* in B if for each b in B there is an element in $b + A$ of maximum height. Such an element is said to be *A-proper*. If A is nice in B, then there is a natural valuation on B/A, and the projection of B on B/A is the cokernel of the inclusion $A \subseteq B$.

We need to establish some facts about submodules of finite rank h-free modules, with particular emphasis on niceness. The reader should keep in mind that if λ is an ordinal, and α and β are in λ, then we cannot necessarily determine whether $\beta = \alpha + n$ for some positive integer n.

LEMMA 1. *Let B be a finite rank h-free module and A a finitely generated submodule of B. Then we can construct a finite subset $e_1, \cdots, e_m$ of a free basis of B, and nonzero generators $a_i = \sum_{j=1}^{m} a_{ij}$ for A such that*

(1) $he_j \leqq he_{j+1} + t_j$ *for some nonnegative integer* t_j

(2) $a_{ij} \in Re_j$

(3) $a_{ij} = 0$ *if* $i > j$

(4) $ha_{ii} \leqq ha_{jk}$ *if* $i \leqq j$ *and* $i \leqq k$

(5) $ha_{ii} < ha_{ik}$ *if* $i < k$.

Proof. If $A = 0$ choose $m = 0$ and we are done. Otherwise let $b_1, b_2, \cdots$ be a free basis for B and suppose that the elements $a_i = \sum_j a_{ij}$ form a set of generators for A, where $a_{ij} \in Rb_j$ and $1 \leqq i \leqq n$. By reindexing we may assume that $ha_{11} \leqq ha_{ij}$ for all i and j. Suppose $ha_{11} = ha_{1k}$ for some $k > 1$. We can relabel, if necessary, so that $hb_1 \leqq hb_k$ and redefine b_1 to be $b_1 + rb_k$ where r is chosen so that $srb_k = a_{ik}$ where $a_{11} = sb_1$. With respect to the new basis $b_1, b_2, \cdots$ we have $ha_{11} < ha_{1k}$. We can repeat this for as many k as necessary until $ha_{11} < ha_{1k}$ for all $k > 1$. Replacing a_i by $a_i - r_i a_1$ we may assume that $a_{i1} = 0$ for $2 \leqq i \leqq n$. Thus $a_i \in \sum_{j \geqq 2} Rb_j$ for $2 \leqq i \leqq n$.

Set $e_1 = b_1$ and fix a_1. By induction on n we can construct $e_2, \cdots, e_m$ and $a_2, \cdots, a_n$ as desired. Note that we still have $ha_{11} \leqq ha_{ij}$ for all i and j, and $ha_{11} < ha_{1k}$ for $k > 1$. Thus we also have $he_1 \leqq he_2 + t_1$ for some nonnegative integer t_1, because $ha_{11} \leqq ha_{22}$ and $a_{22} \neq 0$.

COROLLARY 1. *If B is h-free, and A is a finitely generated submodule of B, then A is h-free.*

Proof. Since A is finitely generated we may assume that B is finite rank. It is readily seen that the generators constructed in Lemma 1 form a free basis for A.

LEMMA 2. *Same set up as Lemma* 1. *Let $A(\beta) = \{a \in A\colon ha \geqq \beta\}$ where $\beta \in \lambda B \cup \{\infty\}$. If $x = \sum x_j$ with $x_j \in Re_j$ and if, for $1 \leqq j \leqq m$,* *we have* $x_j = 0$ *or* $hx_j < \max(ha_{jj}, \beta)$, *then* x *is* $A(\beta)$-*proper. Moreover for any* $b \in B$ *we can find* $a \in A(\beta)$ *so that* $b + a$ *has this form.*

Proof. We may assume $hx \neq \infty$. Let k be the least index so that $hx = hx_k$. If $hx_k < \beta$, then clearly x is $A(\beta)$-proper. We shall show that if $k > m$ or $hx_k < ha_{kk}$, then x is A-proper.

