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MAXIMAL QUOTIENT RINGS OF RING EXTENSIONS

KENNETH LOUDEN

Using torsion theoretic methods we obtain sufficient conditions on a ring extension $R \to S$ so that $Q_{\max}(S) \cong S \otimes_R Q_{\max}(R)$. This is applied to quasi-Frobenius extensions and group rings, generalizing and unifying several known results.

1. Introduction and preliminaries. In **[1]** Burgess showed that, for a ring A and a group G, $AG \otimes_{AH} Q_{\max}(\mathrm{AH}) \subset \mathrm{Q}_{\max}(AG)$ for every central subgroup H of G, with equality if H is of finite index. Later, Kitamura **[6]** showed that, for a Frobenius extension $R \to S$ such that S is finitely generated over R by elements which centralize R, $S \otimes_R Q_{\max}(R) \cong Q_{\max}(S)$. Finally, Formanek **[3]** recently proved that $Q_{\max}(AH) \subset Q_{\max}(AG)$ when H is a subnormal subgroup of G.

We want to show here that a general torsion-theoretic argument leads to a theorem which can be applied to generalize all of the above results.

We note first that all rings have 1, and all modules are on the right unless stated otherwise. We assume that the reader is familiar with torsion theories, for example the contents of **[12]**, whose notation we will generally follow. For other unexplained terminology we refer to **[7]**.

We begin by considering a general ring homomorphism $\varphi: R \to S$ ($\varphi(1) = 1$) with associated "restriction of scalars" functor φ_*: Mod-S $\to$ Mod-R. Recall that $\_\otimes_R S$ is a left adjoint and $\mathrm{Hom}_R(S, -)$ is a right adjoint of φ_*. Let $\mathcal{F}$ be a topology (idempotent topologizing filter) of right ideals of R. We set

$$\tilde{\mathcal{F}} = \{D \underset{r}{<} S \mid \varphi^{-1}(D) \in \mathcal{F}\}.$$

$\tilde{\mathcal{F}}$ is a filter but not necessarily a topology.

DEFINITION. $\mathcal{F}$ is said to be S-*good* if $\tilde{\mathcal{F}}$ is a topology.

An investigation of S-good topologies was made in **[9]**. In particular, the following useful criterion was found.

PROPOSITION 1. (**[9]**, Theorem 2.5(f).) *$\mathcal{F}$ is S-good if and only if $\varphi_*(M \otimes_R S) = (M \otimes_R S)_R$ is $\mathcal{F}$-torsion whenever M_R is $\mathcal{F}$-torsion.*

Associated to the topology $\mathscr{F}$ on R is a quotient functor on Mod-R, which we will denote by Q. When $\mathscr{F}$ is S-good, there is also a quotient functor on Mod-S associated to $\tilde{\mathscr{F}}$, which we will denote by $\tilde{Q}$. The interest in S-good topologies is the following.

PROPOSITION 2. (**[9]**, Theorem 2.7.) *Let $\mathscr{F}$ be an S-good topology on R, and suppose that ${}_RS$ is flat. Then $\varphi_*\tilde{Q}(M) \cong Q(\varphi_*(M))$ canonically for all $M \in$ Mod-S.*

When $M = S$, Proposition 2 says that the module of quotients of S_R with respect to $\mathscr{F}$ is a ring isomorphic to the ring of quotients of S with respect to $\tilde{\mathscr{F}}$.

Modules of quotients are especially nice when they are given by tensor products.

THEOREM 3. *Let $\mathscr{F}$ be an S-good topolopy on R, and suppose that ${}_RS$ is flat and S_R is projective. Then there is an embedding*

$$S \otimes_R Q(R) \subset \tilde{Q}(S)$$

with equality if S_R is finitely generated.

