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WEIGHTED SIDON SETS

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A weighted generalisation of Sidon sets, W-Sidon sets, is introduced and studied for compact abelian groups. Firstly W-Sidon sets are characterised analogously to Sidon sets and variations of these characterisations shown to lead back to Sidon sets. For the circle group W-Sidon sets are constructed which are not $\Lambda(1)$ and hence not Sidon. The algebra of all W's making a set W-Sidon is investigated and Sidon and p-Sidon sets cast in terms of it. Finally analytic properties of W-Sidon sets are pursued and a necessary condition on the growth of W^2 obtained.

Throughout this paper G denotes a compact abelian Hausdorff topological group and X denotes its (discrete) dual group. Both are written multiplicatively with identities e and 1 respectively.

We write $(L^p(G), \|\cdot\|_p)$ for the Lebesgue space derived from the normalised Haar measure on G and $(C(G), \|\cdot\|_\infty)$ for the space of (complex-valued) functions continuous on G with the supremum norm. However for $\Delta \subseteqq X$ and counting measure on Δ we denote the Lebesgue spaces $(l^p(\Delta), \|\cdot\|_p)$ and use $c_0(\Delta)$ for the subset of $l^\infty(\Delta)$ of functions tending to zero at infinity.

If A and B are sets we write B^A for the set of all functions from A to B; if $f \in B^A$ and $C \subseteqq A$ ($\subset$ is reserved for strict inclusion) we write $f \mid C$ for the restriction of f to C; ξ_A is the characteristic function of A; $\mathfrak{F}(A)$ denotes the set of all finite subsets of A; $\mathfrak{P}(A)$ denotes the power set of A; $\nu(A)$ is the cardinality of A; and we write $\square$ for the empty set.

The sets of complex numbers, real numbers, integers and natural numbers will be written $\mathfrak{C}$, $\mathfrak{R}$, $\mathfrak{Z}$, and $\mathfrak{N}$ respectively and we write $\mathfrak{T}$ for the topological group of unimodular complex numbers. If $c \in \mathfrak{C}$, c denotes the constant function with value c, whose domain will be clear from the context.

For $\Delta \subseteqq X$, $\phi \in \mathfrak{C}^\Delta$ and $A \subseteqq \mathfrak{C}^\Delta$ we write ϕA for $\{\phi\psi: \psi \in A\}$.

We denote the Fourier transform of $f \in L^1(G)$ by $\hat{f}$. If E is a Banach space we write E' for its dual. Let $A(G) = \{f \in C(G): \hat{f} \in l^1(X)\}$ be normed by $\|f\|_A = \|\hat{f}\|_1$ and set the space of pseudomeasures on G, $(PM(G), \|\cdot\|_{PM})$, equal to $A(G)'$ so that it contains $(M(G), \|\cdot\|)$, the space of measures on G. For $\pi \in PM(G)$ we write $\hat{\pi}$ for its Fourier transform and $sp\pi$ for its spectrum, i.e. $\{\chi \in X: \hat{\pi}(\chi) \neq 0\}$. If $E \subseteqq PM(G)$ and $\Delta \subseteqq X$ we let $E_\Delta = \{\pi \in E: sp\pi \subseteqq \Delta\}$ and call its members Δ-spectral pseudomeasures. We also write $E^\wedge$ for $\{\hat{\pi}: \pi \in E\}$.

The set of trigonometric polynomials on G will be denoted $T(G)$. A subset Δ of X is called

(i) a Sidon set iff

$\sup\{\sum_{\chi\in\Delta}|\hat{t}(\chi)|: t\in T_\Delta(G) \text{ and } \|t\|_\infty \leqq 1\} < \infty$, and

(ii) a $\Lambda(p)$ set, for $0 < p < \infty$ (written $\Delta \in \Lambda(p)$) iff for some r with $0 < r < p$, $L^p_\Delta(G) = L^r_\Delta(G)$. The reader is referred to [2] for an exposition of Sidon and $\Lambda(p)$ sets.

1. *W*-Sidon sets.

DEFINITIONS 1.0. If $\Delta \subseteq X$ and $W \in \mathfrak{C}^\Delta$ we let

$$\|W\|_\Delta = \sup\{\textstyle\sum_{\chi\in\Delta}|W(\chi)\hat{t}(\chi)|: t\in T_\Delta(G) \text{ and } \|t\|_\infty \leqq 1\}$$

and say Δ is W-Sidon iff this is finite. Set

$$\mathfrak{W}(\Delta) = \{W\in\mathfrak{C}^\Delta: \|W\|_\Delta < \infty\}\,.$$

Evidently $\|W\|_\Delta$ equals the least constant for which, whenever $t\in T_\Delta(G)$, $\sum_{\chi\in\Delta}|W(\chi)\hat{t}(\chi)| \leqq k\|t\|_\infty$.

The letter W is used to suggest a weight function and W-Sidon sets should not be confused with p-Sidon sets ([4]) or V-Sidon sets ([13]).

1.1. Taking $\chi\in\Delta$ as t above we see $\|W\|_\infty \leqq \|W\|_\Delta$. So Δ is Sidon iff $\mathfrak{W}(\Delta) = l^\infty(\Delta)$ and the Sidon constant of Δ equals $\|1\|_\Delta$.

1.2. For any $\Delta \subseteq X$, $l^2(\Delta) \subseteq \mathfrak{W}(\Delta)$.

For if $t\in T(G)$ the Cauchy-Schwarz inequality followed by Parseval's identity shows

$$\sum_{\chi\in\Delta}|W(\chi)\hat{t}(\chi)| \leqq \|W\|_2\|\hat{t}\|_2 = \|W\|_2\|t\|_2 \leqq \|W\|_2\|t\|_\infty\,.$$

Thus $\|W\|_\Delta \leqq \|W\|_2$.

In the W-Sidon theory to follow, sets Δ for which $W\in l^2(\Delta)$ behave very like finite sets in the Sidon theory. We refer to them as trivial W-Sidon sets.

Examples of Δ and W for which $W\notin l^2(\Delta)$ yet Δ is W-Sidon and not Sidon are given in 2.3, and some infinite Δ's which are W-Sidon only for $W\in l^2(\Delta)$ in 3.4.

1.3. In 1.0 we have not referred directly to the group X. The following result excuses this. Let X_1 and X_2 be discrete abelian groups with $\Delta \subseteq X_1$ and X_1 a subgroup of X_2.

THEOREM. *For $W\in\mathfrak{C}^\Delta$, Δ is W-Sidon as a subset of X_1 iff it* *is W-Sidon as a subset of* X_2.

Proof. Suppose that G_i is the dual of X_i for $i \in \{1, 2\}$ and define an equivalence relation α on G_1 by $(x, y) \in \alpha$ iff $\chi(x) = \chi(y)$ for all $\chi \in X_1$. Writing A for $\{x \in G_1: \chi(x) = 1 \text{ for all } \chi \in X_1\}$, the kernel of α, A is a closed subgroup of G_1 and G_1/A is isomorphic to G_2 by [10], 2.1.

For $t \in T_\Delta(G_2)$ define $t^* \in T_\Delta(G_1/A)$ by

$$t^*(\alpha(x)) = \sum_{\chi \in \Delta} \hat{t}(\chi)\chi(x) .$$

By definition of α, the map β: $T_\Delta(G_2) \to T_\Delta(G_1/A)$ given by $\beta(t) = t^*$ is well defined. It is easily seen to be a vector space isomorphism, $\|\cdot\|_\infty$-isometric and to satisfy

$$(\beta(t))\hat{}(\chi) = \hat{t}(\chi) \text{ for all } t \in T_\Delta(G_2) \text{ and } \chi \in \Delta .$$

Consequently

$$\sup\left\{\sum_{\chi \in \Delta} | W(\chi)\hat{t}(\chi) |: t \in T_\Delta(G_2) \text{ with } \|t\|_\infty \leqq 1\right\}$$
$$= \sup\left\{\sum_{\chi \in \Delta} | W(\chi)\hat{u}(\chi) |: u \in T_\Delta(G_1/A) \text{ with } \|u\|_\infty \leqq 1\right\}$$

and the conclusion follows.

