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CO-RADICAL EXTENSION OF *PI* RINGS

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Throughout this paper R will denote any associative ring (without necessarily 1) with a fixed subring A such that for each element x of R, there is a polynomial $g_x(t)$ (depending on x) having integral coefficients so that the element $x - x^2 \cdot g(x)$ must be in A, say, R is a *co-radical extension of the ring A*, or R *is co-radical over A*. In this paper it is shown that if A is *PI* (ring with polynomial identity) then so must be R.

Prime examples of co-radical extensions are the rings R which are co-radical over their centers $Z = Z(R)$ studied by I. N. Herstein, and the algebras R over finite fields where $A = A(R)$ is the subring generated by both the nilpotent and transcendental elements.

Essential to the paper will be both the techniques used by B. Felzenswalb, and by Herstein and L. Rowen in the study of the radical situation, that is, for each $x \in R$, $x^{n(x)} \in A$, and a recent commutativity result asserting that for any ring R the centralizer of the subring A (no *PI* assumption) must be precisely the center $Z = Z(R)$ of R.

Conventions. The center of the ring R is denoted by $Z = Z(R)$. The centralizer of the subring A in the ring R is denoted by $C_R(A)$ ($= \{a, a \in R, xa = ax, \text{ all } xeR\}$). All polynomial $g_x(t)$ considered here are polynomials with integer coefficients.

LEMMA 1. *All nilpotent elements of the ring R must be in A* (no assumption on char (R)).

Proof. Given any $x \in R$ and any $k \geqq 1$ we claim that we can find a polynomial with integer coefficients, $g_{k,x}(t)$, so that $x - x^{2^k} \cdot g_{k,x}(x) \in A$. If $k = 1$, the assertion is just our basic assumption. If true for k, then the assertion is true for $k+1$. In fact let $x_k = x^{2k} \cdot g_{k,x}(x)$. We can find $g_1(t)$ so that $x_k - (x_k)^2 \cdot g_1(x_k) \in A$. Combining these relations we obtain $x - x^{2^{k+1}} \cdot g_{k+1,x}(x) \in A$, where $g_{k+1,x}(t) = g^2_{k,x}(t) \cdot g_1(t^{2^k} \cdot g_{k,x}(t))$. It is now evident that if x is nilpotent, then $x \in A$.

LEMMA 2. *Let $a, b, x, y \in R$ with $xy = 0$ and $aAb = 0$. Then $ayRxb = 0$.*

Proof. It is clear that yRx is nil, so, by Lemma 1, must be contained in A. Thus $ayRxb \subseteq aAb = 0$, and $ayRxb = 0$ follows.

LEMMA 3. *Suppose that R is either torsion-free or has characteristic p, where p is a prime number. If R has no nilpotent ideals, then A has no nilpotent ideals.*

Proof. It suffices to show that $aAa = 0$ with $a^2 = 0$ implies $a = 0$.

Let $x \in R$. Since $x - x^2 \cdot g(x) \in A$, $a(x - x^2 g(x))a = 0$. Amongst all nonconstant polynomials $h(t)$ (with integers or integers modulo p coefficients) satisfying the relation

$$(ax)^r ah(x)a = 0, \quad \text{some} \quad r \geqq 1,$$

choose one $h(t)$ having least number of nonzero coefficients. Let $h(t) = r_0 + r_1 t + \cdots + r_k t^k$, $r_k \neq 0$. We have for some r, $(ax)^r a(h(x) - r_0)a = (ax)^r ah(x)a - (ax)^r r_0 a^2 = 0$. By the choice of $h(t)$, $r_0 = 0$. Suppose that $k > 1$. For $y = (r_1 + r_2 x + \cdots + r_k x^{k-1})a$, $((ax)^r ax)y = (ax)^r ah(x)a = 0$. Setting $z = (ax)^r ax$, we get $z \cdot y = 0$. In view of Lemma 2, $ayRza = 0$, so, $zayRzay = 0$; consequently, $zay = 0$, that is, $(ax)^{r+1}a \times (r_1 + r_2 x + \cdots + r_k x^{k-1})a = 0$. Since $r_1 a^2 = 0$, this gives $(ax)^{r+1}a \times (r_2 x + \cdots + r_k x^{h-1})a = 0$. If $h_1(t) = r_2 t + \cdots + r_k x^{k-1}$, but $r_1 \neq 0$, we get then a polynomial with fewer number of nonzero coefficients, a contradiction. We must conclude that $r_1 = 0$. Repeating eventually $k - 2$ times, we see that $h(t) = r_k t^k$, $r_k \neq 0$, telling us that $r_k (ax)^r \cdot ax^k a = 0$, that is, $(ax)^r \cdot ax^k a = 0$. Choose k minimal for the relation $(ax)^r \cdot ax^k a = 0$, some $r \geqq 1$. If $k \geqq 1$, then repeating the argument above for $z = (ax)^r ax$, $y_1 = x^{k-1}a$ $(zy_1 = 0)$, we get $zay_1 = (ax)^{r+1} ax^{k-1}a$, a contradiction. We must conclude that $(ax)^r axa = 0$, some $r \geqq 1$. From this ax is nilpotent, all $x \in R$.