Consider $x + a$ where $a = \sum r_i a_i$. We wish to show $h(x + a) \leqq hx$. If $hr_i a_i > hx$ for all i we are done. Otherwise let t be the least index such that $hr_t a_t \leqq hx$. Note that $t \neq k$. If $t > k$, then

$$h(x + a) = h\Big(\sum_j\Big(x_j + \sum_{i \leqq j} r_i a_{ij}\Big)\Big) \leqq h\Big(x_k + \sum_{i \leqq k} r_i a_{ik}\Big) = hx_k = hx .$$

If $t < k$, then $h(x + a) \leqq h(x_t + \sum_{i \leqq t} r_i a_{it})$. But $hx_t > hx$ and $hr_i a_{it} > hx$ for $i < t$, while $hr_t a_{tt} = hx$. Thus $h(x + a) \leqq hx$.

The "moreover" is easily proved by induction on m, noting that if $hx_1 \geqq \max(ha_{11}, \beta)$, then we can find r_1 so that $x_1 = r_1 a_{11}$ and hence $x - r_1 a_1$ has first coordinate zero and $r_1 a_1 \in A(\beta)$.

If we could take $\beta = 0$, then Lemma 2 would say directly that A is nice. However we may not be able to lay our hands on the least element of λB. Instead we can take β to be the minimum of the heights of a free basis of B.

COROLLARY 2. *If* B *is* h-*free, and* A *is a finitely generated submodule of* B, *then there is a positive integer* n, *and an* h-*free submodule* F *of* B *such that* $p^n B + A = F + A$ *and each element of* F *is* A-*proper.*

Proof. Choose n so that $hp^n e_i \geqq ha_i$ for $1 \leqq i \leqq m$ and let $F = \sum_{i > m} Rp^n e_i$. By Lemma 2 each element of F is A-proper, and every element of $p^n B$ can be written as $f - a$ with $a \in A$ and $f \in F$.

Since Lemma 2 implies that A is nice in B we may consider B/A as a valuated module in a natural way. Then Corollary 2 may be interpreted in B/A.

COROLLARY 3. *Let* B *be* h-*free and* A *a finitely generated submodule of* B. *Then* $p^n(B/A)$ *is* h-*free for some positive integer* n.

We will need the following relative version of Lemma 2.

THEOREM 1. *Let* F *be* h-*free and* $A \subseteq B$ *be finitely generated submodules of* F. *If* $\beta \in \lambda F$ *and* $B(\beta) = \{b \in B : hb \geqq \beta\}$, *then* $A + B(\beta)$ *is nice in* F.

Proof. Let $x \in F$. We seek an element in $x + A + B(\beta)$ of maximum height. By Lemma 2 we can assume that x is A-proper. If $hx < \beta$ then x is also $(A + B(\beta))$-proper and we are done. So we may assume that $hx \geqq \beta$. By Lemma 2 we can pick y in $x + B(\beta)$ of maximum height. Consider $y + a + b$ where $a \in A$ and $b \in B(\beta)$. If $ha < \beta$ then $h(y + a + b) = ha < hy$. If $ha \geqq \beta$ then $a + b \in B(\beta)$ so $h(y + a + b) \leqq hy$. Thus y is the desired element.

Let $\mathscr{K}$ be the class of finitely related valuated modules which are cokernels of h-free modules, that is, of the form B/A where B is finite rank h-free and A is finitely generated. The following corollary shows that lots of submodules of objects in $\mathscr{K}$ are nice.

COROLLARY 4. *Suppose $M \in \mathscr{K}$ and S is a finitely generated submodule of M. If $\beta \in \lambda M$ let $S(\beta) = \{x \in S\colon hx \geqq \beta\}$. Then $p^n(S(\beta))$ is nice in M. In particular, S is nice in M.*

Proof. Write $M = F/A$ where F is h-free and A is finitely generated. Then $S = C/A$ for some h-free submodule of F. Let $B = p^nC + A$. Then $D = A + B(\beta + n) = A + (p^nC)(\beta + n) = A + p^n(C(\beta))$, so $D/A = p^n(S(\beta))$. But D is nice in F by Theorem 1. Hence $p^n(S(\beta))$ is nice in M. By Choosing β to be the minimum of the heights of the generators of F, and taking $n = 0$, we get S is nice in M.

In the proof of Ulm's theorem we need the following construction (which is trivial in the p-group case).