1.4. To see how W-Sidon sets are affected by group operations on X we extend 1.3 as follows. If ϕ is a function from one discrete abelian group X_1 to another, X_2, (with duals G_i) it induces a map ϕ^* from $T(G_1)$ to $T(G_2)$ by

$$\sum_{\chi \in X_1} \hat{t}(\chi)\chi \longmapsto \sum_{\chi \in X_1} \hat{t}(\chi)\phi(\chi) .$$

When ϕ^* is $\|\cdot\|_\infty$-isometric, ϕ is injective so given $\Delta \subseteq X$ and $W \in \mathfrak{C}^\Delta$ there is a map $W_\phi \in \mathfrak{C}^\phi$ defined by

$$W_\phi(\phi(\chi)) = W(\chi) \text{ for all } \chi \in \Delta .$$

THEOREM. *If* ϕ^* *is* $\|\cdot\|_\infty$*-isometric,* Δ *is W-Sidon iff* $\phi(\Delta)$ *is* W_ϕ*-Sidon.*

Proof. Now ϕ^* maps $T_\Delta(G_1)$ onto $T_{\phi(\Delta)}(G_2)$ and whenever $t \in T_\Delta(G_1)$ and $\chi \in \Delta$,

$$W(\chi)\hat{t}(\chi) = W_\phi(\phi(\chi))(\phi^*t)\hat{}(\phi(\chi)) .$$

Consequently, using 1.3 to move from the group $\phi(X_1)$ to X_2,

$$\begin{aligned}\| W\|_\Delta &= \sup\left\{\sum_{\chi\in\Delta}| W(\chi)\hat{t}(\chi)|: t\in T_\Delta(G_1) \text{ and } \|t\|_\infty \leqq 1\right\}\\ &= \sup\left\{\sum_{\xi\in\phi(\Delta)}| W_\phi(\xi)\hat{u}(\xi)|: u\in T_{\phi(\Delta)}(G_2) \text{ and } \|u\|_\infty \leqq 1\right\}\\ &= \| W_\phi\|_{\phi(\Delta)}\ .\end{aligned}$$

1.5. (i) For example take as ϕ the map $\tau_{\chi_0}: X\to X$ (for $\chi_0\in X$) given by $\tau_{\chi_0}(\chi)=\chi_0\chi$. If $t\in T(G)$,

$$\|\tau^*_{\chi_0}(t)\|_\infty = \left\|\sum_{\chi\in X}\hat{t}(\chi)\chi_0\chi\right\|_\infty = \left\|\sum_{\chi\in X}\hat{t}(\chi)\chi\right\|_\infty$$

whence $\tau^*_{\chi_0}$ is $\|\cdot\|_\infty$-isometric. For any $\Delta\subseteqq X$, $\chi_0\in X$ and $W\in\mathfrak{C}^\Delta$, provided we define $W_0\in\mathfrak{C}^{\chi_0\Delta}$ by $W_0(\chi_0\chi)=W(\chi)$ for all $\chi\in\Delta$, 1.4 guarantees

$$\mathfrak{W}(\chi_0\Delta)=\{W_0: W\in\mathfrak{W}(\Delta)\}\ .$$

(ii) Similarly if we define $\rho: X\to X$ by $\rho(\chi)=\chi^{-1}$ then provided we set $W_\rho\in\mathfrak{C}^{\Delta-1}$ to be $W_\rho(\chi^{-1})=W(\chi)$, 1.4 shows

$$\mathfrak{W}(\Delta^{-1})=\{W_\rho: W\in W(\Delta)\}\ .$$

(iii) Note that for $W\in\mathfrak{C}^{\Delta\cup\chi_0\Delta}$, 1.5(i) does not claim Δ is W-Sidon iff $\chi_0\Delta$ is W-Sidon (and similarly for 1.5(ii)).

If Δ is an infinite proper subgroup of X (it can be chosen for $\mathfrak{Z}$ say) and $\chi_0\in X\backslash\Delta$ then clearly $\chi_0\Delta\cap\Delta=\square$. So we may choose $W\in\mathfrak{C}^{\Delta\cup\chi_0\Delta}$ such that $W\mid\Delta\in l^2(\Delta)$ yet $W\mid\chi_0\Delta\in l^\infty(\chi_0\Delta)\backslash l^2(\chi_0\Delta)$. A premature glance at 3.3 now shows, together with 1.5(i), that $\mathfrak{W}(\Delta)=l^2(\Delta)$ and $\mathfrak{W}(\chi_0\Delta)=l^2(\chi_0\Delta)$. Thus Δ is W-Sidon yet $\chi_0\Delta$ is not W-Sidon (taking restrictions for granted).

1.6. Suppose E is a Banach space contained in $PM(G)$, with norm $\|\cdot\|_E$ stronger than $\|\cdot\|_{PM}$. For $\Delta\subseteqq X$ define $\delta: E\to E^\wedge\mid\Delta$ by $\delta(\pi)=\hat{\pi}\mid\Delta$. Since δ is a vector space morphism, $\ker\delta$ is a subspace of E. This subspace is closed since if $\pi\in E$ and $\{\pi_n: n\in\mathfrak{N}\}\subseteqq\ker\delta$ with $\|\pi-\pi_n\|_E\to 0$ then $\|\hat{\pi}-\hat{\pi}_n\|_\infty\to 0$ hence $\hat{\pi}\mid\Delta=0$.

Thus $E/\ker\delta$ is a Banach space under the quotient norm. Equivalently, $E^\wedge\mid\Delta$ is a Banach space with norm

$$\|\phi\|_\delta=\inf\{\|\pi\|_E: \pi\in E \text{ and } \hat{\pi}\mid\Delta=\phi\}\ .$$

Evidently for all $\pi\in E$,

$$\|\hat{\pi}\|_\infty\leqq\|\hat{\pi}\mid\Delta\|_\delta\leqq\|\pi\|_E\ .$$

(See also 3.7.)

If E is a Banach subalgebra of $PM(G)$ (not necessarily with identity) then so too is $E^\wedge\mid\Delta$.

When considering E' rather than E we write δ' in place of δ.

1.7. Our dependence on Δ-spectral functions makes the following result useful. Refer to [7], Chapter 1, (2.10) for the definition of a homogeneous Banach space on G, replacing $\mathfrak{T}$ there by G.

Suppose E is a homogeneous Banach space on G and E' is the dual of E under a pairing $\langle f, \psi\rangle$ (for $f \in E$ and $\psi \in E'$). If $\psi \in E'$ and $\chi \in X \cap E$ then the Fourier coefficient is defined to be

$$\hat{\psi}(\chi) = \overline{\langle \chi, \psi\rangle}$$

and satisfies $|\hat{\psi}(\chi)| \leqq \|\psi\|_{E'} \|\chi\|_E$.

THEOREM. *Let $\Delta \subseteqq X$, let E be a homogeneous Banach space on G containing Δ and suppose that, restricted to Δ, $\|\cdot\|_E$ is weaker than $\|\cdot\|_A$. Then there is a canonical isomorphism from $(E_\Delta)'$ to $(E')^\wedge | \Delta$ (the latter being normed by $\|\cdot\|_{\delta'}$) whose norm is less than or equal to one.*

Proof. Since

$$\|\hat{f}\|_\infty \leqq \|f\|_1 \leqq \|f\|_E, \text{ for all } f \in E\,,$$

E_Δ is a closed subspace of E. So the canonical map

$$J\colon (E_\Delta)' \longrightarrow E'/(E_\Delta)^0$$

is an isomorphism of norm less than or equal to 1, where $(E_\Delta)^0$, the annihilator of E_Δ, is $\{\psi \in E' \colon \psi(f) = 0 \text{ for all } f \in E_\Delta\}$ (see [8], p. 93).

Now $|\hat{\psi}(\chi)| \leqq \|\psi\|_{E'}$ whenever $\psi \in E'$ and $\chi \in \Delta$ thus by 1.6 it remains to show that $(E_\Delta)^0 = \ker \delta'$. If $\psi \in (E_\Delta)^0$ then $\psi(\chi) = 0$ for all $\chi \in \Delta$ hence $\langle \chi, \psi\rangle = 0$ so that $\hat{\psi} | \Delta = 0$ whence $\psi \in \ker \delta'$. Conversely if $\hat{\psi}(\chi) = 0$ for all $\chi \in \Delta$ then $\psi(t) = 0$ for all $t \in \operatorname{span}(\Delta)$. But $\operatorname{span}(\Delta)$ is dense in E_Δ (by the method of [7], Chapter 1, (2.12)) hence $\psi(f) = 0$ whenever $f \in E_\Delta$, whence $\psi \in (E_\Delta)^0$.

Consequently $(E_\Delta)'$ is isomorphic to $(E')^\wedge | \Delta$ under J followed by the Fourier transform lifted to $E'/\ker \delta'$.

COROLLARY 1.8. *Let $\Delta \subseteqq X$. Then*

(i) *if $1 \leqq p < \infty$, there is a canonical isomorphism from $L^p_\Delta(G)'$ to $L^{p'}(G)^\wedge | \Delta$ whose norm is dominated by 1,*

(ii) *there is a canonical isomorphism from $C_\Delta(G)'$ to $M(G)^\wedge | \Delta$ whose norm is dominated by 1, and*

(iii) *if $1 \leqq p < \infty$, there is a canonical isomorphism from $(L^p(G)^\wedge | \Delta)'$ to $L^{p'}_\Delta(G)$.*

Proof. (i) and (ii) follow immediately from 1.7.

If $1 < p < \infty$, $L_\Delta^{p'}(G)$, being a closed subspace of the reflexive space $L^{p'}(G)$, is also reflexive. So by (i) the dual of $L^p(G)^\wedge | \Delta$ is canonically isomorphic to $L_\Delta^{p'}(G)''$, i.e. to $L_\Delta^{p'}(G)$.