Since aR is nil, xaR is nil. By Lemma 1, $xaR \subseteq A$, all $x \in R$. It follows that $RaR \subseteq A$, whence $aRaRa = 0$, so, $(aR)^3 = 0$. From this $aR = 0$, and $a = 0$ follows. With this the lemma is proved.

Our next lemma deals with the prime case. Here, again, the assumption on char R is automatically verified. Since any prime ring R has evidently no nilpotent ideals Lemma 3 yields that A must be also with no nilpotent ideals, and it is now convenient to get the primeness of the ring A. This is the

LEMMA 4. *If R is prime, then A is also prime.*

Proof. Let $a, b \in A$ with $aAb = 0$. We have $abAab \subseteq aAb = 0$, so, $abAab = 0$, whence $ab = 0$.

Let $x \in A$. For some polynomial $g_x(t)$, $x - x^2 \cdot g_x(x) \in A$, whence $a(x - x^2 \cdot g_x(x))b = 0$. Amongst these nonconstant polynomials with integer or integers modulo p coefficients, choose one $h(t)$ with the least number of nonzero coefficients. Let $h(t) = r_0 + r_1 t + \cdots + r_k t^k$,

$r_k \neq 0$. We have $a(h(x) - r_0)b = 0$. By the choice of $h(x)$, $r_0 = 0$. Suppose that $k > 1$. Let $y = (r_1 + r_2x + \cdots + r_kx^{k-1})b$, and let $z = ax$. We have $zy = ah(x)b = 0$. By Lemma 2, $ayRzb = 0$. Since R is prime, $ay = 0$ or $zb = 0$, which is to say, $a(r_1 + r_2x + \cdots + r_kx^{k-1})b = 0$ or $axb = 0$. If $axb = 0$, then by the choice of $h(t)$, $r_0 = r_1 = \cdots = r_{k-1} = 0$. If $axb \neq 0$, then $a(r_1 + r_2x + \cdots + r_kx^{k-1})b = 0$. Because $ab = 0$, this gives $a(r_2x + \cdots + r_kx^{k-1})b = 0$. By the choice of $h(t)$, $r_1 = 0$. All in all, we have seen that if $k > 1$, then $r_0 = r_1 = 0$. Repeating enough times we get that $r_0 = r_1 = \cdots = r_{k-1} = 0$, so that $h(t) = r_kt^k$, and hence $r_kax^kb = 0$. From this $ax^kb = b = 0$. Choose k minimum for this relation. If $k > 1$, setting $z = ax^{k-1}$, $y_1 = xb$ we get $z_1y_1 = 0$. Repeating the argument above we see that $ay_1 = axb = 0$ or $z_1a = ax^{k-1}a = 0$. By the choice of k, $k = 1$ necessarily. This means that $axb = 0$. Since this holds for all $x \in R$, $a = 0$ or $b = 0$ follows. The lemma is proved.

We are now in a position to show our final result.

THEOREM. *Let R be a ring with a fixed subring A. If R is co-radical over A (in the sense that $x - x^2 \cdot g(x) \in A$ for all $x \in R$, where $g(t)$ is a polynomial with integer coefficients depending on x) and if A satisfies a polynomial identity of coefficients ± 1 of degree d, then R satisfies a polynomial identity of coefficients ± 1 of degree at most d^2.*

Proof. First we reduce to the case where R has nilpotent ideals. In fact let N be the sum of all nilpotent ideals of R. Since N is certainly nil, $N \subseteq A$, follows (Lemma 1). Hence N satisfies the polynomial identity in A. If we could prove that the factor ring R/N satisfies the standard identity of degree d, we would be then done.