COROLLARY 5. *Let $M \in \mathscr{K}$ and S a finitely generated submodule of M. If $x \in M$, then among the S-proper elements y of $x + S$ we can find one for which hpy is maximum.*

Proof. We may assume that x is S-proper. Let $\beta = hx$. Then $x + S(\beta)$ consists of the S-proper elements of $x + S$. Choose an element in $px + p(S(\beta)) = \{py\colon y \in x + S(\beta)\}$ of maximum height by Corollary 4. This gives the desired y.

3. **KT-modules; uniqueness.** A countable *KT-module* G is a countable (discrete) module over a discrete valuation ring R, with a reduced height function h, such that finitely generated submodules of G are in $\mathscr{K}$. We restrict ourselves to countable modules, rather than to countably generated modules, in light of the absence of significant examples of *discrete* countably generated modules that are not countable. If G is a KT-module, and $\alpha < \beta$ are elements of

λG, then we can find an element x in G such that $hx = \beta$, and an element y such that $hy \geqq \alpha$ and $py = x$. If $hy > \alpha$ we have found an element between α and β, that is, we have shown that $\alpha \ll \beta$. If $hy = \alpha$, then we can find $\alpha + 1$ by writing the cyclic submodule Ry as a cokernel of an h-free module. Hence we can always decide whether $\alpha \ll \beta$ or not. Thus by [2; Cor. Thm. 2], given $\alpha < \beta$ in λG, we can find $\alpha + 1$. We say that *λG has successors*. This property, the lack of which caused so many problems in [2], simplifies our task considerably.

Let G be a valuated module of length λ. For each α in λ let

$$F_G(\alpha) = \{x \in G\colon hx \geqq \alpha\}$$

where $x = y$ in $F_G(\alpha)$ if $h(x - y) > \alpha$. The *Ulm invariants* $f_G(\alpha)$ of G are defined by

$$f_G(\alpha) = \{x \in F_G(\alpha)\colon hx \ll hpx\} .$$

In addition we define the *Warfield invariants* $g_G(\alpha)$ by

$$g_G(\alpha) = \varinjlim F_G(\alpha + m)$$

where the connecting maps are induced by multiplication by p.

The set $F_G(\alpha)$, and hence the Ulm and Warfield invariants, are vector spaces over the field R/Rp. We need to establish some of the constructive properties of these spaces.

LEMMA 3. *If $G \in \mathscr{K}$ and $\alpha \in \lambda G$, then $f_G(\alpha)$ is a finite dimensional vector space over R/Rp. Moreover there exists a finite subset X of λG such that $f_G(\alpha) = 0$ if $\alpha \notin X$.*

Proof. Let G be isomorphic to B/A where B and A are as in Lemma 1. Consider $U(\alpha) = \{b \in B\colon hb = \alpha \text{ and } pb = a_{ii} \text{ for some } i\}$. We shall show that $U(\alpha)$ is a basis for $f_G(\alpha)$. By Lemma 2 every linear combination of elements of $U(\alpha)$ with unit coefficients is A-proper, hence has height α in G, so $U(\alpha)$ is a linearly independent subset of $F_G(\alpha)$. Moreover since $h(pb - a_i) > ha_{ii} = \alpha + 1$ we have $U(\alpha) \subseteqq f_G(\alpha)$. Finally, if $x \in B$ represents an element of $f_G(\alpha)$, then we can assume that $x_i = 0$ or $hx_i = \alpha$ for each i. Let $I = \{i\colon x_i \neq 0\}$. By Lemma 2 we may assume that $ha_{ii} > \alpha$ for $i \in I$. If $ha_{ii} > \alpha + 1$ for some $i \in I$, then px would be A-proper, which is impossible since x represents an element of $f_G(\alpha)$. Hence $ha_{ii} = \alpha + 1$ and so $x_i = u_i b_i$ where $b_i \in U(\alpha)$ and u_i is a unit, for each $i \in I$. Clearly by examining the elements a_{ii} we can produce the desired finite set X.

