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ON THE CONVOLUTION ALGEBRAS OF H-INVARIANT MEASURES

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The totality $M(eSe/H)$ of bounded regular Borel measures on the orbit space eSe/H, where S is a locally compact semigroup and H is a compact subgroup with the identity e, forms a Banach space; however, its closed subspace $M_H(ESe/H)$ of H-invariant measures forms even a Banach algebra under a suitable convolution. Furthermore, if w is an idempotent probability measure with compact support on S, then $w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(eSe/H)$ algebraically and in various topologies, where w_H is the normalized Haar measure on some compact subgroup H.

1. Introduction. We denote the Banach space of bounded regular Borel measures and the totality of probability measures on a locally compact (Hausdorff) space X by $M(X)$ and $P(X)$, respectively. Beside the norm topology, $M(X)$ may be equipped with the weak, weak* and vague topologies, which are the topologies of pointwise convergence on $C^b(X)$, $C_0(X)$ and $K(X)$, respectively, where $C^b(X)$ denotes the totality of bounded continuous functions on X, $C_0(X)$ and $K(X)$ the subspaces of functions vanishing at ∞ and functions with compact supports, respectively. In $P(X)$, the weak, weak* and vague topologies coincide (p. 59, **[2]**; **[7]**). Let S be a locally compact semigroup, then $M(S)$ is a Banach algebra and $P(S)$ a topological (Hausdorff) semigroup under the convolution $*$. We refer to **[7]** for the continuity of $*$ in the weak, weak* and vague topologies.

LEMMA 1.1. *Let S be a locally compact semigroup. Then* $\operatorname{supp}(\mu * \nu) \subseteq (\operatorname{supp}(\mu)\operatorname{supp}(\nu))^-$ *for $\mu, \nu \in M(S)$, and equality holds for $\mu, \nu \geqq 0$, where* $\operatorname{supp}(\mu)$ *denotes the support of μ.*

Proof. (Cf. 1.1, p. 686, **[5]**).

LEMMA 1.2. *Let $\alpha: X \to Y$ be a continuous map* (resp. morphism) *between locally compact spaces* (resp. semigroups). *Then $M(\alpha)$: $M(X) \to M(Y)$ given by*

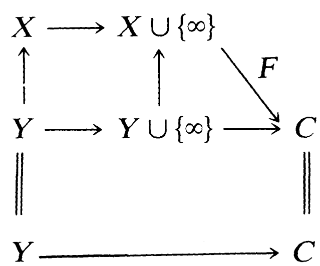
$$[M(\alpha)(\mu)][f) = \mu(f \circ \alpha), \qquad f \in C^b(Y)$$

is a norm-decreasing linear morphism (resp. algebra morphism) *continuous in the weak topology. Moreover, if α is proper, then $M(\alpha)$ is also continuous in both weak* and vague topologies.*

Proof. Straightforward.

LEMMA 1.3. *Let Y be a closed subspace of a locally compact space X. Then every $f \in K(Y)$* (resp. $f \in C_0(Y)$) *has an extension $F \in K(X)$* (resp. $F \in C_0(X)$).

Proof. This follows from (7.40, p. 99, **[1]**) and the following commutative diagram:



$(f(\infty)=0)$.

PROPOSITION 1.4. *Let S be a locally compact semigroup and $e^2 = e \in S$. Then $\delta_e * M(S) * \delta_e = M(eSe)$ is a Banach subalgebra of $M(S)$. In fact, if $i\colon eSe \to S$ is the inclusion map, then $M(i)\colon M(eSe) \to M(S)$ is an embedding.* (Note that, unless mentioned otherwise, our statements are to apply to each of the topologies mentioned before.)

Proof. We first observe from Lemma 1.1 that $\delta_e * M(S) * \delta_e \subseteq M(eSe)$ and that δ_e is the identity for $M(eSe)$, whence $M(eSe) = \delta_e * M(eSe) * \delta_e \subseteq \delta_e * M(S) * \delta_e$ and thus $\delta_e * M(S) * \delta_e = M(eSe)$. Since $\mu \mapsto \delta_e * \mu * \delta_e$ is a Banach space linear retraction, $M(eSe)$ is a linear closed norm retract of $M(S)$. As to the others, we will show the weak embedding only. Let $M(i)(\mu_\alpha) \xrightarrow{w} M(i)(\mu)$ in $M(S)$ and $f \in C^b(eSe)$; then f has an extension $F \in C^b(S)$ given by $F(s) = f(ese)$ and thus $\mu_\alpha(f) = [M(i)(\mu_\alpha)](F) \to [M(i)(\mu)](F) = \mu(f)$. Hence $M(i)$ is an embedding.

For the purpose of this paper it is therefore no loss of generality to assume that S is a monoid with the identity e.