For $p = 1$ we are forced to resort to the method of 1.7. Any $\psi \in (L^1(G)^\wedge | \Delta)'$ lifts to a continuous linear map $\Psi: L^1(G) \to \mathfrak{C}$ which is constant on cosets of $\ker \delta'$ and which may be identified with an element of $L^\infty(G)$, giving $\|\Psi\|_\infty \leqq \|\psi\|$. Consequently if $\chi \in X \backslash \Delta$,

$$\hat{\Psi}(\chi) = \int_G \psi \bar{\chi} = \int_G \psi \cdot 0 = 0$$

so that $\Psi \in L_\Delta^\infty(G)$. This yields a map from $(L^1(G)^\wedge | \Delta)'$ to $L_\Delta^\infty(G)$ and the method of 1.7 completes the proof.

REMARKS 1.9. (i) Obviously $A_\Delta(G)'$ is isometrically isomorphic to $PM(G)^\wedge | \Delta$ as is $L_\Delta^2(G)'$ to $L^2(G)^\wedge | \Delta$.

(ii) In (i) and (iii) above it suffices to take $\Delta = X$ to see the falsity for $p = \infty$. However $L_\Delta^1(G)$ can still be embedded canonically in $(L^\infty(G)^\wedge | \Delta)'$, as can $C_\Delta(G)$ in $(M(G)^\wedge | \Delta)'$.

THEOREM 1.10. *Let $\Delta \subseteqq X$ and $W \in \mathfrak{C}^\Delta$. With the understanding that the constants in* (ii), (iii), (iv) *and* (v) *are the least possible, the following are equivalent*:

(i) *Δ is W-Sidon with $\kappa = \|W\|_\Delta$,*

(ii) *$f \in L_\Delta^\infty(G)$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leqq \kappa \|f\|_\infty$,*

(iii) *$f \in C_\Delta(G)$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| \leqq \kappa \|f\|_\infty$,*

(iv) *for all $\phi \in l^\infty(\Delta)$ there exists $\mu \in M(G)$ with $\hat{\mu}|\Delta = W\phi$ and $\|u\| \leqq \kappa \|\phi\|_\infty$,*

(v) *for all $\phi \in c_0(\Delta)$ there exists $f \in L^1(G)$ with $\hat{f}|\Delta = W\phi$ and $\|f\|_1 \leqq \kappa \|\phi\|_\infty$,*

(vi) *$WL_\Delta^\infty(G)^\wedge | \Delta \subseteqq l^1(\Delta)$* (see section 0 for product notation),

(vii) *$WC_\Delta(G)^\wedge | \Delta \subseteqq l^1(\Delta)$,*

(viii) *$Wl^\infty(\Delta) \subseteqq M(G)^\wedge | \Delta$, and*

(ix) *$Wc_0(\Delta) \subseteqq L^1(G)^\wedge | \Delta$.*

Proof. (i) $\Rightarrow$ (ii) follows by a straightforward modification of (a) $\Rightarrow$ (b) in [10], 5.7.4.

(ii) $\Rightarrow$ (iii) is obvious because $C_\Delta(G) \subseteqq L_\Delta^\infty(G)$.

(iii) $\Rightarrow$ (iv). By hypothesis the map $f \mapsto W\hat{f} | \Delta$ from $C_\Delta(G)$ to $l^1(\Delta)$ is linear and bounded by κ. Let $K: l^\infty(\Delta) \to M(G)^\wedge | \Delta$ denote the canonical isomorphism of 1.8(ii) composed with the adjoint of this map-evidently $\|K\| \leqq \kappa$. For $\chi \in \Delta$,

$$K\phi(\chi) = \sum_{\xi \in X} \phi(\xi)(W(\chi)\hat{\chi})(\xi) = W(\chi)\phi(\chi)\ ,$$

so given $\phi \in l^\infty(\Delta)$, there is $\mu \in M(G)$-namely $\mu \in \delta^{-1}(K\phi)$-with $\hat{\mu} | \Delta = W\phi$

and $\|\mu\| \leqq \kappa\|\phi\|_\infty$.

(iv) $\Rightarrow$ (v) follows by an easy alteration of (d) $\Rightarrow$ (e) in [2], 15.1.4.

(v) $\Rightarrow$ (i). By hypothesis the map $\phi \mapsto W\phi$ from $c_0(\Delta)$ to $L^1(G)^\wedge | \Delta$ is linear and bounded by κ. Let $K: L^\infty_\Delta(G) \to l^1(\Delta)$ denote the composition of its adjoint with the canonical isomorphism of 1.8(iii). Then K is linear and bounded by κ. If $\chi \in \Delta$ and $f \in L^\infty_\Delta(G)$ then

$$(Kf)(\chi) = \int_G W(\chi) f\bar{\chi} = W(\chi)\hat{f}(\chi)$$

hence $Kf = W\hat{f} | \Delta$, so (i) holds.

(ii) $\Rightarrow$ (vi), (iii) $\Rightarrow$ (vii), (iv) $\Rightarrow$ (viii) and (v) $\Rightarrow$ (ix) are obvious. Since the converses fall into similar pairs we show only one of each.

(vii) $\Rightarrow$ (iii). In the following lemma take A to be $l^1(\Delta)$ with α the canonical injection, B to be $C_\Delta(G)$ with $\beta f = W\hat{f} | \Delta$ and C to be $\mathfrak{C}^\Delta$ with the product topology. Now (vii) ensures $\beta(B) \subseteq \alpha(A) \subseteq C$ so by 1.11 to follow, there is a constant κ such that for all $f \in C_\Delta(G)$, there is $\phi \in l^1(\Delta)$ with $W\hat{f} | \Delta = \phi$ and $\|\phi\|_1 \leqq \kappa \|f\|_\infty$. That is, (iii) holds.

(ix) $\Rightarrow$ (v). In the following lemma take A to be $L^1(G)$ with $\alpha(f) = \hat{f} | \Delta$, B to be $c_0(\Delta)$ with $\beta(\phi) = W\phi$ and C to be $\mathfrak{C}^\Delta$ with the product topology. Now (ix) assures us that the hypotheses of 1.11 hold and hence (v) results.

1.11. I am indebted to Professor R. E. Edwards for the following statement:

LEMMA. *If A and B are Banach spaces, C a Hausdorff topological vector space, $\alpha: A \to C$ and $\beta: B \to C$ continuous linear maps and if $\beta(B) \subseteq \alpha(A)$ then there is a constant κ such that for all $b \in B$ there exists $a \in A$ with $\alpha(a) = \beta(b)$ and $\|a\|_A \leqq \kappa \|b\|_B$.*

Proof. Let $\bar{A} = A/\ker \alpha$ and endow it with the quotient topology in which $\|\bar{a}\| = \inf\{\|c\|: c \in \bar{a}\}$ for each $\bar{a} \in \bar{A}$. Since C is Hausdorff, $\{0\}$ is closed in C and since α is continuous $\bar{0} = \alpha^{-1}(\{0\})$ is closed in A. Thus $\bar{A}$ is again a Banach space and α induces a continuous injection $\bar{\alpha}: \bar{A} \to C$ defined by $\bar{\alpha}(\bar{a}) = \alpha(a)$, for $\bar{a} \in \bar{A}$.

Define $\gamma: B \to \bar{A}$ by $\gamma(b) = \bar{\alpha}^{-1} \circ \beta(b)$, for $b \in B$. By hypothesis γ is well defined-it clearly suffices to show it is bounded. Evidently γ is linear, so it remains to show it has a closed graph. If $b_n \to 0$ in B and $\gamma(b_n) \to \bar{a}$ in $\bar{A}$ then $\beta(b_n) \to \beta(0) = 0$ in C. Thus, since $\bar{\alpha}$ is also continuous and linear,

$$\bar{\alpha}(\lim_n \gamma(b_n)) = \bar{\alpha}(\lim_n \bar{\alpha}^{-1} \circ \beta(b_n)) = \lim_n \beta(b_n) = \bar{\alpha}(\bar{a})$$

and so

$$0 = \lim_n \beta(b_n) = \bar{\alpha}(\bar{a}) .$$

Finally by injectivity of $\bar{\alpha}$, $\bar{a} = 0$.

1.12. We shall also use this lemma in another direction.

THEOREM. *Let A and B be Banach spaces, let Δ be a set and suppose $\mathfrak{C}^\Delta$ has the product topology. Let $\alpha: A \to \mathfrak{C}^\Delta$ and $\beta: B \to \mathfrak{C}^\Delta$ be continuous and linear with*

(i) *there is $\lambda > 0$ such that for all $a \in A$ and all $\chi \in \Delta$,*

$$|\alpha(a)(\chi)| \leqq \lambda \|a\|_A, \text{ and}$$

(ii) *there exist $\{b_\chi: \chi \in \Delta\} \subseteqq B$ with*

$$\beta(b_\chi)(\xi) = \begin{cases} 1 & \text{if } \xi = \chi \\ 0 & \text{otherwise} \end{cases}, \text{ and } \sup\{\|b_\chi\|_B: \chi \in \Delta\} < \infty .$$

Suppose finally that $\psi \in \mathfrak{C}^\Delta$ with $\psi\beta(B) \subseteqq \alpha(A)$. Then $\psi \in l^\infty(\Delta)$.

Proof. Applying 1.11 there is a constant κ such that for all $b \in B$, there exists $a \in A$ with $\alpha(a) = \psi\beta(b)$ and $\|a\|_A \leqq \kappa\|b\|_B$. If we write a_χ for an element of A corresponding to b_χ by this process we have

$$|\psi(\chi)| = |\psi(\chi)\beta(b_\chi)(\chi)| = |\alpha(a_\chi)(\chi)| \leqq \lambda \|a_\chi\|_A \leqq \kappa\lambda \|b_\chi\|_B .$$

Consequently $\|\psi\|_\infty < \infty$ as required.