COROLLARY 6. *If G is a countable KT-module, then the Ulm* *and Warfield invariants of G are countable vector spaces over R/Rp which are locally finite dimensional.*

Proof. For the Ulm invariants this follows immediately from Lemma 3. The argument for the Warfield invariants is different because they do not satisfy Lemma 3. However they are obviously countable, and if K is a finitely generated submodule of G, then $g_K(\alpha) \cong g_H(\alpha)$ where $H = p^m K$ as can be seen, for $m = 1$, by considering the sequence

$$pK(\alpha) \subseteqq K(\alpha) \xrightarrow{p} pK(\alpha+1) \subseteqq K(\alpha+1) \xrightarrow{p} \cdots$$

By Corollary 3 we can choose m so that H is h-free. Clearly any finitely generated submodule of $g_H(\alpha)$ is finite dimensional (although $g_H(\alpha)$ may not be because H may have a basis element e for which we cannot tell whether or not $he + n = \alpha$ for some n).

Note that $g_G(\alpha)$ and $g_G(\alpha + 1)$ are naturally isomorphic. The reason for defining g_G on all of λ, rather than just on the limit points in λ, is that we may not know how to write a given element in λ as $\alpha + n$ for a limit α and a nonnegative integer n.

Two countable KT-modules G and K have *isomorphic invariants* if they have a common length λ, and for each α in λ we have isomorphisms between $f_G(\alpha)$ and $f_K(\alpha)$, and isomorphisms between $g_G(\alpha)$ and $g_K(\alpha)$ that respect the natural isomorphisms between $g(\alpha)$ and $g(\alpha + 1)$.

THEOREM 2. *Let G and K be countable KT-modules with isomorphic invariants. Then G and K are isomorphic.*

Proof. We follow Kaplansky's proof of Ulm's theorem [1]. It suffices to show that if φ is a height preserving isomorphism between finitely generated submodules $S \subseteqq G$ and $T \subseteqq K$, and if $x \in G$, then φ can be extended to a height preserving isomorphism into K of the submodule $S + \langle x \rangle$ generated by S and x.

Since $S + \langle x \rangle$ is finitely related, either $p^m x \in S$ for some m, or $p^m x \notin S$ for all m. In the former case we may, by induction on m, assume that $px \in S$. In the latter case we may by induction, and Corollary 3 applied to $(S + \langle x \rangle)/S$, assume that $p^m x$ is S-proper for all m, and that $hp^m x = hx + m$.

Suppose $px \in S$. If $x \in S$ we are done. Otherwise we may assume that x is S-proper, by Corollary 4, since $S + \langle x \rangle \in \mathscr{K}$ and S is finitely generated. By Corollary 5 we may assume that $hpx \geqq hpz$ if z is an S-proper element of $x + S$. The proof now follows the proof of

[2; Thm. 13], the last line of which should read $hy > \alpha$ rather than $hy \geqq \alpha$. The only change being that here we rely on the fact that finitely generated submodules are nice, whereas there the finitely generated subgroups were finite and hence automatically nice.

Suppose that $p^m x$ is S-proper and that $hp^m x = \alpha + m$ for each nonnegative integer m. It is readily seen that this is equivalent to $x \in g_G(\alpha)$ but $x \notin g_S(\alpha)$. It will suffice to find $y \in g_K(\alpha)$ such that $y \notin g_T(\alpha)$. Note that $g_T(\alpha)$ is a detachable subspace of $g_K(\alpha)$, since if z is a nonzero element of $g_K(\alpha)$ then $z \in g_T(\alpha)$ exactly when $p^m z$ is not T-proper for some m, which is decidable by Corollary 3 applied to $(T + \langle x \rangle)/T$. Let σ be the isomorphism from $g_G(\alpha)$ to $g_K(\alpha)$. If $\sigma x \notin g_T(\alpha)$ we are done. Otherwise, by repeated application of φ^{-1} and σ we can construct a finitely generated submodule V of $g_S(\alpha)$ such that $\sigma x \in \varphi V = \sigma V$, or find our y along the way. But $\sigma x \in \sigma V$ implies $x \in V$ which contradicts $x \notin g_S(\alpha)$.

4. **KT-modules; existence.** If G is a countable KT-module of length λ, then λ is a countable ordinal and for each α in λ we have countable vector spaces $f_G(\alpha)$ and $g_G(\alpha)$ over R/Rp which are locally finite dimensional. We turn our attention now to the problem of constructing a KT-module G with prescribed f_G and g_G.

