PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 2, 1976

ISOMORPHISMS BETWEEN HARMONIC AND P-HARMONIC HARDY SPACES ON RIEMANN SURFACES

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In this paper we investigate the relationship between the Hardy space $H^q(R)$ of harmonic functions on a hyperbolic Riemann surface R, and the Hardy space $P^q(R)$ of solutions of the equation $\Delta u = Pu$, where $P \geqq 0$, $P \not\equiv 0$ is a C^1-density on R. Under certain conditions these spaces are shown to be canonically isomorphic, although in general this is not the case. However, specific subspaces are found which are isomorphic and their relationship with other function spaces is discussed.

1. Introduction. Hardy spaces on Riemann surfaces have been studied by Heins [2], Schiff [11], in the setting of Φ-bounded functions by Parreau [8], and in the general context of harmonic spaces by Lumer-Naim [4], among others. The present work, in the setting of a hyperbolic Riemann surface R, examines the Hardy space $P^q(R)$ for the equation $\Delta u = Pu$, $P \geqq 0$, $P \not\equiv 0$, which thus falls within the framework of [4]. Hence, the results contained therein will be applicable to our study of the isomorphic relations between the harmonic Hardy space $H^q(R)$ and $P^q(R)$. In general, no such isomorphic relation between $H^q(R)$ and $P^q(R)$ exists, yet when particular subspaces are considered, an isomorphism is shown to indeed exist between the specified subspaces.

In the fourth section we investigate the conditions under which the inclusion relations are strict or not, between the various spaces which have been introduced. Certain isomorphisms are obtained under new conditions in the final section.

Some of the results obtained have natural generalizations to harmonic spaces and to Φ-bounded P-harmonic functions for a convex increasing Φ, however, these aspects of the theory will not be treated here.

2. Preliminaries. Let R be a hyperbolic Riemann surface and $P \geqq 0$, $P \not\equiv 0$ a C^1-density on R. We denote by $PB(R)$ (resp. $HB(R)$) the space of bounded C^2-solutions on R of the elliptic equation $\Delta u = Pu (\Delta u = 0)$, and by $PB'(R)$ (resp. $HB'(R)$) the quasibounded counterpart. A C^2-solution of $\Delta u = Pu$ is called a *P-harmonic* function, and the space of such functions on an open set U of R is denoted by $P(U)$. $H(U)$ denotes the space of harmonic functions on U. A

C^2-function s is *P-superharmonic* if and only if $\Delta s \leqq Ps$. If s is P-superharmonic, $-s$ is *P-subharmonic*. Refer to Royden **[9]** for details concerning P-harmonic functions.

Associated with R is the Wiener compactification R^* of R, the Wiener ideal boundary $\Gamma = R^* - R$, and the Wiener harmonic boundary $\Delta \subset \Gamma$. We will also make use of the following maximum principle for the class $HB'(R)$, the functions of which have continuous extensions to Δ.

PROPOSITION 1. *Let $u \in HB'(R)$ with*

$$m \leqq u \leqq M$$

on Δ. Then $m \leqq u \leqq M$ on R.

The reader is referred to Chapter IV of Sario-Nakai **[10]** for a comprehensive treatment of the Wiener compactification and the HB'-maximum principle.

By a *regular exhaustion* $\{\Omega\}$ of R we mean an exhaustion of R by relatively compact open subsets Ω of R such that $\partial\Omega$ is analytic.

We proceed to define the Hardy space for $\Delta u = Pu$. Let $\{\Omega\}$ be a regular exhaustion of R and $z_0 \in R$ a fixed point. Denote by $\mu_{z_0}^{\Omega}$ the P-harmonic measure on $\partial\Omega$ relative to z_0 and Ω. Clearly $\int_{\partial\Omega} d\mu_{z_0}^{\Omega} \leqq 1$ for all Ω.

DEFINITION. A function $u \in P^q(R)$, $1 \leqq q < \infty$, if for some constant $M \geqq 0$,

$$\| u \|_{q,\Omega} = \left(\int_{\partial\Omega} | u |^q d\mu_{z_0}^{\Omega} \right)^{1/q} \leqq M$$

for all Ω of a regular exhaustion $\{\Omega\}$ of R.

