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GENERATING LARGE INDECOMPOSABLE CONTINUA

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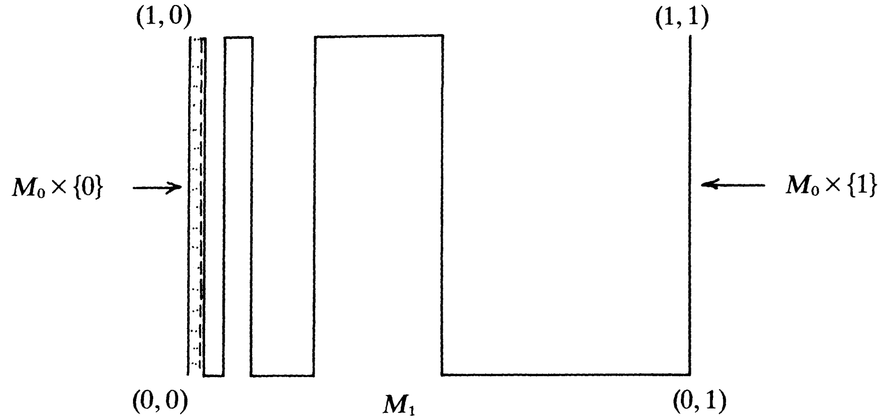
It has been shown by D. P. Bellamy that every metric continuum is homeomorphic to a retract of some metric indecomposable continuum. This result was later extended by G. R. Gordh who proved a similar theorem in the non-metric case. In the present paper a different technique is used to generate such continua.

It is shown that if α is an infinite cardinal number then there is an indecomposable continuum with 2^α composants and if I a (non-metric) continuum then I is homeomorphic to a retract of such a continuum. An indecomposable continuum is constructed such that if C is a composant of it and H is an infinite subset of C then C contains a limit point of H. Finally a non-metric continuum is found so that each proper subcontinuum of it is metric.

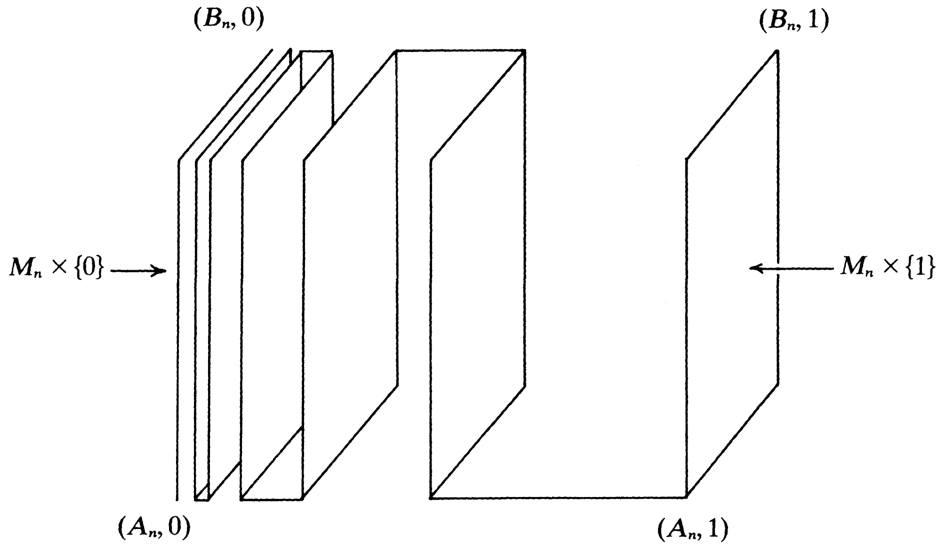
Definitions and notations. A continuum is a compact connected Hausdorff space. Suppose A is a well ordered set, for each $a \in A$ M_a is a topological space, and if $a < b$ in A θ_a^b is a continuous function from M_b onto M_a so that if $a < b < c$ in A then $\theta_a^b \circ \theta_b^c = \theta_a^c$. The space M is the inverse limit $M = \varprojlim\{M_a, \theta\}_{a\in A}$ means that M is the topological space to which the point P belongs if and only if P is a function from A into $\bigcup_{a\in A}\{M_a\}$ so that $P_a \in M_a$ and if $a < b$ in A then $\theta_a^b(P_b) = P_a$. R is a region in M means that there is an element $a \in A$ and an open set $S \subseteq M_a$ so that $R = \{P \mid P_a \in S\}$. $\mathbf{P}_a$ denotes the function from M into M_a so that $\mathbf{P}_a\ (P) = P_a$. If $S = \Pi_{a\in A} S_a$ is a product space then $x = \{x_a\}_{a\in A}$ denotes the point of S so that $x_a \in S_a$, and π_a denotes the function from S into S_a so that $\pi_a(x) = x_a$. If α is an ordinal number $\Pi_{i<\alpha}\,[0, 1]_i$ denotes the cartesian product of α copies of the interval $[0, 1]$. If $M = \varprojlim\{M_a, \theta\}_{a\in A}$ and for each $a \in A$ M_a is a continuum then M is a continuum. Also if for each $a \in A$, M_a is an indecomposable continuum then so is M. For theorems concerning inverse limits the reader should consult **[2]**.

