

CONTINUOUS LINEAR IMAGES OF PSEUDO-COMPLETE LINEAR TOPOLOGICAL SPACES

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The various properties of topological spaces in the classical Baire category theorems which imply the Baire property also imply the stronger property of pseudo-completeness. In contrast to some of these properties and to the Baire property, J. C. Oxtoby has shown that pseudo-completeness is productive. The following main result places pseudo-completeness in the context of linear topological spaces: Let E and F be linear topological spaces and g be a continuous linear mapping of E into F . If E is pseudo-complete, g is almost open, and the completion of $g[E]$ has a continuous metric, then $g[E]$ is complete. The proof of this result uses the difference theorem, but not an open mapping theorem. The hypotheses lead to a discussion of conditions for a linear mapping to be almost open and for a linear topological space to have a continuous metric.

An example shows that, although a translation invariant continuous metric on a linear topological space E extends to a translation invariant continuous pseudo-metric on the completion of E , this extension need not be a metric, even if it induces a normed topology on E . Other examples show that a pseudo-complete linear topological space need not be complete in its natural uniformity, and that the almost open condition of the main result may not be omitted and is not implied by the combination of the other conditions and the conclusion.

Pseudo-complete topological spaces, introduced by Oxtoby [8], successfully bind together the classical Baire category theorems. They have been the subject of recent work; Aarts and Lutzer [1], for example, have shown that a metrizable space is pseudo-complete if and only if it has a dense subspace which is topologically complete. In the context of linear topological spaces, Todd [13] shows that, under certain conditions, a linear topological space which is pseudo-complete is complete in the natural uniformity. Theorem 3.1 extends this result for the image of a pseudo-complete linear topological space under a continuous linear almost open mapping; in contrast, Example 5.5 gives pseudo-complete locally convex linear topological spaces which are not complete, and hence are not Pták spaces.

From open mapping theorems arise restrictions on continuous images of certain types of spaces. As an application of his open mapping theorem, Banach ([2], p. 38), showed that the continuous linear image of a complete metric linear space in a space of the same

kind is either meager or the whole space. Robertson and Robertson [9] extend this result in locally convex linear topological spaces by use of fully complete or Pták spaces. Note that a linear subspace of a locally convex linear topological space which is not both dense and barrelled is a meager subspace. Moreover, Pták spaces include Fréchet spaces (complete metric linear spaces which are locally convex). Thus the following modification of the Robertson and Robertson result includes Banach's result restricted to Fréchet spaces, but with a more precise requirement than meagerness: *If g is a continuous linear mapping of a Pták space E into a Hausdorff locally convex linear topological space F , then g maps E onto F , if $g[E]$ is both dense in F and barrelled.*

Recently, Saxon [11] has shown that each infinite dimensional Banach space contains a dense barrelled subspace which is not Baire, and so, as each must be proper, the above result implies that none is a continuous linear image of a Pták space. This is also true for pseudo-complete locally convex linear topological space in place of Pták space. However, this result is not based on an open mapping theorem, but rather on the difference theorem via Theorem 3.1; the proof of this theorem is the main subject of § 3 which also contains an extension of a result of Aarts and Lutzer. Section 4 discusses the conditions of Theorem 3.1 and states some corollaries to it, while § 2 gives some definitions and earlier results, and § 5 contains examples relating to Theorem 3.1.

2. Definitions and remarks. A topological space is *quasi-regular* if and only if each nonempty open set contains the closure of a nonempty open set. A family \mathcal{B} of nonempty open sets is a *pseudo-base* if and only if each nonempty open set contains an element of \mathcal{B} . Finally, a quasi-regular topological space is *pseudo-complete* if and only if there is a sequence (\mathcal{B}_n) of pseudo-bases such that if the terms of the sequence (U_n) are elements of terms of (\mathcal{B}_n) , respectively, and each U_n contains the closure of U_{n+1} , then the intersection $\bigcap_n U_n$ is nonempty.