That the definition of $P^q(R)$ is independent of the choice of z_0 or the particular exhaustion, is a consequence of the following (cf. Lumer-Naim **[4]**):

PROPOSITION 2. *$u \in P^q(R)$, $1 \leqq q < \infty$, if and only if $|u|^q$ has a P-harmonic majorant.*

A further result of **[4]** is quoted here as it will be useful in the sequel.

PROPOSITION 3. *Every $u \in P^q(R)$* (resp. $H^q(R)$) *is the difference of two positive P-harmonic* (resp. *harmonic*) *functions in $P^q(R)$* (resp. $H^q(R)$), *$1 \leqq q < \infty$, and conversely.*

3. Isomorphisms. Let Ω be a regular subregion of R and ϕ a continuous function on $\partial\Omega$. We denote by P_ϕ^Ω the function belonging to $P(\Omega)\cap C(\bar{\Omega})$ such that $P_\phi^\Omega \mid \partial\Omega = \phi$. The harmonic counterpart to P_ϕ^Ω is denoted by H_ϕ^Ω.

In this section we assume that $1 \leqq q < \infty$.

Define a linear operator $\lambda_P \colon H^q(R) \to P^q(R)$ by

$$\lambda_P h = \lim_{\Omega \to R} P_h^\Omega .$$

To show that λ_P is well-defined, let $h \in H^q(R)$. Then $h = h_1 - h_2$, $h_i \in H^q(R)$, $h_i \geqq 0$, $i = 1, 2$, by Proposition 3. By the P-harmonic maximum principle (cf. Royden **[9]**), $\{P_{h_i}^\Omega\}$ is decreasing and

$$0 \leqq P_{h_i}^\Omega \leqq h_i$$

on Ω. Letting $\Omega \to R$, $P_{h_i}^\Omega$ converges to a function $P_{h_i} \in P(R)$, and therefore

$$P_h^\Omega = P_{h_1}^\Omega - P_{h_2}^\Omega \to P_{h_1} - P_{h_2} = \lambda_P h \in P(R).$$

Moreover,

$$|\lambda_P h|^q \leqq 2^q(|P_{h_1}|^q + |P_{h_2}|^q) \leqq 2^q(h_1^q + h_2^q) \leqq h'$$

for some $h' \in H(R)$. Since the P-subharmonic function $|\lambda_P h|^q$ has the P-superharmonic majorant, h', a standard Perron family argument shows that $|\lambda_P h|^q$ has a P-harmonic majorant, implying $\lambda_P h \in P^q(R)$.

Note that for $h \in H^q(R)$, $h > 0$, $\lambda_P h$ is the greatest P-harmonic minorant of h, and consequently $h - \lambda_P h$ is a P-potential, i.e. a positive P-superharmonic function whose greatest P-harmonic minorant is zero.

Next, let

$$\Lambda_P^q(R) = \{u \in P(R) \colon |u|^q \leqq h \text{ for some } h \in H(R)\}.$$

The inequality $|t|^p \leqq 1 + |t|^q$ for $1 \leqq p \leqq q < \infty$ implies:

PROPOSITION 4. $\Lambda_P^q(R) \subset \Lambda_P^p(R) \subset \Lambda_P^1(R)$.

The previous Perron family considerations also yield:

PROPOSITION 5. $PB(R) \subset \Lambda_P^q(R) \subset P^q(R)$.

We further establish:

PROPOSITION 6. *If* $u \in \Lambda_P^q(R)$, *then* $u = u_1 - u_2$, $0 \leqq u_i \in \Lambda_P^q(R)$, $i = 1, 2$.

Proof. Since $u \in \Lambda_P^1(R)$ by Proposition 4, $|u|$ has a P-harmonic majorant. Denoting by U^+, U^-, the least P-harmonic majorant of $u^+ = u \cup 0$, and $u^- = (-u) \cup 0$ resp., it follows that $u = U^+ - U^-$. Moreover, $u \in \Lambda_P^q(R)$ implies $|u|^q \leqq h \in H(R)$ and therefore $(u^+)^q \leqq h$.