THEOREM 1. *Suppose M is a compact continuum, α is a well ordered set with no last element, M is the inverse limit $M = \varprojlim\{M_a, \theta\}_{a\in\alpha}$ of a collection of Hausdorff continua, and for each $a \in \alpha$ there is a subcontinuum I_a of M_a so that*:

(1) $\theta_a^b(I_b) = M_a$ *for* $a < b$ *in* α, *and*



M_1



M_{n+1}

(2) *if I is a subcontinuum of M_a intersecting I_a and $M_a - I_a$ then I contains I_a.*
Then M is indecomposable.

Proof. Suppose $a \in \alpha$ and P is a point of $M_a - I_a$. Then there is a subcontinuum V of M_a which is irreducible from the point P to I_a. The set $V - I_a \cap V$ is connected and $\overline{V - I_a \cap V} = V$. From condition (2) it follows that $I_a \subseteq V$, so $I_a \subset V = \overline{V - I_a \cap V} \subset \overline{M - I_a}$.

Now suppose M is the union of two proper subcontinua H and K. Let P be a point of H not in K and let Q be a point of K not in H. There exists an element $a \in \alpha$ and mutually exclusive regions R_a and S_a of M_a containing P_a and Q_a respectively so that $R = \{x \mid x_a \in R_a\}$

does not intersect K and $S = \{x \mid x_a \in S_a\}$ does not intersect H. Thus R and S are mutually exclusive open sets in M containing P and Q respectively. It follows from the above and condition (1) that $\theta_a^{(a+1)^{-1}}(R_a)$ and $\theta_a^{(a+1)^{-1}}(S_a)$ are mutually exclusive open sets in M_{a+1} and each intersects both I_{a+1} and $M_{a+1} - I_{a+1}$. So $\mathbf{P}_{a+1}(H)$ and $\mathbf{P}_{a+1}(K)$ both intersect I_{a+1} and $M_{a+1} - I_{a+1}$. So by condition (2) I_{a+1} is a subset of both $\mathbf{P}_{a+1}(H)$ and $\mathbf{P}_{a+1}(K)$. But then $\mathbf{P}_a(K) = M_a = \mathbf{P}_a(H)$, since $\mathbf{P}_a = \theta_a^{a+1} \circ \mathbf{P}_{a+1}$, which is a contradiction. Thus M is indecomposable.