PROPOSITION 1.5. *Suppose that S acts on the left on a locally compact space X. If $\mu \in M(X)$ and $f \in C^b(X)$, then $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$.*

Proof. Let $\epsilon > 0$ be given. By the regularity of $|\mu|$, there exists a compact subset $K \subseteq X$ so that $|\mu|(X\backslash K) < \epsilon$. For this K and a given $s \in S$, let

$$\varphi(t) = \sup\{|f(tx) - f(sx)|: x \in K\}.$$

Then $\varphi(t) \to 0$ as $t \to s$; otherwise, there exist nets $t_\alpha \to s$, and $x_\alpha \to x_0$ in K so that $|f(t_\alpha x_\alpha) - f(sx_\alpha)| > \epsilon$ which contradicts to the continuity of f at sx_0. Hence

$$|f_\mu(t) - f_\mu(s)| \leqq \int_K \varphi(t)|\mu|(dx) + \int_{X\backslash K} 2\|f\||\mu|(dx)$$
$$\leqq \varphi(t)|\mu|(K) + 2\|f\|\epsilon \leqq 3\|f\|\epsilon$$

whenever t is close enough to s. Hence $f_\mu \in C^b(S)$.

2. *H*-invariant measures. Let H be any compact group acting on the left on a locally compact space X. A $\mu \in M(X)$ is called H-invariant if $\int f(hx)\mu(dx) = \int f(x)\mu(dx)$ for all $f \in C^b(X)$, $h \in H$. For convenience, we will denote by $M_H(X)$ the Banach subspace of all H-invariant measures in $M(X)$. We now assume that S acts on the left on X and H is a compact subgroup of units in S. Suppose now that $f \in C^b(X)$ and $\mu \in M_H(X)$. By Proposition 1.4, $f_\mu \in C^b(S)$ is well defined by $f_\mu(s) = \int f(sx)\mu(dx)$. If we set $(fs)(x) = f(sx)$, then we note that $f_\mu(sh) = \int (fs)(hx)\mu(dx) = \mu(fs) = \int f(sx)\mu(dx) = f_\mu(s)$ for all $h \in H$. Hence f_μ is constant on left cosets sH in S. If $S/H = \{sH: s \in S\}$ and $p: S \to S/H$ is given by $p(s) = sH$, then $F \mapsto F \circ p$: $C^b(S/H) \to C^b(S)$ is an isometry onto $C^b_H(S)$ of all functions which are constant on orbits sH. Hence there is a unique function $\widetilde{f_\mu} \in C^b(S/H)$ such that $\widetilde{f_\mu} \circ p = f_\mu$. If now $\mu \in M_H(S/H)$ and $\nu \in M_H(X)$, then we define

$$\mu * \nu(f) = \mu(\widetilde{f_\nu})$$

on $C^b(X)$, which we will write

$$\mu * \nu(f) = \int f(sx)\mu(d\dot{s})\nu(dx), \qquad \dot{s} = p(s).$$

As $\widetilde{(fh)_\nu} = (\widetilde{f_\nu})h$, we have $\mu * \nu(fh) = \mu(\widetilde{(fh)_\nu}) = \mu(\widetilde{f_\nu}h) = \mu(\widetilde{f_\nu}) = \mu * \nu(f)$, whence $\mu * \nu \in M_H(X)$. In particular, if μ, $\nu \in M_H(S/H)$, then $\mu * \nu \in M_H(S/H)$.

LEMMA 2.1. *$M(p)$: $M(S)\to M(S/H)$ is a norm-decreasing continuous linear morphism mapping $w_H * M(S)$ into $M_H(S/H)$ where w_H is the normalized Haar measure on H.*

Proof. We observe first that $w_H * M(S) \subseteq M_H(S)$ by invariance of w_H, and that $M(p)$ maps $M_H(S)$ into $M_H(S/H)$. And since $M(p)$ is continuous in various topologies, then so is any restriction and corestriction of $M(p)$.

LEMMA 2.2. *$M(p)$ induces norm-preserving bijections $M(S) * w_H \to M(S/H)$ and $w_H * M(S) * w_H \to M_H(S/H)$.*

Proof. It suffices to show bijections only (cf. 2.45, p. 20, **[6]**). (1) Surjectivity: Let $f \in C^b(S)$ and set $f_H = \int f(sh)w_H(dh)$. Then $f_H \in C^b_H(S)$ and hence defines a unique $\widetilde{f_H} \in C^b(S/H)$ such that $\widetilde{f_H} \circ p = f_H$. If now $\nu' \in M(S/H)$, then $f \mapsto \nu'(f_H)$ is a bounded linear functional. Hence there is a $\nu \in M(S)$ with $\nu(f) = \nu'(\widetilde{f_H})$. Now $\nu * w_H(f) = \nu(f_H) = \nu'(\widetilde{(f_H)_H}) = \nu'(\widetilde{f_H}) = \nu(f)$. Thus $\nu * w_H = \nu$, i.e. $\nu \in M(S) * w_H$. Now suppose that even $\nu' \in M_H(S/H)$. Then