A subset of a topological space is *rare* if its closure has no interior, and is meager if it may be covered by a countable family of rare sets. A topological space is a Baire space if and only if each nonempty open set is not meager. The closure of a Baire subspace of a topological space is a Baire space. In case E is a linear topological space, E is a Baire space if and only if E is not meager. Oxtoby [8] shows that each pseudo-complete space is a Baire space.

Each topologically complete metrizable space is pseudo-complete. A linear topological space is regular, and so is quasi-regular. Todd [13] shows that a linear metrizable space is pseudo-complete if and

only if it is complete; however, a complete linear topological space need not be pseudo-complete.

A mapping f from a topological space X to a topological space Y is *almost open* if and only if $(fN)^- \cap (fX)$ is a neighborhood of $f(u)$ in the subspace (fX) of Y , whenever N is a neighborhood of u in X . An *open mapping* is defined in the same manner, but without the closure used. The image of a Baire space under an almost open continuous mapping is a Baire space. In case E and F are linear topological spaces and g is a linear mapping of E onto F , the mapping g is almost open if and only if $(gU)^-$ is a neighborhood of 0 in F whenever U is a neighborhood of 0 in E .

A topological space X has a *continuous metric* if and only if there is a metric on the set X which induces a topology on X weaker than the topology of X . It is easy to see that a topological space X has a continuous metric d if d is a metric on X and is continuous as a function from $X \times X$ to the real numbers. Todd [13] shows that a linear topological space E has a continuous translation invariant metric if and only if E has a countable family of open sets which intersects in a singleton.

For other terms, the paper generally follows Horváth [4] and Kelley, Namioka, *et al.* [6], for example, A^{-i} is the interior of the closure of set A ; however, locally convex linear topological space shortens to convex space and the name of a topological property follows "convex" and "linear."

3. Images of a linear pseudo-complete spaces. The main theorem follows.

THEOREM 3.1. *The image of a linear pseudo-complete space under a linear continuous almost open mapping is complete, if its completion has a continuous metric.*

Proof. Let E be a linear topological space which is pseudo-complete, g be a linear continuous almost open mapping from E to a linear topological space F . We may suppose that F is the completion of gE and let d be a continuous metric on F . Finally, suppose (\mathcal{B}_n) is a sequence of pseudo-bases for E as in the definition of pseudo-completeness.

Claim. *There is a sequence (\mathcal{E}_n) with each \mathcal{E}_n a subfamily of \mathcal{B}_n such that, with $\mathcal{E}_0 = \{E\}$, for each $k \geq 0$,*

(0) $U_k = \bigcup \{(gC)^{-i} : C \in \mathcal{E}_k\}$ is dense in F ,

(1) $\{(gC)^{-i} : C \in \mathcal{E}_k\}$ is a disjoint family,

for each C in \mathcal{E}_{k+1} ,

- (2) $d\text{-diam}(gC) < 1/(k + 1)$, and
- (3) some D in \mathcal{E}_k contains C^- .

Proof of claim. Clearly \mathcal{E}_0 satisfies (0) and (1). Suppose $m \geq -1$, and $\mathcal{E}_0, \dots, \mathcal{E}_{m+1}$ satisfy (0) and (1) for $0 \leq k \leq m + 1$ and satisfy (2) and (3) for $0 \leq k \leq m$.

Let $n = m + 1$ and \mathcal{E} be the family of all elements C of \mathcal{B}_{n+1} which satisfy (2) and (3) for $k = n$. Let \mathcal{E}_{n+1} be maximal among subfamilies \mathcal{D} of \mathcal{E} satisfying (1) for \mathcal{E}_k replaced by \mathcal{D} .

Assume $U_{n+1} = \bigcup \{(gC)^{-i} : C \in \mathcal{E}_{n+1}\}$ is not dense in F . Since U_n is dense, there is an element D of \mathcal{E}_n such that $(gD)^{-i} \setminus U_{n+1}^-$ is nonempty. Let V be a nonempty open subset of F with $d\text{-diam}(V) < 1/(n + 1)$ and V^- contained in $(gD)^{-i} \setminus U_{n+1}^-$. As V meets gD , there is B in \mathcal{B}_{n+1} with B^- contained in $D \cap g^{-1}V$.