Consequently

$$(U^+)^q(z_0) = \lim_{\Omega \to R} \left(\int_{\partial\Omega} u^+ \, d\mu_{z_0}^{\Omega} \right)^q \leqq \lim_{\Omega \to R} \int_{\partial\Omega} (u^+)^q d\mu_{z_0}^{\Omega}$$

$$\leqq \lim_{\Omega \to R} \int_{\partial\Omega} h \, d\mu_{z_0}^{\Omega} \leqq h(z_0),$$

where the last inequality follows from the fact that h is P-superharmonic. Hence $U^+ \in \Lambda_P^q(R)$. Similarly $U^- \in \Lambda_P^q(R)$ and thus $u \in \Lambda_P^q(R)$, as desired.

The space $\Lambda_P^q(R)$ is related to $H^q(R)$ in the following way.

THEOREM 1. $\lambda_P(H^q(R)) = \Lambda_P^q(R)$.

Proof. It suffices to show $\lambda_P(H^q(R)) \supset \Lambda_P^q(R)$ since the inclusion $\lambda_P(H^q(R)) \subset \Lambda_P^q(R)$ is obvious from the definition of λ_P.

Let $0 \leqq u \in \Lambda_P^q(R)$. Then $u^q \leqq h$ for some $h \in H(R)$, and since $u \in \Lambda_P^1(R)$, u has a harmonic majorant. Let h_u be the least harmonic majorant of u. Thus for a regular exhaustion $\{\Omega\}$ of R,

$$h_u = \lim_{\Omega \to R} h_u^{\Omega}$$

where $h_u^{\Omega} | \partial\Omega = u | \partial\Omega$, $h_u^{\Omega} \in H(\Omega) \cap C(\bar{\Omega})$. Therefore

$$(h_u^{\Omega})^q | \partial\Omega = u^q | \partial\Omega \leqq h,$$

implying $(h_u^{\Omega})^q \leqq h$ on Ω. Letting $\Omega \to R$ yields

$$h_u^q \leqq h$$

on R, and $h_u \in H^q(R)$.

In view of the fact that $h_u - u$ is a P-potential, u is the greatest P-harmonic minorant of h_u, i.e. $\lambda_P h_u = u$.

For an arbitrary $u \in \Lambda_P^q(R)$, $u = u_1 - u_2$, $0 \leqq u_i \in \Lambda_P^q(R)$, $i = 1, 2$, and we have $h_u = h_{u_1} - h_{u_2}$, which proves the theorem.

We construct a new operator which will turn out to be the inverse of λ_P in certain instances.

Let $u \in P^q(R)$, $u = u_1 - u_2$, $0 \leqq u_i \in P^q(R)$, $i = 1, 2$, and consider the increasing family of harmonic functions $\{H^\Omega_{u_i}\}$. Set

$$H_{u_i} = \lim_{\Omega \to R} H^\Omega_{u_i}$$

whenever the limit exists. In this case

$$H^\Omega_u = H^\Omega_{u_1} - H^\Omega_{u_2} \to H_{u_1} - H_{u_2} \in H(R),$$

and we define

$$\mu_P u = \lim_{\Omega \to R} H^\Omega_u .$$

Observe that for $u > 0$, $\mu_P u$, when it is defined, is the least harmonic majorant of u on R.

PROPOSITION 7. *$\Lambda^q_P(R) \subset$ domain (μ_P) and $\mu_P \colon \Lambda^q_P(R) \to H^q(R)$.*

The proof is analogous to that of Theorem 1.

It is clear that μ_P is linear on $\Lambda^q_P(R)$. Set $M^q_P(R) = \mu_P(\Lambda^q_P(R))$.

THEOREM 2. *$\mu_P \colon \Lambda^q_P(R) \to M^q_P(R)$ is an isomorphism with inverse $\lambda_P \colon M^q_P(R) \to \Lambda^q_P(R)$.*

Proof. Let $u \in \Lambda^q_P(R)$. Then $u = u_1 - u_2$, $0 \leqq u_i \in \Lambda^q_P(R)$, $i = 1, 2$. It follows that $\mu_P u_i \in M^q_P(R)$ and $\mu_P u_i = u_i + p_i$, where p_i is a P-potential. Therefore

$$(\lambda_P \circ \mu_P)(u_i) = \lambda_P(u_i + p_i) = u_i$$

$i = 1, 2$, and hence $\lambda_P \circ \mu_P =$ identity.

$$\begin{tikzcd} M^q_P(R) \arrow[r, "\lambda_P"] & \Lambda^q_P(R) \\ & \Lambda^q_P(R) \arrow[ul, "\mu_P"] \arrow[u, "\lambda_P \circ \mu_P = id."'] \end{tikzcd}$$

We deduce that μ_P is injective, hence an isomorphism, and that λ_P is surjective. Moreover, it is easily seen that λ_P is injective and $\mu_P \circ \lambda_P =$ identity.