1.13. The next result is helpful when showing a set is W-Sidon.

THEOREM. *If $\Delta \subseteqq X$ and $W \in \mathfrak{C}^\Delta$ the following are equivalent:*
(i) *Δ is W-Sidon,*
(ii) *$f \in C_\Delta(G)$ with $\hat{f} \in \mathfrak{R}^X$ implies $\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)| < \infty$, and*
(iii) *whenever $\phi \in l^\infty(\Delta) \cap \mathfrak{R}^X$ there is $\mu \in M(G)$ with $\hat{\mu} \mid \Delta = W\phi$.*

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from 1.10.

(iii) $\Rightarrow$ (i). If $\phi \in l^\infty(\Delta)$ we may write $\phi = \phi_1 + i\phi_2$ where, by (iii), there is $\mu_j \in M(G)$ with $\hat{\mu}_j \mid \Delta = W\phi_j$ for $j \in \{1, 2\}$. Thus taking $\mu = \mu_1 + i\mu_2$ gives $\mu \in M(G)$ and $\hat{\mu} \mid \Delta = W\phi$, so (i) results by 1.10.

1.14. One important respect in which 1.10 differs from the analogous result for Sidon sets is that we only claim inclusions like 1.10(viii) rather than $Wl^\infty(\Delta) = M(G)^\wedge \mid \Delta$. The reasons for this are embodied in:

THEOREM. *Suppose $\Delta \subseteqq X$ and $W \in \mathfrak{C}^\Delta$. Then Δ is Sidon when-*

ever one of the following holds:

(i) $Wl^\infty(\Delta) = M(G)^\wedge \mid \Delta$,

(ii) $Wc_0(\Delta) = L^1(G)^\wedge \mid \Delta$,

(iii) $WC_\Delta(G)^\wedge \mid \Delta = l^1(\Delta)$,

(iv) $WL^\infty_\Delta(G)^\wedge \mid \Delta = l^1(\Delta)$.

Proof. (i) Taking the Dirac measure at e we see $1 \in Wl^\infty(\Delta)$. Thus $l^\infty(\Delta) \subseteqq Wl^\infty(\Delta) \subseteqq l^\infty(\Delta)$ hence $l^\infty(\Delta) = Wl^\infty(\Delta) = M(G)^\wedge \mid \Delta$ so Δ is Sidon.

(ii) By hypothesis we cannot have $W(\chi) = 0$ for any $\chi \in \Delta$, so $W^{-1}L^1(G)^\wedge \mid \Delta = c_0(\Delta)$. Now in 1.12 we take $A \equiv c_0(\Delta)$ with norm $\| \cdot \|_\infty$, α the canonical injection, $B \equiv L^1(G)$ with norm $\| \cdot \|_1$, $\beta(\hat{f}) = \hat{f} \mid \Delta$ and $\psi \equiv W^{-1}$. The hypotheses are readily verified so we conclude that $\| W^{-1} \|_\infty < \infty$. Applying 1.10, whenever $t \in T_\Delta(G)$,

$$\sum_{\chi \in \Delta} | \hat{t}(\chi) | \leqq \| W^{-1} \|_\infty \kappa \| t \|_\infty .$$

So Δ is Sidon.

(iii) Again, W is never zero so we may apply 1.12 taking $A \equiv C_\Delta(G)$, $B \equiv l^1(\Delta)$, $\alpha(f) = \hat{f} \mid \Delta$, β the canonical injection and $\psi \equiv W^{-1}$. As in (ii) we deduce that Δ is Sidon.

(iv) Apply the same method as (iii).

NOTE. The converse to each of these assertions is false. Even if Δ is replaced by $\Delta_0 \equiv \{\chi \in \Delta : W(\chi) \neq 0\}$ and Δ_0 is Sidon, these inclusions are strict if Δ_0 is infinite and $W \in c_0(\Delta)$.

THEOREM 1.15. *Let $\Delta \subseteqq X$, $W \in \mathfrak{C}^\Delta$ and Δ_0 be as above. Assuming the constants in* (ii), (iii) *and* (iv) *to be the least possible, these are equivalent*:

(i) *Δ_0 is Sidon with constant κ,*

(ii) *$f \in L^\infty_{\Delta_0}(G)$ implies $\sum_{\chi \in \Delta} W(\chi)\hat{f}(\chi)\chi \in A_{\Delta_0}(G)$ and $\| W\hat{f} \|_1 \leqq \kappa \| \sum_{\chi \in \Delta} W(\chi)\hat{f}(\chi)\chi \|_\infty$,*

(iii) *$t \in T_{\Delta_0}(G)$ implies $\| W\hat{t} \|_1 \leqq \kappa \| \sum_{\chi \in \Delta} W(\chi)\hat{t}(\chi)\chi \|_\infty$, and*

(iv) *for all $\phi \in l^\infty(\Delta_0)$ there is $\mu \in M(G)$ such that $\hat{\mu} \mid \Delta_0 = W\phi$ and $\| \mu \| \leqq \kappa \| W\phi \|_\infty$.*

Proof. (i) $\Rightarrow$ (ii). If $f \in L^\infty_{\Delta_0}(G)$ then

$$\| W\hat{f} \|_1 = \sum_{\chi \in \Delta_0} | W(\chi)\hat{f}(\chi) | \leqq \| W \|_\infty \| \hat{f} \|_1$$

so that if Δ_0 is Sidon, (ii) follows.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). If $t \in T_{\Delta_0}(G)$ define $u \in T_{\Delta_0}(G)$ by taking

$$\hat{u}(\chi) = W^{-1}(\chi)\hat{t}(\chi) \text{ for all } \chi \in \Delta_0 .$$

They by (iii),

$$\sum_{\chi \in \Delta_0} | \hat{t}(\chi) | = \sum_{\chi \in \Delta_0} | W(\chi)\hat{u}(\chi) | \leqq \kappa \left\| \sum_{\chi \in \Delta_0} W(\chi)\hat{u}(\chi) \right\|_\infty \leqq \kappa \| t \|_\infty$$

so (i) follows.

(i) $\Rightarrow$ (iv). If $\phi \in l^\infty(\Delta)$ and $W \in \mathfrak{W}(\Delta)$ then $\phi W \in l^\infty(\Delta)$ hence (iv) results from (i) and 1.11.

(iv) $\Rightarrow$ (i). If $\psi \in l^\infty(\Delta_0)$ and $\Phi \in \mathfrak{F}(\Delta_0)$ let

$$\psi_\Phi(\chi) = \begin{cases} W^{-1}(\chi)\psi(\chi) \text{ if } \chi \in \Phi \\ 0 \text{ if } \chi \in \Delta_0 \backslash \Phi \end{cases}, \text{ so that } \psi_\Phi \in c_0(\Delta_0) .$$

By hypothesis there is $\mu_\Phi \in M(G)$ with $\hat{\mu}_\Phi \mid \Delta_0 = W\psi_\Phi$ and

$$\| \mu_\Phi \| \leqq \kappa \| W\psi_\Phi \|_\infty \leqq \kappa \| \psi \|_\infty .$$

Thus $\{\mu_\Phi : \Phi \in \mathfrak{F}(\Delta)\}$ is bounded in $M(G)$ hence by Alaoglu's theorem it has a weakly convergent subnet. So there is $\mu \in M(G)$ with $\hat{\mu} \mid \Delta_0 = \psi$, and Δ_0 must be Sidon.

1.16. Many characterisations of Sidon sets have weighted analogues, like 1.10. More of these may be found in [11].

2. Thick W-Sidon sets.

2.0. To find W-Sidon sets which are not Sidon it suffices, by 1.2, to take $\Delta \subseteqq X$ not Sidon and then choose $W \in l^2(\Delta)$ (such Δ exist since infinite subgroups are not Sidon). It is the purpose of this section to exhibit non-Sidon sets Δ which are W-Sidon for some $W \notin l^2(\Delta)$. These sets are in the dual of the circle group and are not even $\Lambda(1)$.

The proof relies on Riesz products and therefore requires a sort of independence condition on Δ. Recall $\Delta^2 = \{\chi\xi : \chi, \xi \in \Delta\}$ whenever $\Delta \subseteqq X$.