Now $gB \cap V^- \subset (gD)^{-i} \setminus U_{n+1}^-$, thus $d\text{-diam}(gB) < 1/(n + 1)$, and, as $B^- \subset D \in \mathcal{E}_n$, B is an element of \mathcal{E} . Moreover, $(gB)^{-i}$ is in the complement of U_{n+1} , so $\mathcal{D} = \mathcal{E}_{n+1} \cup \{B\}$ is a subfamily of \mathcal{E} satisfying (1) for \mathcal{E}_k replaced by \mathcal{D} . From the maximality of \mathcal{E}_{n+1} , B is in \mathcal{E}_{n+1} . Hence $(gB)^{-i} \subset U_{n+1}$, a contradiction as g is almost open, and B is a nonempty open set.

Because g is almost open and continuous and E is a Baire space, the completion F of gE is itself a Baire space. By (0) each U_n is dense and open in the Baire space F , so $A = \bigcap_n U_n$ is a dense G_δ subset of F , and thus, necessarily, nonmeager in F . By the difference theorem (see Kelley-Namioda [6], 10.4, p. 92), the difference set $(A - A)$ is a neighborhood of 0 in F .

We now show each y in A is in gE , and hence gE equals F , which will complete the proof. There are $C_n \in \mathcal{E}_n$ with y in each $(gC_n)^{-i}$. From (2), $\bigcap_n (gC_n)^{-i} = \{y\}$. For n fixed, there is D in \mathcal{E}_n containing C_{n+1}^- by (3). Now, we have

$$y \in (gC_n)^{-i} \cap (gC_{n+1})^{-i} \subset (gC_n)^{-i} \cap (gD)^{-i},$$

so $C_n = D$ by (1), and C_n contains C_{n+1}^- for each n . From pseudo-completeness there is an x in $\bigcap_n C_n$, and so, we have, since g is almost open,

$$g(x) \in \bigcap_n (gC_n) \subset \bigcap_n (gC_n)^{-i} = \{y\}.$$

Therefore y is in gE .

This result generalizes the following result arising from [13]; the proofs are similar.

COROLLARY 3.2. *If the topology of the completion of a linear*

topological space E contains a countable family which intersects in a singleton, then E is complete if it is pseudo-complete.

Proof. Existence of such a countable family is equivalent to the existence of a continuous metric on the completion of E . To apply the theorem, use the injection mapping of E into itself.

The claim in the proof of the theorem does not depend on linearity and may be used in proving the following extension of a theorem of Aarts and Lutzer [1], where now almost open replaces open.

THEOREM 3.3 (Aarts and Lutzer). *Suppose X is pseudo-complete and $f: X \rightarrow Y$ is continuous, onto and almost open. If Y is metrizable, then Y contains a dense subspace which has a complete metric and is zero-dimensional. In particular, Y is pseudo-complete.*

4. **The conditions.** We now consider contexts in which almost open mappings and continuous metrics arise.

A property of barrelled spaces F (see Horvath [4], Prop. 1, p. 296) is that each linear surjection of a convex space E onto F is almost open; in fact, this property characterizes barrelled spaces. A similar statement holds for ultra-barrelled spaces (see W. Robertson [10]) in place of barrelled spaces and linear topological space in place of convex space. Each linear Baire space is ultra-barrelled, and a convex space which is ultra-barrelled is barrelled. Each of these implications is strict ([10]). Interest in linear Baire spaces is further enhanced by the following characterization of Saxon [12]: *A linear topological space is a Baire space if and only if no absorbing balanced set is rare.*

The following two corollaries of Theorem 3.1 follow without difficulty from the above discussion.

COROLLARY 4.1. *The image of a convex pseudo-complete space under a linear continuous mapping is complete, if it is barrelled and its completion has a continuous metric.*

COROLLARY 4.2. *The image of a linear pseudo-complete space under a linear continuous mapping is complete, if it is an ultra-barrelled space and its completion has a continuous metric.*

Since a complete linear metrizable space is pseudo-complete, the following gives the result of Banach discussed in the introduction, but without use of the open mapping theorem.