Proof. Recall that there is a commutative diagram

$$\begin{array}{ccc} & \varphi_*(S) & \\ \swarrow & & \searrow \\ \varphi_*(S) \otimes_R Q(R) & \xrightarrow[\beta]{} & Q(\varphi_*(S)). \end{array}$$

Now $\ker \beta = t(\varphi_*(S) \otimes Q(R))$, the $\mathscr{F}$-torsion submodule of $\varphi_*(S) \otimes Q(R)$ (**[9]**, Proposition 1.1). But $\varphi_*(S) \otimes Q(R)$ is $Q(R)$-projective, hence $\mathscr{F}$-torsionfree. It follows that β is mono. By Proposition 2, $Q(\varphi_*(S)) \cong \varphi_*\tilde{Q}(S)$, giving the required embedding. If S_R is finitely generated, then $Q(\varphi_*(S)) \cong \varphi_*(S) \otimes Q(R)$ [**[4]**, Theorem 4.7), completing the proof.

REMARK. The embedding of Theorem 3 is as left S-right $Q(R)$-modules. When equality holds, $S \otimes Q(R)$ can be made into a ring compatible with the ring structures of S and $Q(R)$ by lifting back the ring structure of $\tilde{Q}(S)$. We do not know if, under the hypotheses of the theorem, $S \otimes Q(R)$ can always be made into a ring such that the embedding is one of rings, though this will be true of our applications.

We will need later the following description of $\tilde{\mathscr{F}}$ in terms of cogenerating injectives.

PROPOSITION 4. *Let $\mathscr{F}$ be an S-good topology on R, and let ${}_RS$ be flat. If I is an injective cogenerator for $\mathscr{F}$, then $I^{\#} = \mathrm{Hom}_R(S, I)$ is an injective cogenerator for $\tilde{\mathscr{F}}$.*

Proof. **[9]**, Lemma 2.4 and Theorem 2.5(b).

2. Main theorem.

We call the topology of dense right ideals of a ring R the *Lambek topology* and denote it by $\mathscr{D}_R$. It is cogenerated by the injective hull $E(R)$ of R, and we say that a module M is $E(R)$-torsionfree if it is $\mathscr{D}_R$-torsionfree. Thus M is $E(R)$-torsionfree if and only if M can be embedded in a direct product of copies of $E(R)$.

THEOREM 5. *Let $\varphi: R \to S$ be a ring homomorphism. Assume ${}_RS$ is flat, S_R is projective, and the Lambek topology $\mathscr{D}_R$ on R is S-good. Then $S \otimes_R Q_{\max}(R) \subset Q_{\max}(S)$, with equality if S_R is finitely generated, $S^* = \mathrm{Hom}_R(S, R)$ is $E(S)$-torsionfree, and the functor $\mathrm{Hom}_R(S, -)$: Mod-R $\to$ Mod-S preserves essential extensions.*

Proof. We use $\tilde{Q}(S)$ as an "intermediate" quotient ring, where Q is the quotient functor associated to $\mathscr{D}_R$ (so $Q_{\max}(R) = Q(R)$). By Theorem 3, $S \otimes Q(R) \subset \tilde{Q}(S)$. We show $\tilde{Q}(S) \subset Q_{\max}(S)$. Indeed, let $\{f_\beta, s_\beta \mid \beta \in A\}$ be a dual basis for the projective module S_R, where A is an index set and $s_\beta \in S$, $f_\beta \in S^*$ for all $\beta \in A$. Define $j: S \to \Pi_A S^*$ by $(j(s)_\beta)(t) = f_\beta(st)$. j is an S-homomorphism. j is also mono, for $j(s) = 0$ implies $f_\beta(s) = 0$ for all β, and so $s = \Sigma s_\beta f_\beta(s) = 0$. By left exactness of $\mathrm{Hom}_R(S, -)$, $S^* \subset E^{\#}$, where $E = E(R)$, so there is an embedding $S \subset \Pi_A E^{\#}$. Since $\Pi_A E^{\#}$ is injective, there is an embedding $E(S) \subset \Pi_A E^{\#}$. Now $\tilde{\mathscr{D}}_R$ is cogenerated by $E^{\#}$, by Proposition 4, hence $\tilde{\mathscr{D}}_R \subset \mathscr{D}_S$ and $\tilde{Q}(S) \subset Q_{\max}(S)$ as desired.