THEOREM 2. *If q is an infinite cardinal number, there is an indecomposable continuum M with 2^q composants.*

Proof. Let α be the first ordinal number so that $|\alpha| = q$. The continuum M will be constructed as an inverse limit of α irreducible continua. Let $M_0 = [0, 1]$. Let M_1 be the subcontinuum of $[0, 1] \times [0, 1]$ so that

$$M_1 = (M_0 \times \{0\}) \cup \left(\bigcup_{i=0}^{\infty} \left[\left(M_0 \times \left\{\frac{1}{2i+1}\right\}\right) \cup \left(\{0\} \times \left[\frac{1}{2i+2}, \frac{1}{2i+1}\right]\right)\right]\right)$$
$$\cup \left(\bigcup_{i=1}^{\infty} \left[\left(M_0 \times \left\{\frac{1}{2i}\right\}\right) \cup \left(\{1\} \times \left[\frac{1}{2i+1}, \frac{1}{2i}\right]\right)\right]\right).$$

The continuum M_1 is the union of countably many copies of M_0 and countably many arcs. If $A_1 = (0, 0)$ and $B_1 = (1, 1)$ then M_1 is irreducible from A_1 to B_1. Let θ_0^1 be the function from M_1 onto M_0 so that $\theta_0^1(P_1, P_2) = P_1$.

Suppose that $b < \alpha$ and that M_a and θ_c^a have been defined for $c < a < b$ so that M_a is a subcontinuum of $\Pi_{i \leqq a} [0, 1]_i$ which is irreducible from the point $A_a = \{0\}_{i \leqq a}$ to the point $B_a = \{1\}_{i \leqq a}$, and θ_c^a is a function from M_a onto M_c so that $\theta_c^a(\{x_i\}_{i \leqq a}) = \{x_i\}_{i \leqq c}$. Suppose that b is not a limit ordinal, $b = a + 1$ for some $a < \alpha$. Let M_b be the subcontinuum of $\Pi_{i \leqq b} [0, 1]_i$ so that

$$[*] \qquad M_b = (M_a \times \{0\}) \cup \left(\bigcup_{i=0}^{\infty} \left[\left(M_a \times \left\{\frac{1}{2i+1}\right\}\right)\right.\right.$$
$$\left.\left. \cup \left(\{A\} \times \left[\frac{1}{2i+2}, \frac{1}{2i+1}\right]\right)\right]\right)$$
$$\cup \left(\bigcup_{i=1}^{\infty} \left[\left(M_a \times \left\{\frac{1}{2i}\right\}\right) \cup \left(\{B_a\} \times \left[\frac{1}{2i+1}, \frac{1}{2i}\right]\right)\right]\right).$$

The continuum M_b is the union of countably many copies of M_a and countably many arcs. M_b is irreducible from any point of $(M_a \times \{0\})$ to the point $(B_a \times \{1\})$. Let $A_b = A_a \times \{0\}$ and $B_b = B_a \times \{1\}$. Let θ_a^b be the function from M_b onto M_a so that if $\{x_i\}_{i \leqq b} \in M_b$ then $\theta_a^b(\{x_i\}_{i \leqq b}) = \{x_i\}_{i \leqq a}$. If $c < a$ define θ_c^b to be the function $\theta_a^b \circ \theta_c^a$.

Suppose that b is a limit ordinal. Let M'_b be the continuum $M'_b = \varprojlim \{M_a, \theta\}_{a<b}$. Let A'_b denote the point P so that $\mathbf{P}_a(P) = A_a$ and let B'_b denote the point P so that $\mathbf{P}_a(P) = B_a$. Then M'_b is irreducible from A'_b to B'_b since for each $a < b$ M_a is irreducible from $\mathbf{P}_a(A'_b)$ to $\mathbf{P}_a(B'_b)$. Let L_b denote the function from M'_b into $\Pi_{i<b}[0, 1]_i$ so that if $P \in M'_b$ then $L_b(P) = \{\pi_i(P_i)\}_{i<b}$ where P_i is the ith coordinate of the point P, $P_i = \mathbf{P}_i(P)$. Note that $\mathbf{P}_i(P) \in M_i \subset \Pi_{k \leqq i}[0, 1]_k$. L_b is a homeomorphism because if P is a point of M'_b and $i < j < b$ then $\pi_a(\mathbf{P}_i(P)) = \pi_a(\mathbf{P}_j(P))$ for all $a \leqq i$; in other words the ath coordinate in the cartesian product $\Pi_{k \leqq i}[0, 1]_k$ of $\mathbf{P}_i(P)$ is the same as the ath coordinate in $\Pi_{k \leqq i}[0, 1]_k$ of $\mathbf{P}_j(P)$. Then $L_b(M'_b) \subset \Pi_{k<b}[0, 1]_k$. M_b is defined by replacing M_a by $L_b(M'_b)$ in [*] above and A_a by $L_b(A'_b)$ and B_a by $L_b(B'_b)$. So M_b is irreducible from any point of $(L_b(M'_b) \times \{0\})$ to the point $(L_b(B'_b) \times \{1\})$. Let $A_b = (L_b(A'_b) \times \{0\})$ and $B_b = (L_b(B'_b) \times \{1\})$. If $a < b$ let θ_a^b be the function from M_b onto M_a so that if $\{x_i\}_{i \leqq b} \in M_b$ then $\theta_a^b(\{x_i\}_{i \leqq b}) = \{x_i\}_{i \leqq a}$. For notational convenience, if b is a limit ordinal let M_{b-1} denote the space $L_b(M'_b)$ and let $\mathbf{P}_{b-1}$ denote the function $f \circ \mathbf{P}_b$ where f projects $L_b(M'_b) \times [0, 1]$ onto $L_b(M'_b) \times \{0\}$.