4. Examples. In this section we give some examples demonstrating the proper inclusion relations between various function spaces that have been considered.

THEOREM 3. *There is a Riemann surface R and a density P for which $M_P^q(R) \subsetneqq H^q(R)$, $1 \leqq q < \infty$.*

Proof. Take a Riemann surface $R \in \mathcal{O}_{PB} - \mathcal{O}_G$, that is R is hyperbolic, yet $PB(R) = \{0\}$ (cf. e.g. Royden **[9]** for the existence of such a surface). Then for any positive constant c, we have $c \in H^q(R)$. However, there is no greatest P-harmonic minorant of c other than zero, and hence $\lambda_P c = 0$.

Since λ_P is injective on $M_P^q(R)$ by Theorem 2, and $\lambda_P 0 = 0$, we deduce that $c \in H^q(R) - M_P^q(R)$.

The P-harmonic Hardy class $P^q(R)$ is related to other function spaces by the following string of inclusions (cf. Lumer-Naim **[4]**):

PROPOSITION 8. $PB(R) \subset \Lambda_P^q(R) \subset P^q(R) \subset PB'(R)$, *for* $1 < q < \infty$.

By the way of contrast with the harmonic case, observe that in the preceding example, $M_P^q(R) = \{0\}$, and thus $HB(R) \not\subset M_P^q(R)$.

Using a similar proof as for that given in the harmonic case (Schiff **[11]**), it can be shown that:

THEOREM 4. *If* $\dim PB(R) = n < \infty$, *then*

$$PB(R) = \Lambda_P^q(R) = P^q(R) = PB'(R),$$

for $1 < q < \infty$.

In general this may not be the case. Some further preliminaries at this juncture are in order.

It has been demonstrated (cf. eg. Nakai **[7]**, also Royden **[9]**), that the Banach spaces $PB(R)$ and $HB(R)$ are canonically isomorphic, that is, there exists a bijective linear isometry T: $PB(R) \to HB(R)$ such that for $u \in PB(R)$, $|u - Tu|$ is dominated by a potential, whenever the following condition is valid:

$$(*) \qquad \int_{R-K} G(z,\zeta) P(\zeta) d\xi \, d\eta < \infty \qquad (\zeta = \xi + i\eta).$$

Here K is a B-negligible subset of R, that is $R^* - \bar{K}$ is a neighborhood of Δ (cf. Nakai **[7]**), $G(z, \zeta)$ is the harmonic Green's function on R, and $(*)$ is assumed to be valid for one (hence for all) $z \in R$.

In view of Theorem 4 and the preceding remarks, we conclude:

PROPOSITION 9. *If* $\dim HB = n < \infty$, *and condition* $(*)$ *is valid, then* $H^q(R)$ *is canonically isomorphic to* $P^q(R)$, $1 < q < \infty$.

Our next two theorems (nos. 5 and 6) require a result from ideal boundary theory, which is preceded by an abbreviated discussion of pertinent material.

The Wiener compactification R^* of R is generated by the Wiener algebra $W(R)$ of bounded, continuous, harmonizable functions on R. We also have the class of (bounded) Wiener potentials denoted by $W_0(R)$. Refer to Sario-Nakai **[10]**, Chapter IV, for details.

In an analogous vein, associated with the equation $\Delta u = Pu$, we have the Wiener P-compactification R_P^*, generated by the Wiener P-algebra $W_P(R)$. See e.g. Constantinescu-Cornea **[1]**, Tanaka **[13]** for details. Denoting the class of Wiener P-potentials by $W_{0P}(R)$, we obtain (cf. also Maeda **[5]**, Tanaka **[12]**):

PROPOSITION 10. *Under condition* $(*)$, $W(R) = W_P(R)$, *and* $W_0(R) = W_{0P}(R)$.

The proof follows from the fact that as we have seen $(*)$ implies $PB(R)$ is canonically isomorphic to $HB(R)$, which implies h_P, the least harmonic majorant of the elliptic measure is identically one (Lahtinen **[3]**), which in turn yields $W(R) = W_P(R)$ and $W_0(R) = W_{0P}(R)$ (Tanaka **[12]**).