COROLLARY 4.3. *The linear continuous image of a linear pseudo-*

complete space in a linear metrizable space is either meager or the whole space.

Proof. Suppose $g: E \rightarrow F$ are as indicated. If gE is not meager in F , then it is dense in F . Moreover, gE is not meager in itself, hence gE is a Baire space, and so is ultra-barrelled. Therefore g is almost open. As F is metrizable, its completion is metrizable, and so the completion of gE has a continuous metric. Hence gE is complete and, in particular, closed in F . Thus gE is F .

As may be seen, meager may be replaced in this corollary by not both dense and ultra-barrelled or, in case the domain space is convex, by not both dense and barrelled. Thus, for example, no proper linear subspace of a Banach space which is both dense and barrelled is the linear continuous image of a convex pseudo-complete space.

Now let us consider contexts in which a linear topological space has a continuous metric. Of course such a metric exists if the linear topological space has a continuous semi-norm or continuous para-norm which is 0 only at the zero of the space. However, the following is more specific.

THEOREM 4.4. *If a linear topological space E has a countable total family of linear continuous functionals, then E has a continuous metric.*

Proof. Suppose (f_n) is a sequence of linear continuous functionals on E which is total, i.e. if $f_n x = 0$ for all n , then $x = 0$. Now $U_{mn} = \{x \in E: |f_n x| < 1/m\}$ gives a countable family of open sets whose intersection $\bigcap_{mn} U_{mn}$ is a singleton. Therefore E has a continuous metric.

(A more direct proof comes from consideration of the Fréchet combination $d(x, y) = \sum_{n \geq 1} 2^{-n} |f_n(x - y)| / (1 + |f_n(x - y)|)$.)

Thus, for example, a linear Hausdorff space with a Schauder basis (see McArthur [7]) has a continuous metric. Also, for E a linear topological space with continuous dual E' , if E' contains a countable total family, then any linear topology on E with the same dual E' has a continuous metric; in particular, the weak topology on E has a continuous metric. Clearly, the countable total family of continuous functionals in Theorem 4.4 may be replaced by a sequence (p_n) of continuous para-norms for which $p_n(x) = 0$, for all n , implies $x = 0$.

Even a continuous norm on a normed linear space need not extend

to a continuous norm on the completion of the space, as may be seen in Example 5.8. However, a translation invariant continuous pseudo-metric on a linear topological space extends uniquely to a continuous pseudo-metric on the completion of the space. The result is essentially a simple observation of uniform spaces, but we shall state it here for linear topological spaces.

THEOREM 4.5. *If linear topological space E has a translation invariant continuous pseudo-metric d' , then d' has a unique extension d to the completion E^\wedge of E which is a translation invariant continuous pseudo-metric on E^\wedge .*

Proof. Clearly $d': E \times E \rightarrow R$ is uniformly continuous. From Kelley ([5], Th. 26, p. 195), d' has a unique continuous extension d to $E^\wedge \times E^\wedge$. It may be verified that d is a translation invariant pseudo-metric on E^\wedge , and so a continuous pseudo-metric on E^\wedge .

Of interest are conditions under which a translation invariant continuous metric which induces a linear topology, may be extended to a continuous metric on the completion of a space. These are essentially characterized by a result of W. Robertson [10], restated here in terms of nets: *Suppose that E is a linear Hausdorff space under each of two topologies ξ and η , with ξ finer than η , and that i is the identity mapping $i: (E, \xi) \subset (E, \eta)$. Then the extension i^\wedge of i is a one-to-one mapping of $(E, \xi)^\wedge$ into $(E, \eta)^\wedge$ if and only if each Cauchy net in (E, ξ) which converges in (E, η) also converges in (E, ξ) .*

With this, we obtain the following.

THEOREM 4.6. *If d is a translation invariant continuous metric on a linear topological space E which induces a linear topology on E , then d has an extension to a continuous metric on the completion E^\wedge of E if and only if each Cauchy net in E which converges in (E, d) also converges in E .*

Proof. In the W. Robertson theorem, let ξ be the topology of E and η be the linear metric topology induced on E by d . With $i: E \subset (E, d)$, the extension $i^\wedge: E^\wedge \rightarrow (E, d)^\wedge$ of i is one-to-one if and only if each Cauchy net in E which converges in (E, d) is convergent in E .