Let $M = \varprojlim \{M_a, \theta\}_{a<\alpha}$. If for each a, $I_a = M_{a-1} \times \{0\}$ then M and the collection $\{I_a\}_{a<\alpha}$ satisfy the hypothesis of Theorem 1 because M_a is irreducible from the point B_a to each point of I_a. Thus M is indecomposable. If $P \in M$ let P_γ denote $\mathbf{P}_\gamma(P)$. Let L denote the projection L_α as defined above.

Suppose x is a point of M and w_x is the set to which P belongs if and only if there exists a $\beta < \alpha$ so that if $\beta < \gamma < \alpha$ then $\pi_a(P_\gamma) = \pi_a(x_\gamma)$ for all a so that $\beta < a \leqq \gamma$. Equivalently: w_x is the point set to which P belongs if and only if there exists a $\beta < \alpha$ so that $\pi_a(L(P)) = \pi_a(L(x))$ for all $a > \beta$. The set w_x will be shown to be the composant of M containing x.

Suppose $P \in w_x$. Then there exists a $\beta < \alpha$ so that $\pi_a(L(P)) = \pi_a(L(x))$ for all $a > \beta$. Then $\{y \mid y \in M$ and $(y_\gamma)_a = (x_\gamma)_a$ for all a such that $\beta < a \leqq \gamma\}$ is a proper subcontinuum of M containing x and P. The following lemma implies that w_x is a composant.

LEMMA A. *If I is a proper subcontinuum of M containing the point x then there exists a $\beta < \alpha$ so that if $\beta < \gamma < \alpha$ then $\pi_a(\mathbf{P}_\gamma(I)) = \pi_a(x_\gamma)$ for all a so that $\beta < a \leqq \gamma$; (or, there exists a $\beta < \alpha$ so that $\pi_a(L(I)) = \pi_a(L(x))$ for all a so that $\beta < a < \alpha$.)*

Proof. Suppose that I is a subcontinuum of M containing the point x. Then there exists an element $\beta < \alpha$ so that $\mathbf{P}_\beta(I) \neq M_\beta$. Suppose that the lemma is false. Then there exists a first element $a_1 > \beta$ so that $\pi_{a_1}(L(I))$ is non-trivial. Likewise there is a first element a_2 after a_1 and a first element a_3 after a_2 so that $\pi_{a_2}(L(I))$ and $\pi_{a_3}(L(I))$ are non-trivial, $\beta < a_1 < a_2 < a_3$.

Let $\gamma > a_3$. Suppose $0 \in \pi_{a_i}(\mathbf{P}_\gamma(I))$ for some $i = 1, 2, 3$. Then there is a number t distinct from 0 in $\pi_{a_i}(\mathbf{P}_\gamma(I))$. But $\mathbf{P}_\gamma(I)$ intersects $M_{a_i-1} \times \{0\}$ and $M_{a_i} - (M_{a_i-1} \times \{0\})$, so $M_{a_i-1} \times \{0\} \subset \mathbf{P}_{a_i}(I)$. Thus $M_\beta \subset \mathbf{P}_\beta(I)$ which is a contradiction.