Since $W_0(R)$ and $W_{0P}(R)$ completely determine Δ and Δ_P (the Wiener P-harmonic boundary) respectively, we have the following useful result.

COROLLARY. *Under condition* $(*)$, $\Delta_P = \Delta$.

In the sequel we make use of the fact that every continuous P-potential on R vanishes on Δ_P (cf. Constantinescu-Cornea **[1]**).

THEOREM 5. *On the unit disk* U, *any density* P *satisfying condition* $(*)$ *yields* $PB(U) \subsetneqq \Lambda^q(U)$, $1 < q < \infty$.

Proof. Choose a function $h \in H^q(U) - HB(U)$. Let P be a density satisfying $(*)$, so that $\Delta_P = \Delta$.

For $u = \lambda h \in \Lambda^q(U)$, $h - u$ is a continuous P-potential, implying $h - u \mid \Delta_P = 0$. Hence $h \mid \Delta = u \mid \Delta$, and by Proposition 1, since

$H^q(R) \subset HB'(R)$, h is unbounded on Δ. It follows that u is unbounded and $u \in \Lambda^q(U) - PB(U)$.

COROLLARY. $M_P^q(U) \not\subset HB(U)$, $1 < q < \infty$.

Proof. If $0 < u \in \Lambda_P^q(U) - PB(U)$, then $h' = \mu_P u \in M_P^q(U)$ and $h' \notin HB(U)$.

Thus, in general, there is no inclusion relation between $HB(R)$ and $M_P^q(R)$.

In contradistinction to Theorem 3, $(*)$ is a sufficient condition for $M^q(R)$ to be identical with $H^q(R)$.

THEOREM 6. λ_P: $H^q(R) \to \Lambda_P^q(R)$ *is an isomorphism*, i.e. $M^q(R) = H^q(R)$, *whenever condition* $(*)$ *is valid*, $1 < q < \infty$.

Proof. Suppose $\lambda_P h = 0$ for some $h \in H^q(R)$. Since $h = h_1 - h_2$, $h_i \geqq 0$, $h_i \in H^q(R)$, $i = 1, 2$, we have $\lambda_P h_1 = \lambda_P h_2$. As $h_i - \lambda_P h_i$ is a P-potential, and $\Delta_P = \Delta$, it follows that $h_i | \Delta = \lambda_P h_i | \Delta$. Hence $h_1 | \Delta = h_2 | \Delta$, i.e. $h | \Delta = 0$. Since $H^q(R) \subset HB'(R)$ for $1 < q < \infty$ (cf. Lumer-Naim **[4]**), the maximum principle for $HB'(R)$ implies $h \equiv 0$ on R, and λ_P is injective on $H^q(R)$.

Our final example concerns the inclusion relation $\Lambda_P^q(R) \subset P^q(R)$.

PROPOSITION 11. *There is a Riemann surface R and a density P such that* $\Lambda_P^1(R) \subsetneqq P^1(R)$.

Proof. Let R be the punctured unit disk U_0, and for $|z| = r$, $P = P(r) = 1/r^2$. Then the function $u = u(r) = 1/r$ satisfies $\Delta u = Pu$ on U_0. Moreover, any positive harmonic function h on U_0 must be asymptotic to $-a \log r$, at the origin, for some $a \in [0, \infty)$ (cf. eg. Sario-Nakai **[10]**). Consequently, for any $h \in H(U_0)$, $|u| = 1/r \not\leqq h$ on U_0, i.e. $u \notin \Lambda_P^1(U_0)$. However, $u \in P^1(U_0)$.

Whether there exists a pair (R, P) such that $\Lambda_P^q(R) \subsetneqq P^q(R)$ for $1 < q < \infty$, is as yet unresolved.