DEFINITION. Let λ be a countable ordinal with successors. For each α in λ let $f(\alpha)$ and $g(\alpha)$ be countable, locally finite dimensional, vector spaces over R/Rp. Then f and g are a pair of *Zippin functions* if

(1) If $\alpha + 1 \in \lambda$, then we have an isomorphism between $g(\alpha)$ and $g(\alpha + 1)$.

(2) Given $\alpha < \beta$ we can find γ such that $\alpha \leqq \gamma < \beta$ and either $\gamma + 1 = \beta$ or $f(\gamma) \neq 0$.

(3) Given α we can find $\gamma \geqq \alpha$ such that either $f(\gamma) \neq 0$ or $g(\alpha) \neq 0$.

LEMMA 4. *If G is a countable KT-module, then f_G and g_G are a pair of Zippin functions.*

Proof. We must verify properties 2 and 3 in the definition of Zippin functions. Suppose $\alpha < \beta$. Then we can find x in G such that $hx = \beta$ and y in G such that $py = x$ and $hy \geqq \alpha$. Let $\gamma = hy$. If $\gamma + 1 < \beta$ then y is a nonzero element of $f_G(\gamma)$. To prove 3, suppose we are given α. We can find x such that $hx = \alpha$. By Corollary 3 we can find m such that $Rp^m x$ is h-free. So either $hp^{n+1}x = hp^n x + 1$ for all $n \geqq 0$, in which case x is a nonzero element of $g_G(\alpha)$, or $hp^{n+1}x > hp^n x + 1$ for some n, in which case $p^n x$ is a nonzero element of $f_G(hp^n x)$.

Let f and g be Zippin functions on λ. We say that $S \in \mathscr{K}$ is *admissible* with respect to f and g if the valuation h on S takes values in λ, and if for each $\alpha \in hS$ we have imbeddings $f_S(\alpha) \to f(\alpha)$ and $g_S(\alpha) \to g(\alpha)$ that respect the isomorphisms $g(\alpha) \to g(\alpha + 1)$ and $g_S(\alpha) \to g_S(\alpha + 1)$. The fundamental construction step is the following.

LEMMA 5. *Let f and g be a pair of Zippin functions on the countable ordinal λ. Let $S \in \mathscr{K}$ be admissible with respect to f and g. Suppose $x \in S$ and $\alpha < hx$. Then we can construct $T \in \mathscr{K}$ containing S such that T is admissible with respect to f and g, and such that $py = x$ for some $y \in T$ such that $hy \geqq \alpha$.*

Proof. Since pS is finitely generated it is nice by Corollary 4 so we may assume that x is pS-proper. If $\alpha + 1 \neq hx$ we can increase α, by property 2 of Zippin functions, so that $f(\alpha) \neq 0$. If we still have $\alpha + 1 \neq hx$ and in addition $f_S(\alpha) \neq 0$, then we replace α by $\alpha + 1$ and start all over again. Since $f_S(\alpha) \neq 0$ for only finitely many α, by Lemma 3, we eventually have either

$$\alpha + 1 = hx \text{ or } f(\alpha) \neq 0 \text{ and } f_S(\alpha) = 0 .$$

Let Re be rank-one h-free generated by e where $he = \alpha$. Let $S = B/A$ where B is h-free and A is finitely generated. Write $x = b + A$ where $hb = hx$. Let $B' = B \oplus Re$ and $A' = A + R(b - pe)$. Set $T = B'/A'$ and let $y = a + A'$, so $py = x$. Clearly any A-proper element of B is A'-proper, so the imbedding of S in T preserves heights. Moreover it is readily seen that $h(s + uy) = \min(hs, \alpha)$ if u is a unit in R. It remains to show that T is admissible with respect to f and g.