Suppose $1 \in \pi_{a_2}(\mathbf{P}_\gamma(I))$. Then there is a number $t < 1$ in $\pi_{a_2}(\mathbf{P}_\gamma(I))$. But there is a number $r \geqq t$ so that $\{A_{a_2-1}\} \times [r, 1] \subseteq \mathbf{P}_{a_2}(I)$, this follows from the construction of M_{a_2}. Then $0 \in \pi_{a_1}(\mathbf{P}_{a_2}(I))$ since $A_{a_2-1} = \{0\}_{i<a_2}$ and this is a contradiction. So $1 \notin \pi_{a_2}(\mathbf{P}_\gamma(I))$. Similarly $1 \notin \pi_{a_3}(\mathbf{P}_\gamma(I))$.

Suppose $0 < t_1 < t_2 < 1$ and $[t_1, t_2] \subset \pi_{a_3}(\mathbf{P}_\gamma(I))$. But $\mathbf{P}_{a_3}(I)$ does not intersect any of the sets $\{\{A_{a_3-1}\} \times [1/(2i+2), 1/(2i+1)]\}_{i=0}^{\infty}$ or any of the sets $\{\{B_{a_3-1}\} \times [1/(2i+1), 1/2i]\}_{i=1}^{\infty}$, or else either 0 or 1 would belong to $\pi_{a_2}(\mathbf{P}_{a_2}(I))$. Then $\mathbf{P}_{a_3}(I)$ must be a subset of $M_{a_3} \times \{1/k\}$ for some integer $k > 1$. But $\pi_a(\mathbf{P}_{a_3}(I)) = \pi_a(\mathbf{P}_\gamma(I))$ for $a \leqq a_3$ so $\pi_{a_3}(\mathbf{P}_\gamma(I)) = 1/k$ which is a contradiction. So the lemma must be true.

LEMMA B. *Suppose q is a cardinal number and α is the first ordinal number so that $q = |\alpha|$. Then there exists a collection G of functions from α into the set $\{0, 1\}$ of cardinality 2^q so that if f and g belong to G then the set $\{x \mid x \in \alpha$ and $f(x) \neq g(x)\}$ is cofinal in α.*

Proof. Let T be a bijection from $\alpha \times \alpha$ onto α. If $a \in \alpha$ then the set $T(\{a\} \times \alpha)$ is cofinal in α. Suppose that S is a subset of α, let f_S be the function from α into $\{0, 1\}$ so that $f_S(t) = 1$ if and only if $t \in T(S \times \alpha)$. Let $G = \{f_S \mid S$ is a subset of $\alpha\}$. Suppose S_1 and S_2 are two distinct subsets of α and a is an element of S_1 not in S_2. Then $f_{S_1}(T(\{a\} \times \alpha)) = 1$ and $f_{S_2}(T(\{a\} \times \alpha)) = 0$ so $\{x \mid x \in \alpha$ and $f_{S_1}(x) \neq f_{S_2}(x)\}$ contains the set $T(\{a\} \times \alpha)$ which is cofinal in α. Thus $|G| = 2^q$ and the lemma is proven.

The continuum M was constructed so that every function from α into the set $\{0, 1\}$ belongs to $L(M)$. If q is a cardinal number and α is the first ordinal number so that $q = 2^{|\alpha|}$ then, by Lemma B, the number of composants of M is at least $2^{|\alpha|}$. If c denotes the cardinality of $[0, 1]$ then M has cardinality at most $c^{|\alpha|}$. But $2^{|\alpha|} = c^{|\alpha|}$, so M has $2^{|\alpha|}$ composants.

Notation: If λ is a limit ordinal let M_λ denote the indecomposable continuum obtained by the construction of Theorem 2 with $\lambda = \alpha$.