Suppose the extension i^\wedge is one-to-one. In any case, $(E, d)^\wedge$ is a linear metrizable space and the extension d^\wedge of d from (E, d) to $(E, d)^\wedge$ is a translation invariant metric inducing the topology of $(E, d)^\wedge$. As i^\wedge is continuous and assumed one-to-one, $d^\wedge(x, y) = d(i^\wedge x, i^\wedge y)$ is

a continuous metric on E^\wedge , which is an extension of d on E to E^\wedge .

Now suppose d has an extension to a continuous metric d^- on E^\wedge , and let (x_α) be a Cauchy net in E which converges to x in (E, d) . The net (x_α) converges to some y in E^\wedge , and so

$$d^-(x, y) = \lim_\alpha d^-(x, x_\alpha) = \lim_\alpha d(x, x_\alpha) = 0.$$

Thus $x = y$, and (x_α) converges in E .

5. **Examples and an open question.** In [8], Oxtoby showed that the Cartesian product of any family of pseudo-complete spaces is pseudo-complete. In this section, we shall see that any product of an uncountable family of nontrivial pseudocomplete spaces contain proper dense subspaces which are pseudocomplete. Other examples show the necessity of the almost open requirement of Theorem 3.1 and the difficulty of obtaining it even with all other properties of Theorem 3.1 satisfied. The final example shows that a continuous norm need not have a continuous extension to a norm. The paper ends with an open question.

For the first objective, we shall need several technical propositions.

PROPOSITION 5.1. *For each sequence (\mathcal{C}_n) of pseudo-bases of a quasi-regular space X , there is a sequence (\mathcal{B}_n) of pseudo-bases with each \mathcal{B}_n a subfamily of \mathcal{C}_n and the family of closure of elements of \mathcal{B}_{n+1} a refinement of \mathcal{B}_n .*

Proof. We define (\mathcal{B}_n) by induction. Let \mathcal{B}_1 equal \mathcal{C}_1 , and suppose we have $\mathcal{B}_1, \dots, \mathcal{B}_n$ satisfying the requirements. Let $\mathcal{B}_{n+1} = \{C \in \mathcal{C}_{n+1} : \text{some } H \text{ in } \mathcal{B}_n \text{ contains } C^-\}$. Let U be a nonempty open subset of X . Now U contains some element B of \mathcal{B}_n , and, as X is quasi-regular, B contains the closure of some nonempty open set V . \mathcal{C}_{n+1} is a pseudo-base, so some element C of \mathcal{C}_{n+1} is contained in V . Now $U \supset B \supset V^- \supset V \supset C$, hence U contains C and B contains C^- , thus C is in \mathcal{B}_{n+1} . Therefore \mathcal{B}_{n+1} is a pseudo-base for X , and the family of closures of elements of \mathcal{B}_{n+1} refines \mathcal{B}_n .

PROPOSITION 5.2. *If X is pseudo-complete and nonempty, then there is a sequence (\mathcal{B}_n) of pseudo-bases of X as in the definition of pseudo-completeness and such that each \mathcal{B}_n contains X .*

Proof. Let (\mathcal{C}_n) be a sequence of pseudo-bases for X which satisfies the requirements of pseudo-completeness for X . Suppose (\mathcal{B}'_n) is a sequence of pseudo-bases related to (\mathcal{C}_n) as in Proposition 5.1, and let $\mathcal{B}_n = \mathcal{B}'_n \cup \{X\}$. We need only see, for any sequence (U_n)