Now assume the remaining conditions. Then $S \otimes Q(R) \cong \tilde{Q}(S)$ by Theorem 3. To complete the proof we show $E^{\#} \subset \Pi E(S)$. By assumption we have $S^* \subset \Pi E(S)$, and since $\mathrm{Hom}(S, \_)$ preserves essential extensions, $E^{\#}$ is an injective hull of S^*. Since $\Pi E(S)$ is injective, $E^{\#} \subset \Pi E(S)$.

3. Applications.

DEFINITION. A topology $\mathscr{F}$ on R is *automorphism invariant* if $\sigma(D) = \{\sigma(d) \mid d \in D\} \in \mathscr{F}$ for all $D \in \mathscr{F}$ and $\sigma \in \mathrm{Aut}(R)$.

Of course, not all topologies are automorphism invariant, but we do have the following.

LEMMA 6. (**[9]**, Example 1 after Corollary 3.6.) *The Lambek topology is automorphism invariant.*

LEMMA 7. *If $\mathcal{F}$ is automorphism invariant, then every automorphism of R extends to an automorphism of $Q(R)$.*

Proof. Let $t(R)$ be the torsion submodule of R with respect to $\mathcal{F}$. Since $\mathcal{F}$ is automorphism invariant, $\sigma(t(R)) = t(R)$ for all $\sigma \in \text{Aut}(R)$. Let $f: D \to R/t(R)$ represent an element of $Q(R)$. Define $\sigma(f): \sigma(D) \to R/t(R)$ by $\sigma(f)(\sigma(d)) = \sigma(f(d))$. Since $\sigma(t(R)) = t(R)$, $\sigma(f)$ is well-defined. It is straightforward to check that $\sigma(f)$ is a homomorphism and that this defines σ on $Q(R)$ to be an automorphism.

DEFINITION. A bimodule ${}_RM_R$ is said to be *generated by normalizing elements* if there are sets $\{m_i \mid i \in I\} \subset M$ and $\{\sigma_i \mid i \in I\} \subset \text{Aut}(R)$ such that $M = \Sigma_{i \in I} Rm_i$ and $m_i r = \sigma_i(r) m_i$ for all $i \in I$, $r \in R$.

LEMMA 8. *Let $\varphi: R \to S$ be a ring homomorphism such that S is generated over R by normalizing elements. Then an automorphism invariant topology on R is S-good.*

Proof. We use the criterion of Proposition 1. Let $\mathcal{F}$ be an automorphism invariant topology on R, and let M_R be $\mathcal{F}$-torsion. Let $\{s_i \mid i \in I\}$ be a set of normalizing generators of S with automorphisms $\{\sigma_i \mid i \in I\}$. Then any element of $M \otimes_R S$ may be written in the form $\Sigma\, m_i \otimes s_i$, the sum taken over finitely many $i \in I$. Let $D_i \in \mathcal{F}$ be such that $m_i D_i = 0$, and set $D = \bigcap_i \sigma_i^{-1}(D_i)$. Then $D \in \mathcal{F}$ since the intersection is finite and $\mathcal{F}$ is automorphism invariant. But $(\Sigma\, m_i \otimes s_i)D = \Sigma\, m_i \otimes \sigma_i(D)s_i \subset \Sigma\, m_i \otimes D_i s_i = \Sigma\, m_i D_i \otimes s_i = 0$. Hence $M \otimes S$ is $\mathcal{F}$-torsion, and $\mathcal{F}$ is S-good.

