PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No 1, 1976

SOME REMARKS ON THE CENTER OF THE UNIVERSAL ENVELOPING ALGEBRA OF A CLASSICAL SIMPLE LIE ALGEBRA

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This paper is concerned with explicitly producing generating sets of the centers of the universal enveloping algebras of classical simple Lie algebras.

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field K of characteristic zero, let U be its universal enveloping algebra, and let Z be the center of U. If l is the dimension of a Cartan subalgebra H of L, then it is known that Z is a polynomial ring in l independent variables. In this paper a set of l algebraically independent generators of Z is produced rather explicitly for the classical algebras of type A, B, C, D by casewise considerations.

It is straightforward to show that generating Z is equivalent to generating the L-invariants I_L^* in the symmetric algebra S_{L^*} of L^*. In addition, there is a homomorphism from S_{L^*} onto S_{H^*} which embeds I_L^* into the Weyl-invariants I_W. Due to Chevalley this embedding is also a surjection. For the classical simple Lie algebras the action of the Weyl group W on S_{H^*} is describable in a sufficiently convenient fashion so as to permit easy construction of generators of I_W. It is shown here that certain generating sets of I_W can be explicitly lifted back to I_L^* via trace functions on the first fundamental representation of L. As a result of this construction of the generators of I_W and the lifting process, the following well-known results are proven rather directly for the classical algebras:

1. $I_L^* \cong I_W$ (Chevalley), and
2. Z and I_W are polynomial rings in l algebraically independent variables.

The center Z of U plays a fundamental role in the finite-dimensional representation theory of L. Since any irreducible representation is determined up to isomorphism by its character, if $z_1, \cdots, z_l$ were generators of Z and if M and N were non-isomorphic irreducible L-modules, then for some i one must have $(z_i)_M \neq (z_i)_N$ (due to Schur's lemma they are scalars). The central element $(z_i - (z_i)_N)/((z_i)_M - (z_i)_N)$ would act as one on M and zero on N. For any list of pairwise non-isomorphic irreducible L-modules one could thus find a central element acting as one on one of them, and as zero on the rest. Such elements could be used to isolate the isotypic components in a reducible representation of L. Hence there is good reason to produce generators of Z as explicitly as possible.

Section 1 is concerned with showing that generating Z is equivalent to generating I_L^* and leads up to §§2–5 where the Chevalley isomorphism $I_L^* \cong I_W$ is proven by explicitly lifting generating sets of I_W back to I_L^*.

1. Generation of Z. There are well known actions of L on the symmetric algebras S_L and S_{L^*} by graded derivations extending the adjoint representation of L and its contragredient, and if W is the Weyl group of L with respect to the Cartan subalgebra H, it acts on S_{H^*} by graded automorphisms. The standard symmetrization map $\eta: S_L \to U$ given by $(x_1 \cdots x_r)^\eta = (1/r!)\Sigma_{\alpha \in S_r} x_{\alpha(1)} \cdots x_{\alpha(r)}$ for a monomial of degree r in S_L, induces a linear isomorphism between the L-invariants I_L in S_L and the L-invariants Z in U since it is an L-module isomorphism. While this induced map between I_L and Z is not an algebra isomorphism it is known to have the following redeeming qualities:

LEMMA. *Suppose S is a finite set of homogeneous invariant elements in S_L generating I_L. Then S^η generates Z, and if S is algebraically independent so is S^η.*

Proof. Let U have its usual filtration and let U_p be the subspace of all elements of filter less than or equal to p. Observe that due to the Poincaré-Birkhoff-Witt theorem, if $x_1, \cdots, x_r$ are homogeneous elements of S_L of degrees $d_1, \cdots, d_r$ and $d = \Sigma_i d_i$, then $(x_1 \cdots x_r)^\eta = x_1^\eta \cdots x_r^\eta + t$ where t is in U_{d-1}.

(i) Since L acts by graded derivations, I_L is homogeneous. Recalling that η induces a linear isomorphism between I_L and Z, proceed by induction on the filter of a central element to show it is in the subalgebra of U generated by S^η. Let $S = \{x_1, \cdots, x_r\}$. Now η takes constants to constants so it suffices to check the induction step. Every element in Z is a linear combination of images of homogeneous elements in I_L, so it suffices to show that if $x_{i_1} \cdots x_{i_k}$ is a monomial in I_L then $(x_{i_1} \cdots x_{i_k})^\eta$ is in the subalgebra generated by S^η. The remarks in the first paragraph complete the proof.

(ii) Set $y_i = x_i^\eta$. Suppose the y_i are algebraically dependent and let p be a nonzero polynomial in $K[Y_1, \cdots, Y_r]$ such that $p(y_1, \cdots, y_r) = 0$. Write $p = q + t$ where q is the homogeneous part of p of highest total degree d. Since η takes $q(x_1, \cdots, x_r)$ onto $q(y_1, \cdots, y_r)$ plus an element $u(y_1, \cdots, y_r)$ whose filter is less than d, there is a polynomial h of degree less than d such that η takes $h(x_1, \cdots, x_r)$ onto $t(y_1, \cdots, y_r) - u(y_1, \cdots, y_r)$. Since η is an isomorphism $(q + h)(x_1, \cdots, x_r) = 0$. This contradicts the independence of the x_i since $q + h \neq 0$.