of elements of the terms of (\mathcal{B}_n) respectively, with each U_n containing U_{n+1}^- , that the intersection $\bigcap_n U_n$ is nonempty. Should each U_n be X , then $\bigcap_n U_n = X \neq \emptyset$. Otherwise, for some m , $U_m \neq X$. As each U_n contains U_{n+1} , each $U_k \neq X$ for $k \geq m$, and we may suppose $m > 2$. Now each U_k is in \mathcal{B}'_k for $k \geq m$. As the family of closures of elements of each \mathcal{B}'_{n+1} refines \mathcal{B}'_n , there is an element V_{m-1} of \mathcal{B}'_{m-1} with V_{m-1} containing U_m^- , and, by induction, elements V_k of \mathcal{B}'_k for $k < m - 1$, so that V_k contains V_{k+1}^- . Finally (V_n) with $V_n = U_n$, for $n \geq m$, satisfies $V_n \in \mathcal{B}'_n$ and $V_n \supset V_{n+1}^-$ so that $\bigcap_n U_n = \bigcap_{n \geq m} U_n$ contains $(\bigcap_{k < m} V_k) \cap (\bigcap_{k \geq m} U_k) = \bigcap_n V_n \neq \emptyset$.

The following theorem and, especially, its corollary may be compared with similar facts concerning barrelled, linear Baire and other closely related spaces in Corollary 4.12 of [14].

THEOREM 5.3. *Let X be a product of a family $\{X_j: j \in J\}$ of pseudo-complete spaces, and x_0 be in X . If X_0 is a subspace of X containing all x with $x(j) = x_0(j)$ for all but a countable subset of J , then X_0 is pseudo-complete.*

Proof. For each j in J , let $(\mathcal{B}_n(j))$ be a sequence of pseudo-bases for X_j as in the definition of pseudo-completeness for X_j . We may suppose each $\mathcal{B}_n(j)$ contains X_j . Let each \mathcal{C}_n consist of products of $B_j \in \mathcal{B}_n(j)$ where all but a finite number of B_j are X_j , and take \mathcal{B}_n equal to $\{C \cap X_0: C \in \mathcal{C}_n\}$.

Clearly each \mathcal{B}_n is a pseudo-base for the dense subspace X_0 of X . Moreover, X_0 is quasi-regular. Now suppose $U_n \in \mathcal{B}_n$ and U_n contains $U_{n+1}^- \cap X_0$ for each n . There are $B_n^j \in \mathcal{B}_n(j)$ with $U_n = (\prod_j B_n^j) \cap X_0$, and so, for each fixed n ,

$$\begin{aligned} (\prod_j B_n^j) \cap X_0 &= U_n \supset U_{n+1}^- \cap X_0 = [(\prod_j B_{n+1}^j) \cap X_0]^- \cap X_0 \\ &= (\prod_j B_{n+1}^j)^- \cap X_0 = (\prod_j B_{n+1}^-) \cap X_0 . \end{aligned}$$

Let K be a countable subset of J such that for each j in $J \setminus K$, B_n^j is X_j for all n . For any u in B_{n+1}^h , define x in X_0 by $x(h) = u$, choose any $x(j)$ in B_{n+1}^j for j in $K \setminus \{h\}$, and $x(j) = x_0(j)$ otherwise. Thus x is in U_n , and so $u = x(h)$ is in B_n^h . Hence B_n^j contains B_{n+1}^j for all n and j . Now $\bigcap_n B_n^j$ is nonempty for each j . Therefore there is an x in X such that $x(j)$ is in $\bigcap_n B_n^j$ for all j , and, additionally, $x(j) = x_0(j)$ for j in $J \setminus K$. Thus x is in X_0 . Finally, x is in the intersection $\bigcap_n U_n$ so that (\mathcal{B}_n) satisfies the requirements of pseudo-completeness for X_0 .

The following is a direct consequence of this theorem.

COROLLARY 5.4. *Let E be a product of a family of linear pseudo-complete spaces. If E_0 is a linear subspace of E which contains all elements of E whose coordinates are zero except for a countable subset, then E_0 is a linear pseudo-complete space.*

An example of [13] is a convex space which has a continuous norm and which is complete yet not pseudo-complete. The existence of linear pseudo-complete spaces which are not complete was left open. The following exhibits such spaces.