LEMMA 9. *Let $\mathcal{F}$ be an automorphism invariant topology on R with faithful ring of quotients $Q(R)$, and let ${}_RM_R$ be generated by normalizing elements. Assume further that ${}_RM$ and M_R are flat and that $M \otimes_R Q(R)$ and $Q(R) \otimes_R M$ are $\mathcal{F}$-torsionfree. Then $M \otimes_R Q(R) \cong Q(R) \otimes_R M$, and $M \otimes_R Q(R)$ becomes a $Q(R)$-bimodule generated by normalizing elements.*

Proof. Let $\{m_i \mid i \in I\}$ be a set of normalizing generators for M with associated automorphisms $\{\sigma_i \mid i \in I\}$. Define $\beta: M \otimes Q(R) \to$

$Q(R) \otimes M$ by $\beta(\Sigma m_i \otimes q_i) = \Sigma \sigma_i(q_i) \otimes m_i$, where σ_i is extended to $Q(R)$ via Lemma 7. We show that β is a well-defined automorphism. It is then easy to see that this makes $M \otimes Q(R)$ into a $Q(R)$-bimodule with normalizing generators $\{m_i \otimes 1 \mid i \in I\}$. So suppose $\Sigma m_i \otimes q_i = 0$; and let $D \in \mathcal{F}$ be such that $q_i D \subset R$ for all i. Then $(\Sigma m_i \otimes q_i)D = (\Sigma m_i q_i D) \otimes 1 = 0$. Now M_R is flat and $R \subset Q(R)$, so $M \subset M \otimes Q(R)$ via the map $m \mapsto m \otimes 1$. Hence $\Sigma m_i q_i D = 0$, and so $0 = 1 \otimes \Sigma m_i q_i D = \Sigma \sigma_i(q_i)\sigma_i(D) \otimes m_i = (\Sigma \sigma_i(q_i) \otimes m_i)D$, so $\Sigma \sigma_i(q_i) \otimes m_i = 0$ since $Q(R) \otimes M$ is torsionfree. It follows that β is well-defined. By symmetry, β^{-1} is also well-defined, so β is an isomorphism.

LEMMA 10. *Let $\varphi: R \to S$ be a ring homomorphism such that S is finitely generated over R by normalizing elements. Then* $\mathrm{Hom}_R(S, -)$ *preserves essential extensions.*

Proof. Let $M \subset_{ess} N$, and let $f: S \to N$ be a nonzero R-homomorphism. Let $\{s_i \mid i = 1, \cdots, n\}$ be a set of normalizing generators for S with automorphisms $\{\sigma_i \mid i = 1, \cdots, n\}$. Arrange the s_i so that $f(s_1) \neq 0$. Then there is an $r_1 \in R$ such that $0 \neq f(s_1)r_1 = f(s_1 r_1) = f(\sigma_1(r_1)s_1) = (f\sigma_1(r_1))(s_1) \in M$. If $(f\sigma_1(r_1))(s_i) = 0$ for $i = 2, \cdots, n$ we are done. If not, suppose $(f\sigma_1(r_1))(s_2) = f(s_2)\sigma_2^{-1}\sigma_1(r_1) \neq 0$. Then there exists an $r_2 \in R$ such that $0 \neq f(s_2)\sigma_2^{-1}\sigma_1(r_1)r_2 = (f\sigma_1(r_1)\sigma_2(r_2))(s_2) \in M$. Note that then $(f\sigma_1(r_1)\sigma_2(r_2))(s_1) = f(s_1)r_1\sigma_1^{-1}\sigma_2(r_2) \in M$. By finite induction, we are done.

We are now ready for the applications. Recall (**[11]**) that a ring monomorphism $R \to S$ is a (left) *quasi-Frobenius* (or *QF*) *extension* if ${}_RS$ is finitely generated projective and there exists a module ${}_SM_R$ such that ${}_SS_R \oplus {}_SM_R \cong \bigoplus_1^n {}_S^*S_R$, where ${}^*S = \mathrm{Hom}({}_RS, {}_RR)$.