THEOREM 2.1. *Suppose* $\Delta = \bigcup \{\Delta_n : n \in \mathfrak{N}\}$ *where* $0 < \nu(\Delta_n) < \aleph_0$ *and*

(i) $1 \notin \Delta_0$,

(ii) $\Delta_n^{-1} = \Delta_n$,

(iii) $\Delta_{n+1} \subseteqq X \backslash \bigcup \{\Delta_0^{\varepsilon_0}\Delta_1^{\varepsilon_1} \cdots \Delta_n^{\varepsilon_n} : \varepsilon_i \in \{0, 1, 2\}$ *for* $0 \leqq i \leqq n$ *and at most one* ε_i *equal to* $2\}$, *and*

(iv) $\Delta_{n+1}^2 \subseteqq X \backslash \bigcup \{\Delta_0^{\varepsilon_0}\Delta_1^{\varepsilon_1} \cdots \Delta_n^{\varepsilon_n} : \varepsilon_i \in \{0, 1\}$ *for* $0 \leqq i \leqq n$ *and* $\sum_{i=0}^n \varepsilon_i \geqq 1\}$

Define $W: \Delta \to (0, 1]$ *to equal* $\nu(\Delta_n)^{-1}$ *on* Δ_n. *The conclusion is that* Δ *is W-Sidon.*

Proof. Suppose $\phi \in \mathfrak{R}^\Delta$ with $\|\phi\|_\infty \leqq 1$. For $n \in \mathfrak{N}$ define $t_n \in T(G)$ by

$$t_n = (2\nu(\Delta_n))^{-1}\Big(\sum_{\substack{\chi \in \Delta_n \\ \chi^2 \neq 1}} \phi(\chi)(\chi + \bar{\chi}) + \sum_{\substack{\chi \in \Delta_n \\ \chi^2 = 1}} \phi(\chi)\chi\Big) .$$

It is easy to see that

(2.1.1) $\quad t_n$ is real-valued

(2.1.2) $\quad \|t_n\|_\infty \leqq 1$

(2.1.3) $\quad$ and, by (ii), $\hat{t}_n(\chi) = \begin{cases} (2\nu(\Delta_n))^{-1}\phi(\chi) \text{ if } \chi \in \Delta_n \\ 0 \text{ if } \chi \in X \backslash \Delta_n \end{cases}.$

Next for $N \in \mathfrak{N}$ set $P_N = \prod_{n=0}^N (1 + t_n)$ so that $P_N = 1 + \sum_{n=0}^N t_n + Q_N$ where

(2.1.4)
$$Q_N = \sum_{0 \leqq n_1 < n_2 \leqq N} t_{n_1}t_{n_2} + \sum_{0 \leqq_1 < n_2 \leqq_3 \leqq N} t_{n_1}t_{n_2}t_{n_3} + \cdots + t_0 t_1 \cdots t_N .$$

(2.1.5) $\quad$ Now $\hat{P}_N \mid \Delta_n = \hat{t}_n \mid \Delta_n$ if $0 \leqq n \leqq N$

provided that whenever $0 \leqq n \leqq N$,

$$\Delta_n \subseteqq X \backslash [sp(1) \cup \bigcup \{\Delta_m : 0 \leqq m \leqq N \text{ and } m \neq n\} \cup sp(Q_N)] .$$

Consequently the lemma to follow ensures this for each $N \in \mathfrak{N}$.

By (2.1.1), (2.1.2) and (2.1.3), for each N, if we have

$$1 \notin \bigcup \{\Delta_n : 0 \leqq n \leqq N) \cup sp(Q_N)$$

then

(2.1.6)
$$\|P_N\|_1 = \int_G P_N = 1 + \sum_{n=0}^N \int_G t_n + \int_G Q_N = 1 .$$

Again, the lemma assures us of this.

So by (2.1.6), $\{P_N : N \in \mathfrak{N}\}$ is bounded in $M(G)$ and thus has a weak cluster point $\tau \in M(G)$; let $\mu = 2\tau$. Then for each $n \in \mathfrak{N}$ and $\chi \in \Delta_n$,

$$\begin{aligned} \hat{\mu}(\chi) = 2\hat{\tau}(\chi) &= 2\hat{t}_n(\chi) \text{ by (2.1.5)} \\ &= \nu(\Delta_n)^{-1}\phi(\chi) \text{ by (2.1.3)} \\ &= W(\chi)\phi(\chi) \text{ by definition of } W . \end{aligned}$$

Thus $\hat{\mu} \mid \Delta = W\phi$ so by 1.13(iii), Δ is W-Sidon.

LEMMA 2.2. *Suppose* $\{\Delta_n : n \in \mathfrak{N}\} \subseteq \mathfrak{P}(X)$ *satisfies conditions* (i) *to* (iv) *of the previous theorem. Then with* Q_N *given by* (2.1.4), *for each* $N \in \mathfrak{N}$,

(i) $0 \leqq n \leqq N$ *implies*

$$\Delta_n \subseteq X \backslash [\{1\} \cup \bigcup \{\Delta_m : 0 \leqq m \leqq N \text{ and } m \neq n\} \cup sp(Q_N)]\,, \text{ and}$$

(ii) $1 \notin \bigcup \{\Delta_n : 0 \leqq n \leqq N\} \cup sp(Q_N)$.

Proof. By (2.1.4) and (2.1.3),

$$sp(Q_N) \subseteq \bigcup \left\{ \Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \cdots \Delta_N^{\varepsilon_N} : \varepsilon_i \in \{0, 1\} \text{ for } 0 \leqq i \leqq N \text{ and } \sum_{i=0}^{N} \varepsilon_i \geqq 2 \right\} .$$

For brevity define

$$A(N, n) = \{1\} \cup \bigcup \{\Delta_m : 0 \leqq m \leqq N \text{ and } m \neq n\} \text{ for } 0 \leqq n \leqq N\,,$$

and

$$B(N, j) = \bigcup \{\Delta_0^{\varepsilon_0} \Delta_1^{\varepsilon_1} \cdots \Delta_N^{\varepsilon_N} : \varepsilon_i \in \{0, 1\} \text{ and } \sum_{i=0}^{N} \varepsilon_i \geqq j\} \text{ for } j \in \{1, 2\} .$$

In these terms we have to prove, for each $N \in \mathfrak{N}$, $0 \leqq n \leqq N$ implies $\Delta_n \subseteq X \backslash [A(N, n) \cup B(N, 2)]$, and

$$1 \notin \bigcup \{\Delta_n : 0 \leqq n \leqq N\} \cup B(N, 2) .$$

A straightforward induction, relying heavily on 2.1(ii), completes the argument.

THEOREM 2.3. *There is a subset* Δ *of* $\mathfrak{Z}$ *which is W-Sidon for some* $W \in l^\infty(\Delta) \backslash l^2(\Delta)$ *yet which is not* $\Delta(1)$.

Proof. Take $m_0 \neq 0$ and let $\Delta_0 = \{\pm m_0\}$. Supposing $\Delta_0, \cdots \Delta_n$ have been defined so as to satisfy the hypotheses of 2.1, let $m \in \mathfrak{N}$ be the supremum of the finite set

$$\bigcup \{\varepsilon_0 \Delta_0 + \cdots + \varepsilon_n \Delta_n : \varepsilon_i \in \{0, 1, 2\} \text{ with at most one } \varepsilon_i = 2\} .$$

Now if $n = 0$ set $\Delta_1 = \{\pm(m + 1)\}$ and if $n \geqq 1$ take

$$\Delta_{n+1} = \{\pm j(m + 1) : 1 \leqq j \leqq [(n + 1)/2]\} .$$

Since $\Delta_{n+1} + \Delta_{n+1}$ is also disjoint from the finite set above, it is disjoint from

$$\bigcup \left\{ \varepsilon_0 \Delta_0 + \cdots + \varepsilon_n \Delta_n : \varepsilon_i \in \{0, 1\} \text{ with } \sum_{i=0}^{n} \varepsilon_i \geqq 1 \right\} .$$

Consequently 2.1 shows $\Delta \equiv \bigcup \{\Delta_n : n \in \mathfrak{N}\}$ is W-Sidon where

$$\sum_{\chi \in \Delta} |W(\chi)|^2 \geqq \sum_{n \in \mathfrak{N}} (1+n)^{-1} = \infty$$

so $W \notin l^2(\Delta)$.

By construction Δ contains arbitrarily long arithmetic progressions hence it is not $\Lambda(1)$ by [9], (4.1).

2.4. Using multiplier notation from 4.2, by 3.3 to follow,

$$l^2(\Delta) = (C_\Delta(G), A_\Delta(G))$$

whenever Δ is a subgroup of X. If $\Delta \subseteqq X$, Parseval's identity shows

$$l^2(\Delta) \subseteqq (C_\Delta(G), A_\Delta(G)) .$$

To find Δ for which this inclusion is strict it suffices to take Δ an infinite Sidon set so that $1 \in (C_\Delta(G), A_\Delta(G)) \backslash l^2(\Delta)$. However 2.3 provides examples of non-Sidon sets Δ in $\mathfrak{Z}$ for which the strict inclusion holds. It also indicates the impossibility of extending [1], Theorem 1 to arbitrary subsets of X.

3. The algebra of weight functions.

3.0. From 1.10 we may read off more expressions for $||W||_\Delta$:

$$\begin{aligned} ||W||_\Delta &= \sup\{\sum_{\chi \in \Delta} |W(\chi)\hat{f}(\chi)|: f \in C_\Delta(G) \text{ with } ||f||_\infty \leqq 1\} \\ &= \sup\{\inf\{||f||_1: f \in L^1(G) \text{ with } \hat{f} \mid \Delta = W\phi\}: \phi \in c_0(\Delta) \text{ and } ||\phi||_\infty \leqq 1\} \\ &= \sup\{\inf ||\mu||: \mu \in M(G) \text{ with } \hat{\mu} \mid \Delta = W\phi\}: \phi \in l^\infty(\Delta) \text{ and } ||\phi||_\infty \leqq 1\} . \end{aligned}$$

THEOREM 3.1. *$\mathfrak{W}(\Delta)$ is a commutative Banach algebra under $||\cdot||_\Delta$ and pointwise operations. It has an identity iff Δ is Sidon.*

Proof. The following straightforward formulae establish that $||\cdot||_\Delta$ makes $\mathfrak{W}(\Delta)$ into a commutative normed algebra under pointwise operations.