EXAMPLE 5.5. Suppose E is an uncountable product of nontrivial linear complete metric spaces, and let E_0 be the linear subspace of E consisting of all elements x of E whose coordinates are zero except for a countable subset. The linear topological space E_0 is pseudo-complete by the corollary above; however, it is a dense proper subspace of E , and so not a complete linear topological space. Thus pseudo-completeness does not imply completeness.

Suppose the factor spaces in the above example are all convex, then, as Pták spaces are complete, we have examples of convex pseudo-complete spaces which are not Pták spaces.

The following example suggests the importance of the almost open requirement in the main theorem.

EXAMPLE 5.6. Let g be the injection of $E = (\mathcal{L}, \|\cdot\|_1)$ into $F = (\mathcal{L}, \|\cdot\|_\infty)$. E is a Banach space, and so is pseudo-complete; F is a normed space, and so its completion has a continuous metric; finally, g is a linear continuous mapping which is both one-to-one and onto. However, the image $gE = F$ is not complete. To see, independently of Theorem 3.1, that g is not almost open, observe that E is a Pták space, and so each linear continuous mapping of E into a convex Hausdorff space is open if it is almost open.

Even with what appear to be strong conditions, in particular, all the conditions, except for almost open, of the hypothesis and conclusion of Theorem 3.1, a mapping need not be almost open. This is born out in the following.

EXAMPLE 5.7. Let $E = (E, \|\cdot\|)$ be an infinite dimensional Banach space for which there is a sequence of linear continuous functionals which is total. Let $F = (E, \sigma)$ where σ is the convex topology for E with local base consisting of all polars $(f_n)^0 = \{x \in E: |f_n x| \leq 1, n \in \omega\}$ of sequences $(f_n) \subset E'$ which norm converge to 0 in E' . This is the weak-bounded topology or bw-topology for E (Day [3], Def. 2,

p. 41). It may be seen that σ is weaker than the norm topology and stronger than the weak topology for E . Thus, the injection g of pseudo-complete space E into F is linear, one-to-one, onto and continuous. Now F is complete or not according as E is or is not reflexive ([3], Th. 2, p. 57). When E is reflexive, the completion of F clearly has a continuous metric. However, g is never almost open. For let S be the closed unit ball of E , then $S = gS$ is closed in F since it is closed in the weak topology for E . Suppose $(f_n)^0$ is a neighborhood of 0 in F , as above, and choose N such that $\|f_n\| \leq 1/2$ for $n \geq N$, and x in E such that $\|x\| = 2$ and $f_n x = 0$ for $1 \leq n \leq N$. Thus x is in $(f_n)^0$, yet not in S , and so S is not a neighborhood of 0 in F .

The following example shows that a translation invariant continuous metric on a linear topological space need not have a continuous extension which is a metric on the completion.

EXAMPLE 5.8. Let E be an infinite dimensional Banach space under the norm p , and suppose $\|\cdot\|$ is a norm on E which induces a linear topology strictly finer than that induced by p , so that p is a continuous norm on $(E, \|\cdot\|)$. Now $(E, \|\cdot\|)$ is not complete, hence there is a Cauchy sequence (x_n) in $(E, \|\cdot\|)$ which does not converge. Suppose it converges to y in $(E, \|\cdot\|)^\wedge$. It also converges to some x in (E, p) . Let p^- be the continuous extension of p to $(E, \|\cdot\|)^\wedge$. Clearly, $p^-(y - x) = \lim_n p^-(x_n - x) = \lim_n p(x_n - x) = 0$, yet $y - x \neq 0$, so that p^- is not a norm on $(E, \|\cdot\|)^\wedge$. Now $p(u - v)$ gives a translation invariant continuous metric on $(E, \|\cdot\|)$; its extension to $(E, \|\cdot\|)^\wedge$ is $p^-(u - v)$, which is not a metric.

Open question. Since there are convex pseudo-complete spaces which are not Pták spaces, there is a linear continuous mapping g of such a space E onto a convex Hausdorff space F such that g is almost open, but not open. Of more interest in the context of Theorem 3.1 would be for F , in addition, to be metrizable or for its completion to have a continuous metric. Such an example remains to be found.

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