COROLLARY 2.1. *If X is a continuum then X is homeomorphic to a retract of an indecomposable continuum with an arbitrarily large number of composants.*

Proof. It follows from the construction in [3] that X is homeomorphic to a retract of an irreducible continuum Y. Then if Y is irreducible from the point A to the point B merely replace M_0 by Y and $\{0\}$ and $\{1\}$ by A and B respectively in the above construction.

COROLLARY 2.2. *There exists a non-metric continuum each proper subcontinuum of which is metric.*

Proof. Consider M_{ω_1}, where ω_1 is the first uncountable ordinal. By Lemma A, if I is a proper subcontinuum of M there is a point $x \in M$ and an element $\beta < \omega_1$ so that $\pi_a(L(I)) = \pi_a(x)$ for all a so that $\beta < a < \omega_1$. Thus $L(I)$ is embedded in $\Pi_{a \leqq \beta}[0, 1]_a \times (\{\pi_a(L(x))\}_{a<\beta})$. So I is homeomorphic to a subset of the cartesian product of countably many intervals and hence is metric. For each $a < \omega_1$ let x_a be the point of $\Pi_{\iota<\omega}[0, 1]_\iota$ which is 1 at the ath coordinate and is 0 elsewhere. Then the set $\{x \mid x = x_a, a < \alpha\}$ is an uncountable set of points in $L(M)$ which contains none of its limit points. Thus $L(M)$ is not metric.

Observation 1. If X is a non-metric continuum and every proper subcontinuum of X is metric then X is indecomposable.

Observation 2. The continuum M_{ω_1} has $2^{\aleph_1}$ composants, and $c \leqq 2^{\aleph_1} \leqq 2^c$. Thus the continuum could have c or 2^c composants depending on which axioms of set theory are assumed. It is also possible that neither equality holds.

COROLLARY 2.3. *There exists a continuum M every proper subcontinuum of which is less numerous than M.*

Proof. Let α be the first ordinal number so that $2^c < 2^{|\alpha|}$, where c is the cardinality of the interval $[0, 1]$. Then if $\beta < \alpha$, $2^{|\beta|} < 2^{|\alpha|}$. Consider the continuum M_α constructed above. M_α contains at least $2^{|\alpha|}$ points. By Lemma A, if I is a proper subcontinuum of M there exists a point $x \in M$ and an element $\beta < \alpha$ so that $\pi_a(L(I)) = \pi_a(x)$ for all a so that $\beta < a < \alpha$. Thus $L(I)$ is embedded in $\Pi_{a \leqq \beta}[0, 1]_a \times (\{\pi_a(L(x))\}_{\beta<\alpha})$. So I has at most $c^{|\beta|}$ points and $c^{|\beta|} \leqq 2^c < 2^{|\alpha|}$. Again observe that any continuum having this property must be indecomposable.

THEOREM 3. *Suppose q is a cardinal number, α is the first ordinal number so that $|\alpha| = q$, and C is a composant of M_α. If $H \subset C$ and $|H| < \alpha$ then $\bar{H} \subset C$.*

Proof. Suppose $H \subset w_x$. It follows from the definition of w_x that there exists a $\beta < \alpha$ so that if $P \in H$ then $\pi_a(L(P)) = \pi_a(L(x))$ for all a

so that $\beta < a < \alpha$. Suppose $Q \in M - w_x$. Then there exists a $\delta > \beta$ so that $\pi_\delta(L(Q)) \neq \pi_\beta(L(x))$. Let S_δ be a region in $[0,1]_\delta$ containing $\pi_\delta(L(\theta))$ and not $\pi_\delta(L(x))$. Then $R = \{Z \mid \pi_\delta(Z) \in S\}$ is an open set in $L(M)$ containing $L(Q)$ but no point of $L(H)$. So $Q \notin \bar{H}$. So $\bar{H} \subset w_x$.

DEFINITION. The subset H of the Hausdorff space X is said to be conditionally compact if and only if it is true that every infinite subset of H has a limit point in H.

COROLLARY 3.1. *There exists a conditionally compact indecomposable connected Hausdorff space with only one composant.*

Proof. By Theorem 3 any composant of M_{ω_1} is such a space.

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