Since the Killing form of L is nondegenerate there is an induced isomorphism between L and L^* which extends to an L-module algebra isomorphism between S_L and S_{L^*}. Hence there is an induced algebra isomorphism between I_L and I_L^*. Viewing S_{L^*} as the ring of polynomial functions on L, one gets by restriction to H an epimorphism $\rho: S_{L^*} \to S_{H^*}$ which injects I_L^* into I_W ([2], 126). The remainder is concerned with producing algebraically independent generating sets of I_W and exhibiting how they lift back to I_L^*. Chevalley's isomorphism ($I_L^* \cong I_W$) is thus proven as well as the theorems that Z and I_W are polynomial rings in l independent variables.

2. Simple algebras of type A. Let L be simple of type A_l. View L as the Lie algebra of trace zero endomorphisms of $V = K^{l+1}$, and identify L with its matrices with respect to standard basis vectors $e_1, \cdots, e_{l+1}$. Let H be the Cartan subalgebra of diagonal matrices of trace zero and let $\epsilon_1, \cdots, \epsilon_{l+1}$ be functionals on H given by $(e_i)h = \epsilon_i(h)e_i$ for h in H. Then the ϵ_j generate $H^*, \Sigma_j \epsilon_j = 0$, and W acts as the symmetric group on the ϵ_j. ([3], 136 and [1] 205–207, 250–251). Let A be an $l+1$-dimensional auxiliary space with basis $\bar{\epsilon}_1, \cdots, \bar{\epsilon}_{l+1}$ on which W acts as the symmetric group. There is a W-epimorphism $A \to H^*$ taking $\bar{\epsilon}_i$ to ϵ_i which extends to a W-epimorpism $S_A \to S_{H^*}$. Hence there is an induced epimorphism $\bar{I}_W \to I_W$ where $\bar{I}_W$ is the set of W-invariants in S_A. Now $\bar{I}_W$ is generated by the (algebraically independent) elementary symmetric functions in $\bar{\epsilon}_1, \cdots, \bar{\epsilon}_{l+1}$. The kernel of $\bar{I}_W \to I_W$ is easily seen to be generated by $\Sigma_j \bar{\epsilon}_j$, thus I_W is generated by the algebraically independent elementary symmetric functions $s_2, \cdots, s_{l+1}$ in $\epsilon_1, \cdots, \epsilon_{l+1}$ — the analysis being identical to the situation $K[X_1, \cdots, X_{l+1}] \to K[X_1, \cdots, X_l]$ where X_{l+1} goes to zero. Unfortunately the symmetric functions do not lift easily. Due to Newton's identities, however, $I_W = K[p_2, \cdots, p_{l+1}]$ where $p_i = \epsilon_1^i + \cdots + \epsilon_{l+1}^i$ and the p_i do lift easily. They are algebraically independent since they generate a ring known to have transcendence degree equal to l. Now let F_i in I_L^* be given by $F_i(x) = \mathrm{tr}(x_V)^i$. Then $F_i^\rho = p_i$ and $\rho: I_L^* \to I_W$ is surjective. Under the isomorphisms $Z \simeq I_L \simeq I_L^*$ the element z_k of Z corresponding to F_k is given by

$$(1) \qquad z_k = \sum_{i_1 \cdots, i_k = 1}^{n} \mathrm{tr}(u_{i_1} \cdots u_{i_k})_V \, u^{i_1} \cdots u^{i_k}$$

where $\{u_i\}, \{u^i\}$ are dual bases of L with respect to its Killing form. By Lemma 1 and the discussion in §1 $Z = K[z_2, \cdots, z_{l+1}]$ and the z_k are algebraically independent.

3. Simple algebras of type B. Let L be a simple algebra of type B_l. Let V be a $(2l+1)$-dimensional space with basis $e_1, \cdots, e_{2l+1}$, and define a non-degenerate symmetric form on V by $B(e_1, e_1) = 1 = B(e_i, e_{i+l}) = B(e_{i+l}, e_i)$ $i = 2, \cdots, l+1$ and $B(e_j, e_k) = 0$ otherwise. View L as the Lie algebra of all endomorphisms of V which are skew with respect to this form and identify L with its matrices with respect to the e_i. Let H be the Cartan subalgebra of diagonal matrices in L, and let $\epsilon_1, \cdots, \epsilon_l$ be functionals on H given by $(e_i)h = \epsilon_i(h)e_i$ for h in H (**[3]**, 138). Then $\{\epsilon_k\}_k$ is a basis of H^* and W is the semidirect product of the symmetric group S_l on $\epsilon_1, \cdots, \epsilon_l$ with $(\mathbf{Z}/2\mathbf{Z})^l$ acting by $\epsilon_i \rightarrow (\pm 1)_i \epsilon_i$. Thus I_W consists of symmetric functions in $\epsilon_1^2, \cdots, \epsilon_l^2$ (**[2]**, 202 and 252). By Newton's identities $I_W = k[p_1, \cdots, p_l]$ where $p_i = \epsilon_1^{2i} + \cdots + \epsilon_l^{2i}$. Since $\epsilon_1^2, \cdots, \epsilon_l^2$ are algebraically independent, so are the p_i. Let F_i in I_L^* be given by $F_i(x) = \operatorname{tr}(x_V)^{2i}$. Then $F_i^\rho = 2p_i$ and ρ is onto. $Z = K[z_2, z_4, \cdots, z_{2l}]$ where the z_{2k} are as in (1).