Suppose $W_1, W_2 \in \mathfrak{W}(\Delta)$, $\alpha \in \mathfrak{C}$ and $t \in T_\Delta(G)$ with $||t||_\infty \leqq 1$. Then

$$\begin{aligned} \sum_{\chi \in \Delta} |(W_1(\chi) + W_2(\chi))\hat{t}(\chi)| &\leqq \sum_{\chi \in \Delta} |W_1(\chi)\hat{t}(\chi)| + \sum_{\chi \in \Delta} |W_2(\chi)\hat{t}(\chi)| \\ &\leqq ||W_1||_\Delta + ||W_2||_\Delta ; \end{aligned}$$

$$\sum_{\chi \in \Delta} |\alpha W_1(\chi)\hat{t}(\chi) = |\alpha| \sum_{\chi \in \Delta} |W_1(\chi)\hat{t}(\chi)| \leqq |\alpha| \, ||W_1||_\Delta ;$$

$$\sum_{\chi \in \Delta} |W_1(\chi)W_2(\chi)\hat{t}(\chi)| \leqq ||W_1||_\infty \sum_{\chi \in \Delta} |W_2(\chi)\hat{t}(\chi)| \leqq ||W_1||_\Delta ||W_2||_\Delta \text{ by 1.1 ;}$$

and if $||W||_\Delta = 0$ then $||W||_\infty = 0$ hence $W = 0$.

Suppose $\{W_n: n \in \mathfrak{N}\} \subseteqq \mathfrak{W}(\Delta)$ is a Cauchy sequence. Then by 1.1 again, $||W_n - W_m||_\infty \to 0$ hence there is $W \in l^\infty(\Delta)$ for which $||W - W_n||_\infty \to 0$.

If $\varepsilon > 0$, there is $N \in \mathfrak{N}$ such that $n \geqq N$ implies, for all $t \in T_\Delta(G)$ with $\|t\|_\infty \leqq 1$,

$$\sum_{\chi \in \Delta} |(W_n(\chi) - W_m(\chi))\hat{t}(\chi)| < \varepsilon .$$

Letting $m \to \infty$, the same inequality holds with W replacing W_m. So $n \geqq N$ implies $\|W_n - W\|_\Delta < \varepsilon$. Furthermore

$$\|W\|_\Delta - \|W_N\|_\Delta \leqq \|W - W_N\|_\Delta < \varepsilon$$

hence $\|W\|_\Delta < \varepsilon + \|W_N\|_\Delta < \infty$. Thus $W_n \to W$ in $\mathfrak{W}(\Delta)$.

Finally $\mathfrak{W}(\Delta)$ has an identity iff $1 \in \mathfrak{W}(\Delta)$ iff Δ is Sidon.

3.2. From 1.1 we have: Δ is Sidon iff $\mathfrak{W}(\Delta) = l^\infty(\Delta)$. Our next few results consider $\mathfrak{W}(\Delta)$ contained in $c_0(\Delta)$.

THEOREM. *If* $L^1(G)^\wedge | \Delta \subseteqq \mathfrak{W}(\Delta)$ (in particular, if $\mathfrak{W}(\Delta) = c_0(\Delta)$) *then* Δ *is Sidon.*

Proof. Suppose $f \in C_\Delta(G)$—we show $\|\hat{f}\|_1 < \infty$ by using the boundedness principle 1.11. Take therein $A \equiv l^1(\Delta)$ with α the identity, $B \equiv L^1(G)$ with $\beta(g) = \hat{f}\hat{g} | \Delta$ and $C \equiv \mathfrak{C}^\Delta$ with the product topology. Then for some constant κ, for all $g \in L^1(G)$, there is $\phi \in l^1(\Delta)$ such that $\phi = \hat{f}\hat{g} | \Delta$ and $\sum_{\chi \in \Delta} |\phi(\chi)| \leqq \kappa \|g\|_1$. In other words, $\sum_{\chi \in \Delta} |\hat{f}(\chi)\hat{g}(\chi)| \leqq \kappa \|g\|_1$.

Allowing g to vary over an approximate identity,

$$\sum_{\chi \in \Delta} |\hat{f}(\chi)| < \infty$$

as required.

3.3. At the other end of the spectrum we can have equality in 1.2.

THEOREM. *If* Δ *is a subgroup of* X *then* $\mathfrak{W}(\Delta) = l^2(\Delta)$.

Proof. Obviously $l^2(\Delta) \subseteqq \mathfrak{W}(\Delta)$ by 1.2.

If $W \in \mathfrak{W}(\Delta)$ then by 1.3 we may suppose $\Delta = X$. Now by 1.10(iii) and [1], 2.1(a), it follows that $W \in l^2(\Delta)$. This completes the proof.

REMARKS 3.4. From 3.3 it follows that if Δ is cofinite in some subgroup of X then $\mathfrak{W}(\Delta) = l^2(\Delta)$.

Similarly by [10], 8.7.8, if Δ is cofinite in the positive cone of the ordered dual of a compact connected abelian group then $\mathfrak{W}(\Delta) = l^2(\Delta)$.

THEOREM 3.5. *For $\Delta \subseteqq X$, $\mathfrak{W}(\Delta)$ is an ideal in $M(G)^\wedge \mid \Delta$ which is improper iff Δ is Sidon. For each $W \in \mathfrak{W}(\Delta)$, $\|W\|_\delta \leqq \|W\|_\Delta$* (see 1.6 for notation).

Proof. If $W \in \mathfrak{W}(\Delta)$ by applying 1.10(iv) to $\phi = 1$, there is $\nu \in M(G)$ with $\hat{\nu} \mid \Delta = W$ and $\|\nu\| \leqq \|W\|_\Delta$. So $\mathfrak{W}(\Delta) \subseteqq M(G)^\wedge \mid \Delta$ and for all $W \in \mathfrak{W}(\Delta)$, $\|W\|_\delta \leqq \|W\|_\Delta$.

Obviously the algebraic operations on these spaces coincide and if $\mu \in M(G)$, for all $t \in T_\Delta(G)$ with $\|t\|_\infty \leqq 1$,

$$\sum_{\chi \in \Delta} | W(\chi)\hat{\mu}(\chi)\hat{t}(\chi) | \leqq \|\hat{\mu}\|_\infty \|W\|_\Delta .$$

Thus $W\hat{\mu} \mid \Delta \in \mathfrak{W}(\Delta)$ which, by 3.1, is consequently an ideal in $M(G)^\wedge \mid \Delta$ which is improper iff Δ is Sidon.

NOTE. By 3.3, $\mathfrak{W}(\Delta)$ need not be closed in $M(G)^\wedge \mid \Delta$.

3.6. As algebras, for $\Delta \subseteqq X$,

$$l^2(\Delta) \subseteqq \mathfrak{W}(\Delta) \subseteqq M(G)^\wedge \mid \Delta \subseteqq l^\infty(\Delta) .$$

Each is endowed with a norm-they are $\|\cdot\|_2$, $\|\cdot\|_\Delta$, $\|\cdot\|_\delta$ and $\|\cdot\|_\infty$ respectively. When Δ is a subgroup of X, $\|\cdot\|_2$ and $\|\cdot\|_\Delta$ are actually equivalent (by 3.3 and the open mapping theorem or [1], (2.1)(b)) on $\mathfrak{W}(\Delta)$.

A different proof of the inequality $\|\cdot\|_\delta \leqq \|\cdot\|_\Delta$ (established above) follows by the method in [10], 1.9.1 which yields the characterisation: for $W \in \mathfrak{W}(\Delta)$,

$$\|W\|_\delta = \sup \{ | \sum_{\chi \in \Delta} W(\chi)\hat{t}(\chi) | : t \in T_\Delta(G) \text{ and } \|t\|_\infty \leqq 1 \} .$$

This shows why, in 1.0, we kept the modulus signs inside the sum.

We now consider when pairs of these norms are equivalent.

THEOREM 3.7. *For $\Delta \subseteqq X$ these are equivalent:*

(i) *Δ is Sidon,*

(ii) *$\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$ are equivalent on $\mathfrak{W}(\Delta)$,*

(iii) *$\|\cdot\|_\delta$ and $\|\cdot\|_\Delta$ are equivalent on $M(G)^\wedge \mid \Delta$,*

(vi) *$\|\cdot\|_\delta$ and $\|\cdot\|_\infty$ are equivalent on $M(G)^\wedge \mid \Delta$.*

Proof. (a) If Δ is Sidon and $W \in \mathfrak{W}(\Delta)$ and $t \in T_\Delta(G)$ with $\|t\|_\infty \leqq 1$ then

$$\sum_{\chi \in \Delta} | W(\chi)\hat{t}(\chi) | \leqq \|W\|_\infty \sum_{\chi \in \Delta} | \hat{t}(\chi) | \leqq \|W\|_\infty \|\mathbf{1}\|_\Delta .$$

Thus whenever $W \in \mathfrak{W}(\Delta) = M(G)^\wedge \mid \Delta$,

$$\|W\|_\infty \leqq \|W\|_\delta \leqq \|W\|_\Delta \leqq \|\mathbf{1}\|_\Delta \|W\|_\infty \leqq \|\mathbf{1}\|_\Delta \|W\|_\delta ,$$

so the norms are pairwise equivalent.