We must define, for each β in λ, the imbeddings $f_T(\beta) \to f(\beta)$ and $g_T(\beta) \to g(\beta)$. Now the imbedding $S \subseteq T$ induces natural maps $f_S(\beta) \to f_T(\beta)$ and $g_S(\beta) \to g_T(\beta)$. The latter map is an isomorphism for all β because $pT \subseteq S$, so we have the map from $g_T(\beta)$ to $g(\beta)$. We shall show that the former map is an isomorphism if $\beta \neq \alpha$. Suppose $s + ry$ is an element of $f_T(\beta)$. If $r = pr'$, then $s + ry = s + r'x \in f_S(\beta)$. If r is a unit then $\beta \leqq h(s + ry) = \min(hs, \alpha)$ so $\beta = hs < \alpha$. Thus $s + ry$ is equal to s in $F_T(\beta)$ so $s + ry$ is equal to an element of $f_S(\beta)$.

It remains to take care of $f_T(\alpha)$. Suppose $\alpha + 1 = hx$ and $s + ry$ is in $f_T(\alpha)$. If $r = pr'$, then $s + ry = s + r'x \in f_S(\alpha)$. If r is a unit, then $\alpha = h(s + ry) = \min(hs, \alpha)$ so $hs \geqq \alpha$. Now $\alpha + 1 < hp(s + ry) = h(ps + x)$. But x was assumed to be pS-proper, so this case cannot arise. Finally suppose $f(\alpha) \neq 0$ and $f_S(\alpha) = 0$ and $hx > \alpha + 1$. Then

$f_T(\alpha)$ is generated by y, for if $s + ry \in f_T(\alpha)$, then r is a unit and $hs \geqq \alpha$ and $h(ps + rx) > \alpha + 1$, so $hps > \alpha + 1$. Hence, since $f_S(x) = 0$, we have $hs > \alpha$ so $s + ry$ is equal to ry in $F_T(\alpha)$. Since $f(\alpha) \neq 0$ we simply map $y \in f_T(\alpha)$ to any nonzero element of $f(\alpha)$ and we have the desired map $f_T(\alpha) \to f(\alpha)$.

THEOREM 3. *Let f and g be a pair of Zippin functions on the countable ordinal λ. Let $S \in \mathscr{K}$ be admissible with respect to f and g. Then S can be imbedded in a KT-module G such that $f_G(\alpha) \cong f(\alpha)$ and $g_G(\alpha) \cong g(\alpha)$ for each α in λ, and so that the height function on G induces the valuation on S.*

Proof. If μ is a finite subset of λ, and X is a finite subset of S, then by repeated applications of Lemma 5 we can imbed S in an object $E(S, X, \mu)$ of $\mathscr{K}$, and extend h so that $E(S, X, \mu)$ is admissible with respect to f and g, and so that if $\alpha \in \mu$ and $x \in X$ with $\alpha < hx$, then there is a y in $E(S, X, \mu)$ satisfying $py = x$ and $hy \geqq \alpha$. We shall construct G as the union of a chain $S_0 \subseteqq S_1 \subseteqq \cdots$ so that $S = S_0$ and so that the S_i are admissible with respect to f and g, and have a common height function h. Let $x_0, x_1, \cdots$ be an enumeration of $\bigcup_{\alpha \in \lambda} f(\alpha)$, let $y_0, y_1, \cdots$ be an enumeration of $\bigcup_{\alpha \in \lambda} g(\alpha)$, and let $\alpha_0, \alpha_1, \cdots$ be an enumeration of λ. Let $\mu_n = \{\alpha_1, \cdots, \alpha_n\}$. Let $S_0 = S$ and let $\{s_{i0}\}$ be an enumeration of S_0. The construction, for $n = 1, 2, \cdots$, proceeds as follows:

(1) Suppose $n = 3m + 1$ and $y_m \in g(\alpha)$. Set $S_n = S_{n-1}$ if y_m is in the image of $g_{S_{n-1}}(\alpha)$; otherwise set $S_n = S_{n-1} \oplus \langle z \rangle$ where $\langle z \rangle$ is a rank-1 h-free module of type α. Map $z \in g_{S_n}(\alpha)$ onto y_m.

(2) Suppose $n = 3m + 2$ and $x_m \in f(\alpha)$. Set $S_n = S_{n-1}$ if x_m is in the image of $f_{S_{n-1}}(\alpha)$; otherwise set $S_n = S_{n-1} \oplus \langle z \rangle$ where z has order p and height α. Map $z \in f_{S_n}(\alpha)$ onto x_m.