4. Simple algebras of type C. Let L be simple of type C_l. Let V be a $2l$-dimensional space with basis $e_1, \cdots, e_{2l}$, and define a nondegenerate skew form on V by $B(e_i, e_{i+l}) = 1 = -B(e_{i+l}, e_i)$ $i = 1, \cdots, l$ and $B(e_j, e_k) = 0$ otherwise. View L as the Lie algebra of endomorphisms which are skew with respect to this form, and identify L with its matrices with respect to the e_i. Let H be the Cartan subalgebra of diagonal matrices in L, and let $\epsilon_1, \cdots, \epsilon_l$ be functionals on H given by $(e_i)h = \epsilon_i(h)e_i$ when h is H (**[3]**, 139). Then $\epsilon_1, \cdots, \epsilon_l$ is a basis of H^*, W acts just as in the preceding case, and I_W consists of symmetric functions in $\epsilon_1^2, \cdots, \epsilon_l^2$ (**[2]**, 204 & 254). As before one sees ρ is onto and $Z = K[z_2, \cdots, z_{2l}]$ where z_{2k} is as in (1).

5. Simple algebras of type D. Let L be simple of type D_l. Let V be a $2l$-dimensional space with basis $e_1, \cdots, e_{2l}$, and define a nondegenerate symmetric form on V by $B(e_i, e_{i+l}) = 1 = B(e_{i+l}, e_i)$ $i = 1, \cdots, l$ and $B(e_j, e_k) = 0$ otherwise. View L as the Lie algebra of endomorphisms of V which are skew with respect to this form and identify L with it matrices with respect to the e_j. Let H be the Cartan subalgebra of diagonal matrices in L and let $\epsilon_1, \cdots, \epsilon_l$ be functionals on H given by $(e_i)h = \epsilon_i(h)e_i$ when h is in H (**[3]**, 140). Then $\epsilon_1, \cdots, \epsilon_l$ is a basis of H^* and W is the semi-direct product of the symmetric group S_l acting as before with $(\mathbf{Z}/2\mathbf{Z})^{l-1}$ acting by $\epsilon_i \rightarrow (\pm 1)_i \epsilon_i$ where $\Pi_i(\pm 1)_i = 1$ (**[2]**, 208 and 256). Thus I_W consists of polynomials in the elementary symmetric functions in $\epsilon_1^2, \cdots, \epsilon_l^2$ and the function $\epsilon_1 \cdots \epsilon_l$. Let s_k be the kth elementary symmetric function in the ϵ_i^2 and let $t = \epsilon_1 \cdots \epsilon_l$. Since $s_l = t^2$, one has $I_W = K[s_1, \cdots, s_{l-1}, t]$. If $s_1, \cdots, s_{l-1}, t$ were algebraically dependent, by an even-odd degree argument there would be a relation in which every monomial has t to an even power, or every monomial has t to an odd power. If the relation is of the second type multiply it by t to make it of the first type. But a relation of the first type is impossible since the elementary symmetric function in the ϵ_i^2 are algebraically independent. Thus I_W is a polynomial ring in l independent variables. By Newton's identities $I_W = K[p_1, \cdots, p_{l-1}, t]$ where $p_i = \epsilon_1^{2i} + \cdots + \epsilon_l^{2i}$. These generators are also algebraically independent since there are l of them in a ring known to have transcendence degree equal to l. As before $2p_i$ lifts back to I_L^* as $\mathrm{tr}(\ )_V^{2i}$, and it is easy to check that $t = \epsilon_1 \cdots \epsilon_l$ lifts back to I_L^* as $pf(\ )_V$ — the pfaffian. Thus ρ is onto and $Z = K[z_2, z_4, \cdots, z_{2l-2}, w]$ where the z_{2k} are as in (1) and w corresponds to $pf(\ )_V$ under $Z \simeq I_L \simeq I_L^*$.

REMARK. Dual bases of L with respect to its Killing Form can be explicitly constructed ([**3**], 246; $h^i = h_{\lambda_i}$ where the λ_i are the fundamental weights). According to part VI of Planche I–IV ([**1**], 250–258) the coefficients q_{ij} of the equations $\lambda_i = \Sigma_j q_{ij}\alpha_j$ $(\alpha_1, \cdots, \alpha_l$ a simple root system) are known, thus enabling one to express h^i explicitly as a $\mathbf{Q}$-linear combination of the h_i.

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Received May 15, 1974. The research presented here was supported in part by the National Science Foundation under grant GP–43814.

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