$$w_H * \nu(f) = \int f(hx)w_H(dh)\nu(dx) = \int \nu(fh)w_H(dh)$$

$$= \int \nu'(\widetilde{(fh)_H})w_H(dh) = \int \nu'(\widetilde{f_H}h)w_H(dh) = \nu'(\widetilde{f_H})$$

since $\nu' \in M_H(S/H)$. The last term equals $\nu(f_H) = \nu(f)$. Thus $w_H * \nu = \nu$, i.e. $\nu \in w_H * M(S) * w_H$. Now, for $f \in C^b(S/H)$, $[M(p)(\nu)](f) = \nu(f\circ p) = \nu'(\widetilde{(f\circ p)_H})$. But $(f\circ p)_H \circ p = (f \circ p)_H = f\circ p$, whence $f = \widetilde{(f\circ p)_H}$; thus $\nu'(\widetilde{(f\circ p)_H}) = \nu'(f)$. This shows $M(p)(\nu) = \nu'$ in both cases, i.e. $M(S/H)$ is in the image of $M(S) * w_H$ and $M_H(S/H)$ is in the image of $w_H * M(S) * w_H$ under $M(p)$. (2) Injectivity: For μ, $\nu \in M(S) * w_H$, we note that $M(p)(\mu) = M(p)(\nu)$ implies $\mu(f) = [M(p)(\mu)](\widetilde{f_H}) = [M(p)(\nu)](\widetilde{f_H}) = \nu(f)$ for $f \in C^b(S)$, hence $\mu = \nu$.

LEMMA 2.3. *$M(p)$: $w_H * M(S) * w_H \to M_H(S/H)$ is an algebra morphism.*

Proof. First of all, we observe the following facts: (1) For $\mu \in w_H * M(S) * w_H$ and $f \in C^b(S)$, $\mu(f) = [M(p)(\mu)](f_H)$. (2) For $\nu \in w_H * M(S) * w_H$ and $f \in C^b(S/H)$, $f_\nu \in C^b_H(S)$ is well defined by

$$f_{\dot\nu}(x)=\int f(x\dot y)[M(p)(\nu)](dy)=\int f(x\dot y)\dot\nu(dy)$$

$$=\int f\circ p(xy)\nu(dy), \text{ with } \dot\nu = M(p)(\nu).$$

Then, if μ, $\nu\in w_H * M(S) * w_H$ and $f\in C^b(S/H)$, we have

$$[M(p)(\mu*\nu)](f)=\mu*\nu(f\circ p)=\int f\circ p(xy)\mu(dx)\nu(dy)$$

$$=\int f(x\dot y)\mu(dx)[M(p)(\nu)](dy)$$

$$=\mu(f_{\dot\nu})=[M(p)(\mu)](\widetilde{(f_{\dot\nu})_H})$$

$$=[M(p)(\mu)](\tilde f_{\dot\nu})=[M(p)(\mu)*M(p)(\nu)](f).$$

PROPOSITION 2.4. *$M(p)$: $w_H * M(S) * w_H \to M_H(S/H)$ is a norm-preserving algebra isomorphism.*

Proof. It remains to show that $M(p)|\, w_H * M(S) * w_H$ is open which follows from the facts that $\mu(f)=[M(p)(\mu)](f_H)$ for all $\mu\in w_H * M(S) * w_H$, and that $f\in K(S)$ (resp. $f\in C_0(S)$) implies $f_H\in K(S)$ (resp. $f_H\in C_0(S)$) and thus $\tilde f_H\in K(S/H)$ (resp. $\tilde f_H\in C_0(S/H)$).

COROLLARY 2.5. *Let H be normal in S* (2.1, p. 17, **[3]**). *Then $M(p)$: $M(S)\to M(S/H)$ is a continuous algebra morphism mapping $w_H * M(S) * w_H$ isomorphically onto $M_H(S/H)$.*

COROLLARY 2.6. *Let $P_H(S/H)$ denote the totality of H-invariant probability measures in $P(S/H)$. Then $M(p)$: $w_H * P(S) * w_H \to P_H(S/H)$ is an isomorphism.*

In the remainder, we assume that w is an idempotent probability measure with compact support on S; then $w=\mu_E * w_H * \mu_F$ **[4]**.

LEMMA 2.7. *The maps $w * M(S) * w \overset{\alpha}{\underset{\beta}{\rightleftarrows}} w_H * M(S) * w_H$ defined via $\alpha(\mu)=w_H*\mu*w_H$ and $\beta(\nu)=w*\nu*w$ are mutually inverse norm-preserving continuous algebra morphisms so that $\alpha(w)=w_H$ and $\beta(w_H)=w$.*

Proof. The proof in (3.1–2, **[8]**) yields this.

PROPOSITION 2.8.

$$w * M(S) * w \cong w_H * M(S) * w_H \cong M_H(S/H)$$

algebraically and topologically.

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