5. Extensions. In the sequel we assume $PB(R) \neq \{0\}$.

The Wiener algebra $W(R)$ has the following extension. Let $\mathscr{W}(R)$ be the class of continuous harmonizable functions f on R such that there exists a continuous superharmonic function s_f with a discrete $\{s_f = \infty\}$ and with $|f| \leqq s_f$. Denote by $\mathscr{W}_P(R)$ the P-harmonic analogue (cf. Constantinescu-Cornea **[1]**). Clearly $W(R) \subset \mathscr{W}(R)$ and $W_P(R) \subset \mathscr{W}_P(R)$. Furthermore, define

$$\mathscr{W}_0(R) = \{f \in \mathscr{W}(R) \colon f | \Delta = 0\},$$

and

$$\mathcal{W}_{0P}(R) = \{f \in \mathcal{W}_P(R): f \mid \Delta_P = 0\}.$$

We now establish:

LEMMA. *If $\mathcal{W}(R) = \mathcal{W}_P(R)$, then $\Lambda_P^q(R) = P^q(R)$, $1 \leqq q < \infty$.*

Proof. Suppose not, and let $u \in P^q(R) - \Lambda_P^q(R)$, for some q, $1 \leqq q < \infty$. Then $|u|^q \leqq v$ for some $v \in P(R)$. Since u is also continuous, positive P-subharmonic, $|u|^q \in \mathcal{W}_P(R)$. If moreover, $|u|^q \in \mathcal{W}(R)$, then $|u|^q \leqq s$ for some superharmonic function s on R. Since $|u|^q$ is subharmonic, $|u|^q \leqq h$ for some $h \in H(R)$, implying $u \in \Lambda_P^q(R)$, a contradiction. Thus, $|u|^q \not\in \mathcal{W}(R)$ which yields $\mathcal{W}(R) \neq \mathcal{W}_P(R)$, and the result follows.

THEOREM 7. *If $\mathcal{W}(R) = \mathcal{W}_P(R)$ and $\mathcal{W}_0(R) = \mathcal{W}_{0P}(R)$, then $H^q(R)$ is isomorphic to $P^q(R)$, $1 < q < \infty$.*

Proof. It is easily seen that the hypothesis implies $W(R) = W_P(R)$ and $W_0(R) = W_{0P}(R)$, and consequently, $R^* = R_P^*$ and $\Delta = \Delta_P$. Under the condition that $\Delta = \Delta_P$, Theorem 6 obtains, that is $\lambda_P: H^q(R) \to \Lambda_P^q(R)$ is an isomorphism, for $1 < q < \infty$. The theorem now follows from the lemma.

THEOREM 8. *If $\mathcal{W}(R) = \mathcal{W}_P(R)$ and $\mathcal{W}_0(R) = \mathcal{W}_{0P}(R)$, then $HB'(R)$ is isomorphic to $PB'(R)$.*

Proof. For $u \in PB'(R) \subset \mathcal{W}_P(R) = \mathcal{W}(R)$, we have the decomposition

$$u = v + g$$

where $v \in HB'(R)$ and $g \in \mathcal{W}_0(R)$ (cf. Sario-Nakai **[10]**). We define $T: PB'(R) \to HB'(R)$ by $Tu = v$. Since $\Delta = \Delta_P$ and $u \mid \Delta = Tu \mid \Delta$, it is a simple matter to verify that T is the desired isomorphism.

As we have seen, condition $(*)$ implies $W(R) = W_P(R)$, and the question naturally arises whether or not $(*)$ implies $\mathcal{W}(R) = \mathcal{W}_P(R)$. That this is not the case is seen as follows.

Let $R_\alpha = \{0 < |z| < \alpha < 1\}$, and $u = u(r) = (\log r)^2$, with $r = |z|$. Then $\Delta u = Pu$ for $P = P(r) = 2/(r \log r)^2$. Since u is continuous, P-harmonizable on R_α, $u \in \mathcal{W}_P(R_\alpha)$. Moreover, if u had a superharmonic majorant on R_α, then the subharmonicity of u would imply u had a harmonic majorant h on R_α. As in Proposition 11, h would be asymptotic to $-a \log r$ at $z = 0$, for $a \in [0, \infty)$, which would violate the fact that h must dominate u. We conclude that $u \notin \mathcal{W}(R_\alpha)$.

Finally, a simple calculation shows that

$$\int_{R_\alpha} P\, dx\, dy = \frac{-4\pi}{\log \alpha} < \infty,$$

and hence $\int_{R_\alpha} G(z, \zeta) P(\zeta) d\xi d\eta < \infty$. Therefore (*) is valid, but $\mathcal{W}(R_\alpha) \neq \mathcal{W}_P(R_\alpha)$.

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Received September 29, 1975.

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