Our second reduction will be to show that if R is a prime ring as in Theorem, then R satisfies the standard identity of degree d. The reduction follows immediately from the well-known fact that if R has no nilpotent ideals, then R is a subdirect product of prime rings. Summarizing, the theorem reduces to showing that if R is a prime co-radical extension of a PI-subring A of degree d, then R satisfies the standard identity of degree d.

By Lemma 4, A is a prime PI-ring. By a well-known result, A satisfies the standard identity of degree d. Perform the standard multilinear polynomial $p(t_1, t_2, \cdots, t_d) = [t_1, \cdots, t_d]$ on $b_1, b_2, \cdots, b_d$, where the b_i's are in the subring $\tilde{A}$ generated by A and the center Z of R. Since a typical element b in $\tilde{A}$ is a linear combination of elements in A with coefficients λ integers or in Z, by the multilinearity of the polynomial $[t_1, \cdots, t_d]$, we get $[b_1, \cdots, b_d] = \Sigma_{i=(i_1,\cdots,i_d)} \lambda_i [a_{i_1}, \cdots, a_{i_d}]$, so, $\tilde{A}$ satisfies the standard identity of degree d. Since R is evidently co-radical over $\tilde{A}$, $\tilde{A}$ is prime. This shows that without loss of generality $A \supseteq Z$.

Let $B = \{b, ba = ab$, all $a \in A\} = C_R(A)$. Given any $x \in R$, we have $x-x^2\cdot g(x)\in A$, so, b commutes with $x-x^2g(x)$. By **[1**, Theorem 1], $b\in Z$, all $b\in B$. Thus $C_R(A)=Z$. In particular the center of A is precisely the center Z of R.

Now the center of a prime *PI* ring is not trivial. Since A is a prime *PI* ring, Z is then not trivial. Localize R with respect to $Z^*=Z-\{0\}$. Let $\bar{R}$ be the ring of fractions of R, let $\bar{A}$ be the expansion of A, and let $\bar{B}$ be the expansion of $C_R(A)$. Since we localized with respect to Z, the center of A, it is known that $\bar{A}$ is a simple finite dimensional algebra over its center $\bar{Z}=(Z\cdot z^{-1}, z\neq 0, z\in Z)$. (Formanek, Posner). Let $\bar{x}=x\cdot z^{-1}\in C_{\bar{R}}(\bar{A})$, $x\in R$. It is clear that x centralizes $\bar{A}$, whence $x\in C_R(A)=Z$, which forces $\bar{x}$ to be in $\bar{Z}$. Thus, again, $C_{\bar{R}}(\bar{A})=\bar{Z}$. All in all, $\bar{A}$ is a central simple finite dimensional algebra over the field $\bar{Z}$, the center of the over-algebra $\bar{R}$, and $C_{\bar{R}}(\bar{A})=\bar{Z}$. It follows that $\bar{R}\approx\bar{A}\otimes_{\bar{Z}}C_{\bar{R}}(\bar{A})=\bar{A}\otimes_{\bar{Z}}\bar{Z}\approx\bar{A}$, and $\bar{R}$ satisfies the standard identity of degree d. Hence $R\subseteq\bar{R}$ satisfies the standard identity of degree d. With this the theorem is now proved.

To conclude let us observe that the transfer properties in Lemmas 2 and 3 can be reversed and established in the general set up (à la C. Faith) of algebras R over commutative rings with 1, in which, given any $x\in R$, $x-x^2\cdot g_x(x)\in A$, where $g_x(t)$ is a polynomial over Φ. But obviously Theorem is false under this setting. What makes the case $\Phi=\mathbf{Z}$, (the integers) work is, as the reader has already guessed, that for such choice of Φ we have at our disposal the commutativity fact that if $a\in R$ commutes with all $x-x^2\cdot g_x(x)$, x ranging over R, then $a\in Z(R)$ in, at least, the prime case. Under the latter assumption, we can certainly extend Theorem from rings to Φ-rings R.

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