(3) Suppose $n = 3m$. Set $S_n = E(S_{n-1}, X_{n-1}, \mu_n)$ where $X_{n-1} = \{s_{ij}\colon i + j \leqq n - 1\}$.

(4) Let $\{s_{in}\}$ be an enumeration of S_n.

The decisions in steps 1 and 2 can be made because $f(\alpha)$ and $g(\alpha)$ are locally finite dimensional. Since $G = \bigcup S_n$ is a union of a countable chain of objects in $\mathscr{K}$, it will be a KT-module if the valuation h on G is a height function. Step 3 assures that property 4 of height functions is satisfied. We must also show that $hG = \lambda \cup \{\infty\}$. Note that steps 1 and 2 provide isomorphisms between $f_G(\alpha)$ and $f(\alpha)$, and between $g_G(\alpha)$ and $g(\alpha)$.

Suppose $\alpha \in \lambda$. Then there is $\gamma \in \lambda$ such that $\gamma \geqq \alpha$ and either $f(\gamma) \neq 0$ or $g(\alpha) \neq 0$. If $g(\alpha) \neq 0$, then $hx = \alpha$ for some $x \in G$. If $f(\gamma) \neq 0$, then we can find $x_1 \in G$ such that $hx_1 = \gamma$. If $\gamma = \alpha$ we are done. Otherwise we can find $x_2 \in G$ such that $\alpha \leqq hx_2 < hx_1$.

Continuing in this way we will find an $x_n \in G$ such that $\alpha = hx_n$ by [2; Thm. 2].

5. **Classical considerations.** Is there anything here of interest to the classical mathematician? One result is that Warfield's definition of KT-modules in [5] defines the same class of modules, in the countable case, as are defined here in section 3. To see this we note that if λ is a limit ordinal, then a countable reduced module A is a *λ-elementary KT-module*, in the sense of Warfield, if and only if

$$\begin{aligned} &g_A(\lambda + n) \quad \text{is one-dimensional for } n = 0, 1, 2, \cdots \\ &g_A(\mu) = 0 \quad \text{otherwise} . \end{aligned}$$

By Theorem 3 there is a KT-module B, in the sense of this paper, with the same invariants as any given countable λ-elementary KT-module A. Hence by Warfield's Ulm's theorem [5; Thm. 3] we have $A \cong B$ so any countable λ-elementary KT-module in the sense of Warfield is a KT-module in the sense of this paper. It follows easily that the two notions of a countable KT-module are classically equivalent. Thus we have the following

THEOREM 5. *A countable reduced module A is a* (classical) *KT-module if and only if every finitely generated submodule of A is the cokernel of a finite rank h-free module.*

We close with a few classical observations which may have gotten lost in the preoccupation with constructive problems. Lemma 1 provides a canonical form for submodules of finite rank h-free modules and demonstrates that finitely generated submodules of h-free modules are h-free. Lemma 2 shows that such submodules are also nice. The proof of Lemma 3 shows how to read the Ulm and Warfield invariants of B/A from the canonical form of $A \subseteq B$ given in Lemma 1. Theorem 3 gives precise conditions for a finitely generated valuated module S to be imbeddable in a countable KT-module with prescribed invariants (in view of Theorem 5 which says that S must be in $\mathscr{K}$). Theorem 5 implies that every finitely generated submodule of a KT-module is nice.

REFERENCES

1. I. Kaplansky, *Infinite abelian groups*, Univ. of Mich. Press, 1969.
2. F. Richman, *The constructive theory of countable abelian p-groups*, Pac. J. Math. **45** (1973), 621-637.
3. F. Richman, *Constructive aspects of Noetherian rings*, Proc. Amer. Math. Soc. **44** (1974), 436-441.

4. E. A. Walker, *Ulm's theorem for totally projective groups,* Proc. Amer. Math. Soc. **37** (1973), 387-392.
5. R. B. Warfield, Jr., *Classification theorems for p-groups and modules over a discrete valuation ring,* Bull. Amer. Math. Soc. **78** (1972), 88-92.

Received May 29, 1975. This research was supported by NSF Grant GP-28379.

NEW MEXICO STATE UNIVERSITY