THEOREM 11. *Let $R \to S$ be a left QF extension such that S is finitely generated by normalizing elements over R. Then $Q_{\max}(S) \cong S \otimes_R Q_{\max}(R)$. If $R \to S$ is two-sided QF, then $S \otimes_R Q_{\max}(R) \cong Q_{\max}(R) \otimes_R S$, $Q_{\max}(R) \to Q_{\max}(S)$ is two-sided QF, and $Q_{\max}(S)$ is finitely generated by normalizing elements over $Q_{\max}(R)$.*

Proof. We write Q for $Q_{\max}$. By Lemmas 6 and 8 the Lambek topology $\mathcal{D}_R$ on R is S-good. Since ${}_RS$ is projective, it is flat. By **[11]**, Satz 2, S_R is finitely generated projective. By Lemma 10, $\mathrm{Hom}_R(S, -)$ preserves essential extensions. Finally, ${}_RS_S^* \oplus {}_RM_S^* \cong \bigoplus_1^n {}_RS_S$ (**[11]**, Satz 2), so S^* is $E(S)$-torsionfree. Theorem 5 applies to give $Q(S) \cong S \otimes Q(R)$. Now assume $R \to S$ is also right *QF*. Then ${}_S^*S_R \oplus {}_SN_R \cong \bigoplus_1^m {}_SS_R$, so in particular ${}_R^*S_R$ is a direct summand of $\bigoplus_1^m {}_RS_R$. Hence ${}_R^*S_R$ is generated over R by normalizing elements, because ${}_RS_R$ and thus also

$\bigoplus_1^m {}_RS_R$ is. This implies that $\mathrm{Hom}_R({}^*S, Q(R))$ is $\mathscr{D}_R$-torsionfree, as follows. Let $f\colon {}^*S \to Q(R)$ be an R-homomorphism with $fD = 0$ for some $D \in \mathscr{D}_R$. Let $\{m_i\}$ be a set of normalizing generators of ${}^*S$ with automorphisms $\{\sigma_i\}$, and let $m = \Sigma\, m_i r_i \in {}^*S$ be arbitrary. Then $D_m = \bigcap r_i^{-1}\sigma_i^{-1}(D) \in \mathscr{D}_R$ and $f(m)D_m = f(mD_m) \subset \Sigma f(Dm_i) = 0$, so $f(m) = 0$ and $f = 0$. Now, $\mathrm{Hom}_R({}^*S, Q(R)) \cong Q(R) \otimes_R S$ by **[10]**, V.4.1 and V.4.2. Hence $Q(R) \otimes S$ is $\mathscr{D}_R$-torsionfree. $S \otimes Q(R)$ is also $\mathscr{D}_R$-torsionfree, since S_R is projective. Lemma 9 applies to give $S \otimes Q(R) \cong Q(R) \otimes S$ and that $Q(S)$ is finitely generated by normalizing elements over $Q(R)$.

It remains to show that $Q(R) \to Q(S)$ is *QF*. It is easy to see that this is a monomorphism and that $Q(S)$ is right and left projective over $Q(R)$. Further applications of Lemma 9 give $S^* \otimes Q(R) \cong Q(R) \otimes S^*$ and $M^* \otimes Q(R) \cong Q(R) \otimes M^*$, so these both have a structure of right $Q(S)$-left $Q(R)$-bimodule. Finally,

$$Q(R) \otimes S^* \cong \mathrm{Hom}_R(S, Q(R)) \cong \mathrm{Hom}_{Q(R)}(S \otimes Q(R), Q(R))$$
$$\cong \mathrm{Hom}_{Q(R)}(Q(S), Q(R)) = Q(S)^*$$

(the $*$ as $Q(R)$-module), and

$$Q(S)^* \oplus (Q(R) \otimes M^*) \cong \bigoplus_1^n Q(R) \otimes S \cong \bigoplus_1^n Q(S),$$

so $Q(R) \to Q(S)$ is left *QF*, by **[11]**, Satz 2. The right *QF*-ness follows similarly.

We remark that the ring structure on $S \otimes Q_{\max}(R)$ obtained from $Q_{\max}(S)$ can be defined directly in the expected way, namely $(\Sigma_i s_i \otimes q_i) \times (\Sigma_j s_j \otimes p_j) = \Sigma_{i,j} s_i s_j \otimes \sigma_j^{-1}(q_i)p_j$, where the s_i are normalizing generators of S over R with automorphisms σ_i.