(b) If Δ is not Sidon then by 3.2, $l^2(\Delta) \subseteqq \mathfrak{W}(\Delta) \subset c_0(\Delta)$. Since $l^2(\Delta)$ contains all finite linear combinations of characteristic functions of singleton subsets of Δ and these are dense in $c_0(\Delta)$, $\mathfrak{W}(\Delta)$ cannot be closed in $c_0(\Delta)$. Thus $\mathfrak{W}(\Delta)$ cannot be complete under the restriction of $\|\cdot\|_\infty$. So by 3.1, $\|\cdot\|_\infty$ and $\|\cdot\|_\Delta$ cannot be equivalent on $\mathfrak{W}(\Delta)$.

(c) If $\|\cdot\|_\delta$ and $\|\cdot\|_\Delta$ are equivalent on $M(G)^\wedge \mid \Delta$ then $\mathfrak{W}(\Delta) = M(G)^\wedge \mid \Delta$ hence by 3.5, Δ is Sidon.

(d) If $\|\cdot\|_\delta$ and $\|\cdot\|_\infty$ are equivalent on $M(G)^\wedge \mid \Delta$ then it is complete under $\|\cdot\|_\infty$ and hence $c_0(\Delta) \subseteqq M(G)^\wedge \mid \Delta$. So by 1.9(ii), $C_\Delta(G)^\wedge \mid \Delta \subseteqq l^1(\Delta)$ and so Δ is Sidon.

REMARKS 3.8. (i) As a Banach algebra, $\mathfrak{W}(\Delta)$ is neither separable nor a B^*-algebra in general. The former follows by 1.1 and the latter by 3.3.

(ii) Considering $C_\Delta(G)^\wedge \mid \Delta$ as a sequence space, $\mathfrak{W}(\Delta)$ is its α-dual (see [8], § 30). However 3.3 shows that $C_\Delta(G)^\wedge \mid \Delta$ is not, in general, a perfect sequence space.

3.9. Refer to [4], 1.1 for the definition of a p-Sidon set.

THEOREM. *Let $\Delta \subseteqq X$ and $1 \leqq p < 2$. Then Δ is p-Sidon iff $l^{p'}(\Delta) \subseteqq \mathfrak{W}(\Delta)$.*

Proof. For $p = 1$ this is just 1.1 (it is trivial when $p = 2$). If $1 < p < 2$ and Δ is p-Sidon then by [4], 1.2(ii), $f \in C_\Delta(G)$ implies $\hat{f} \mid \Delta \in l^p(\Delta)$. So if $W \in l^{p'}(\Delta)$, Hölder's inequality shows

$$\sum_{\chi \in \Delta} | W(\chi)\hat{f}(\chi) | < \infty$$

hence by 1.10, $W \in \mathfrak{W}(\Delta)$.

Conversely if $l^{p'}(\Delta) \subseteqq \mathfrak{W}(\Delta)$ then by 3.5, $l^{p'}(\Delta) \subseteqq M(G)^\wedge \mid \Delta$. So by [4], 1.2(iv), Δ is p-Sidon.

From this follows, by the Hausdorff-Young theorem, a converse of 3.2 for $p > 1$.

COROLLARY. *If $1 < p < 2$ and Δ is p-Sidon then $L^p(G)^\wedge \mid \Delta \subseteqq \mathfrak{W}(\Delta)$.*

4. Multipliers and W-Sidon sets.

4.0. When Δ is Sidon, spaces of Δ-spectral functions collapse. Not only is $L^\infty_\Delta(G) = A_\Delta(G)$ but $M_\Delta(G) = \bigcap \{L^p_\Delta(G) \colon 1 \leqq p < \infty\}$. In this section we investigate analogues for W-Sidon sets.

In this context it is natural to consider the trigonometric series $\sum_{\chi \in \Delta} W(\chi)\hat{\mu}(\chi)\chi$ for $\mu \in M_\Delta(G)$ (see for instance 1.15.) To ensure such objects make sense we define, for $\Delta \subseteq X$,

$$T: l^\infty(\Delta) \times PM_\Delta(G) \longrightarrow PM_\Delta(G)$$

by

$$T(\phi, \pi) = \sum_{\chi \in \Delta} \phi(\chi)\hat{\pi}(\chi)\chi .$$

When ϕ is fixed we shall use the single variable notation T_ϕ even for its restriction to some subset of $PM_\Delta(G)$.

If $\phi \in l^\infty(\Delta)$ let $\pi_\phi \in PM_\Delta(G)$ be given by

$$\hat{\pi}_\phi(\chi) = \begin{cases} \phi(\chi) & \text{if } \chi \in \Delta \\ 0 & \text{if } \chi \in X \backslash \Delta \end{cases} .$$

Then $T(\phi, \pi) = \pi_\phi * \pi$, for all $\pi \in PM_\Delta(G)$, so T is just convolution from $PM_\Delta(G) \times PM_\Delta(G)$ into $PM_\Delta(G)$. From this it is evident that T is bilinear, continuous and behaves nicely under translation and convolution.

THEOREM 4.1. *If Δ is W-Sidon and $t \in T_\Delta(G)$ then*

$$(4.1.1) \qquad \| T_W t \|_p \leqq 2 \| W \|_\Delta p^{1/2} \| t \|_2 \quad \text{if} \quad 2 < p < \infty$$

and

$$(4.1.2) \qquad \| T_W t \|_2 \leqq 8 \| W \|_\Delta \| t \|_1 .$$

Proof. We modify Rudin's proof for Sidon sets. For an exposition of the Rademacher functions $\{r_n: n \in \mathfrak{N}\}$ refer to [2], Chapter 14. By redefining r_n on a set of measure zero so that is is right continuous at each dyadic rational and left continuous at 1, we ensure $r_n \in \{\pm 1\}^{[0,1]}$.

For $t \in T_\Delta(G)$ let $j \in X^{\mathfrak{N}}$ be an injection with $sp(t) \subseteq j(\mathfrak{N})$, and define $R: X \rightarrow \{\pm 1\}^{[0,1]}$ by

$$R_\chi = \begin{cases} r_j^{-1}(\chi) & \text{if } \chi \in j(\mathfrak{N}) \\ r_0 & \text{if } \chi \in X \backslash j(\mathfrak{N}) \end{cases} .$$

Now let $f: G \times [0, 1] \rightarrow \mathfrak{C}$ be given by

$$f(x, \rho) = \sum_{\chi \in X} \hat{t}(\chi) R_\chi(\rho)\chi(x) .$$

Using single variable notation we have $f_\rho \in T_\Delta(G)$ for all $\rho \in [0, 1]$ and for all $x \in G$, $f_x = \sum_{n \in \mathfrak{N}} \hat{t}(j(n))j(n)(x)r_n$ which is a Rademacher series.

Since f is a finite sum of functions which are measurable on $G \times [0, 1]$ each dominated by the constant $\|t\|_\infty$, f is integrable and we may use Fubini's theorem.

Suppose $\rho \in [0, 1]$. By 1.10(iv), there is $\mu_\rho \in M(G)$ such that $\hat{\mu}_\rho(\chi) = W(\chi)R_\chi(\rho)$, for all $\chi \in \varDelta$ and $\|\mu_\rho\| \leqq \|W\|_\varDelta \|R.\ (\rho)\|_\infty = \|W\|_\varDelta$. So for $\chi \in \varDelta$,

$$\hat{\mu}_\rho(\chi)\hat{f}_\rho(\chi) = W(\chi)R_\chi(\rho)\hat{t}(\chi)R_\chi(\rho) = W(\chi)\hat{t}(\chi) = (T_W t)\hat{\ }(\chi) :$$

and if $\chi \in X \backslash \varDelta$,

$$(T_W t)\hat{\ }(\chi) = 0 = \hat{f}_\rho(\chi) .$$

Thus $T_W t = \mu_\rho * f_\rho$ hence $\|T_W t\|_p \leqq \|\mu_\rho\| \|f_\rho\|_p \leqq \|W\|_\varDelta \|f_\rho\|_p$.