Theorem 11 applies in particular to Frobenius extensions, projective separable algebras, and Azumaya algebras. We note that centralizing generators are a special case of normalizing generators, so for instance an algebra over a commutative ring is always generated by normalizing elements with all the automorphisms equal to the identity automorphism. Group rings, however, provide examples of rings with normalizing, but not centralizing, generators.

COROLLARY 12. *Let A be a ring, G a group, and H a normal subgroup of G. Then $AG \otimes_{AH} Q_{\max}(AH)$ is a ring, and there is a ring embedding $AG \otimes_{AH} Q_{\max}(AH) \subset Q_{\max}(AG)$ with equality if H is of finite index.*

Proof. By Lemmas 6 and 7, the G-action on AH defined by $x^g = g^{-1}xg$ for $x \in AH$ and $g \in G$ extends to $Q_{\max}(AH)$. The ring structure on $AG \otimes Q_{\max}(AH)$ is now defined by $(\Sigma g_i \otimes q_i)(\Sigma g_j \otimes p_j) = \Sigma g_i g_j \otimes q_i^{g_j} p_j$. AG is generated by normalizing elements over AH, so Lemmas 6 and 8 and Theorem 5 give $AG \otimes Q_{\max}(AH) \subset Q_{\max}(AG)$. It is easy to see that this is a ring embedding. Finally, if H is of finite index then $AH \to AG$ is a Frobenius extension, so Theorem 11 applies to give equality.

COROLLARY 13. *If A is a ring, G is a group, and H is a subgroup of G of finite index, then $Q_{\max}(AG) \cong AG \otimes_{AH} Q_{\max}(AH)$.*

Proof. There is a $K \lhd G$ and of finite index with $K \subset H$. By Corollary 12, $Q_{\max}(AG) \cong AG \otimes_{AK} Q_{\max}(AK) \cong AG \otimes_{AH} (AH \otimes_{AK} Q_{\max}(AK)) \cong AG \otimes_{AH} Q_{\max}(AH)$.

DEFINITION. A subgroup H of G is *subnormal-by-finite* if there is a subnormal subgroup H_1 of G such that $H \subset H_1$ and H is of finite index in H_1.

COROLLARY 14. *If H is a subnormal-by-finite subgroup of G, then $Q_{\max}(AH) \subset Q_{\max}(AG)$.*

Proof. Suppose first that H is normal in G. Then $AG \otimes Q_{\max}(AH) \subset Q_{\max}(AG)$ by Corollary 12. Since AG_{AH} is free and hence faithfully flat, we have $Q_{\max}(AH) \subset AG \otimes Q_{\max}(AH)$ via $q \mapsto 1 \otimes q$. Thus $Q_{\max}(AH) \subset Q_{\max}(AG)$. This extends immediately to the case when H is subnormal. Finally, let H_1 be subnormal such that H is a subgroup of H_1 of finite index. Then an application of Corollary 13 gives $Q_{\max}(AH) \subset Q_{\max}(AH_1)$, and the result follows.

Corollary 14 generalizes a result of Formanek **[3]**.

Conjecture. If H is a normal subgroup of G, then $Q_{\max}(AG) \cong AG \otimes Q_{\max}(AH)$ if and only if H is of finite index.

REMARKS.

1. When $H = 1$ the conjecture is a generalization of the recently proved self-injectivity theorem for group rings, which says that AG is self-injective if and only if A is self-injective and G is finite (see for example **[2]**). The conjecture for $H = 1$ also follows from that theorem when the singular ideal of AG is zero. Some further partial results when $H = 1$ appear in **[8]**. The general case when $H \neq 1$ seems to be more difficult, even in special cases.

2. A corresponding conjecture for the classical quotient ring, avoiding questions of Ore-ness, is the following: If H is a normal subgroup of G, and AH and AG are right Ore, then $Q_{cl}(AG) = AG \otimes_{AH} Q_{cl}(AH)$ if and only if H is of locally finite index (i.e. G/H is a locally finite group). This is a generalization of a conjecture of Herstein (**[5]**, page 36). A discussion and special cases appear in **[8]**.

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