So when $p = 2m$ (for some $m \in \mathfrak{N}$),

$$\int_G |T_W t|^{2m} \leqq \|W\|_\varDelta^{2m} \int_G |f_\rho|^{2m} . \tag{4.1.3}$$

But a property of Rademacher series ([2], 14.2.1) ensures that for all $x \in G$,

$$\int_0^1 |f_x|^{2m} \leqq (4m)^m \Big(\sum_{\chi \in X} |\hat{t}(\chi)\chi(x)|^2 \Big)^m .$$

So using Fubini's theorem to integrate (4.1.3) along [0, 1],

$$\int_G |T_W t|^{2m} \leqq \|W\|_\varDelta^{2m} (4m)^m \Big(\sum_{\chi \in \varDelta} |\hat{t}(\chi)|^2 \Big)^m . \tag{4.1.4}$$

Now given any $p \in (2, \infty)$ choose $m \in \mathfrak{N}$ such that $2(m-1) < p \leqq 2m$ and $1 < m \leqq p$. Then (4.1.4) guarantees

$$\|T_W t\|_p \leqq \|T_W t\|_{2m} \leqq 2\|W\|_\varDelta m^{1/2} \|t\|_2 \leqq 2\|W\|_\varDelta p^{1/2} \|t\|_2$$

which yields (4.1.1).

To prove (4.1.2) we argue similarly, except that for $t \in T_\varDelta(G)$ we redefine $f(x, \rho) = \sum_{\chi \in \varDelta} W(\chi)\hat{t}(\chi)R_\chi(\rho)\chi(x)$.

NOTATION 4.2. When $E, F \subseteq PM(G)$ and $\varDelta \subseteq X$ we shall write $(E_\varDelta, F_\varDelta)$ for the set of all $\phi \in \mathfrak{C}^\varDelta$ such that $\pi \in E_\varDelta$ implies $\phi\hat{\pi} \mid \varDelta \in F_\varDelta^\wedge \mid \varDelta$. Writing (E, F) for (E_X, F_X) we return to the standard multiplier notation.

4.3. Exploiting the conclusions of 4.1 we have

THEOREM. *If* $1 \leqq p, q \leqq \infty$ *with* $p \neq \infty$ *and* $q \neq 1$, *these are equivalent*:

(i) $\sup\{\|T_W t\|_q : t \in T_\varDelta(G)$ *and* $\|t\|_p \leqq 1\} < \infty$,

(ii) $f \in L^p_\Delta(G)$ *implies* $T_W f \in L^q_\Delta(G)$,
(iii) $W \in (L^p_\Delta(G), L^q_\Delta(G))$, *and*
(iv) $WL^{q'}(G)^\wedge|_\Delta \subseteq L^{p'}(G)^\wedge | \Delta$.

Proof. (i) $\Rightarrow$ (ii). Let $\{t_\alpha\} \subseteq T(G)$ be an approximate identity (see [6], (28·53)). If $f \in L^p_\Delta(G)$ then $t_\alpha * f \in T_\Delta(G)$ hence by (i), for some $\kappa > 0$

$$\| T_W(t_\alpha * f) \|_q \leqq \kappa \| t_\alpha * f \|_p \leqq \kappa \| f \|_p .$$

By the weak compactness of norm balls in $L^q(G)$ $(q \neq 1)$ there exists $g \in L^q(G)$ with $\| g \|_q \leqq \kappa \| f \|_p$ and $\hat{g} = W\hat{f}$. So by the uniqueness theorem, $T_W f = g \in L^q_\Delta(G)$.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv). By hypothesis and the boundedness result 1.11, $T_W: L^p_\Delta(G) \to L^q_\Delta(G)$ is bounded and linear. So by 1.8 and 1.9 there is a bounded linear map $K: L^{q'}(G)^\wedge | \Delta \to L^{p'}(G)^\wedge | \Delta$ for which, whenever $f \in L^{q'}(G)$ and $\chi \in \Delta$, $K(\hat{f} | \Delta)(\chi) = W(\chi)\hat{f}(\chi)$.

(iv) $\Rightarrow$ (i) follows similarly.

4.4. It is usually hard to identify (E_Δ, F_Δ) even when (E, F) is known (for $E, F \subseteq PM(G)$) so we pause to combine the approach of 3.1 with the result above.

COROLLARY. *Let* $1 \leqq p, q \leqq \infty$ *with* $p \neq \infty$ *and* $p \neq 1$. *Then* $W \in (L^p_\Delta(G), L^q_\Delta(G))$ *iff* $\sup\{\inf\{\| g \|_{p'}: g \in L^{p'}(G)$ *and* $\hat{g} | \Delta = W\hat{f} | \Delta\}: f \in L^{q'}(G)$ *and* $\| f \|_{q'} \leqq 1\} \equiv \sup\{\| T_W t \|_q: t \in T_\Delta(G)$ *with* $\| t \|_p \leqq 1\} < \infty$. $(L^p_\Delta(G), L^q_\Delta(G))$ *is a Banach space and when* $p \leqq q$ *it is a commutative Banach algebra which has an identity iff* $\Delta \in \Lambda(q)$.

REMARKS. (i). Although $(L^p_\Delta(G), L^q_\Delta(G))$ is unknown in general, special cases yield: $W \in (L^2_\Delta(G), L^2_\Delta(G))$ iff $W \in l^\infty(\Delta)$; and for $1 \leqq p < \infty$, $W \in (L^p_\Delta(G), L^\infty_\Delta(G))$ iff $W \in L^{p'}(G)^\wedge | \Delta$ by [2], 16.7.5.

(ii). Conditions sufficient to ensure membership to $(L^p(\mathfrak{T}), L^q(\mathfrak{T}))$ are known and yield:

$$\text{if } 1 < p \leqq 2 < q < \infty \text{ and } W \in \mathfrak{C}^\Delta \text{ with}$$
$$\sup\{| W(n) | (1 + | n |)^{1/p - 1/q}: n \in \Delta\} < \infty$$

then $W \in (L^p_\Delta(\mathfrak{T}), L^q_\Delta(\mathfrak{T}))$—see [2], 16.4.6(3). More involved conditions apply when $q = p$.

4.5. When $p = 1$, 4.3 can be extended 'at each end'.

COROLLARY. *For $1 < q < \infty$ these are equivalent:*
(i) $W \in (L_\Delta^1(G), L_\Delta^q(G))$,
(ii) $WM_\Delta(G)^\wedge | \Delta \subseteq L_\Delta^q(G)^\wedge | \Delta$,
(iii) $WL^{q'}(G)^\wedge | \Delta \subseteq L^\infty(G)^\wedge | \Delta$,
(iv) $WL^{q'}(G)^\wedge | \Delta \subseteq C(G)^\wedge | \Delta$.

Proof. (i) $\Rightarrow$ (ii) follows as in 4.3(i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (iii). Since (ii) $\Rightarrow$ (i), 4.3 implies this.
(iii) $\Rightarrow$ (iv). If $f \in L^{q'}(G)$, by [6], (32·30), there exist $g \in L^1(G)$ and $f_0 \in L^{q'}(G)$ with $f = g * f_0$. By (iii) there is $h_0 \in L^\infty(G)$ with $W\hat{f}_0 | \Delta = \hat{h}_0 | \Delta$. Setting $h = g * h_0$ gives $h \in C(G)$ and

$$\hat{h} | \Delta = \hat{g}\hat{h}_0 | \Delta = \hat{g} W \hat{f}_0 | \Delta = W\hat{f} | \Delta$$

as required.

4.6. More can also be said when $p = 2$.

THEOREM. *For $1 < q \leqq \infty$, $W \in (L_\Delta^2(G), L_\Delta^q(G))$ iff for all $f \in L^{q'}(G)$,*

$$\left(\sum_{\chi \in \Delta} | W(\chi)\hat{f}(\chi) |^2\right)^{1/2} \leqq \kappa \| f \|_{q'} \tag{4.6.1}$$

for some constant κ.

Proof. ($\Rightarrow$) uses the adjoint of T_W as in 4.3(iii) $\Rightarrow$ (iv).
($\Leftarrow$). Parseval's identity with the hypothesis shows $WL^{q'}(G)^\wedge | \Delta \subseteq L^2(G)^\wedge | \Delta$ hence by 4.3(iv), $W \in (L_\Delta^2(G), L_\Delta^q(G))$.

NOTE. By choosing an approximate identity the method above shows $W \in (L_\Delta^2(G), L_\Delta^\infty(G))$ iff $W \in l^2(\Delta)$, as noted in 4.4(i).

Since $(L_\Delta^1(G)), L_\Delta^\infty(G)) \subseteq (L_\Delta^2(G), L_\Delta^\infty(G))$ we have thus dealt with the case $q = \infty$ of 4.5. Alternatively,

$$(L_\Delta^p(G), L_\Delta^\infty(G)) \subseteq l^2(\Delta) \text{ when } 1 \leqq p \leqq 2.$$

See also 4.8.

4.7. Summarising what we have gleaned about W-Sidon sets by virtue of 4.1:

COROLLARY. *If Δ is W-Sidon then*
(i) *for all $\mu \in M_\Delta(G)$, $T_W\mu \in L_\Delta^2(G)$ and $\| T_W\mu \|_2 \leqq 8 \| W \|_\Delta \| \mu \|$,*
(ii) *for all $f \in L_\Delta^2(G)$, $T_W f \in L_\Delta^p(G)$ whenever $2 < p < \infty$ and $\| T_W f \|_p \leqq 2 \| W \|_\Delta p^{1/2} \| f \|_2$,*
(iii) *for all $\mu \in M_\Delta(G)$, $T_{W^2}\mu \in L_\Delta^p(G)$ whenever $2 < p < \infty$ and $\| T_{W^2}\mu \|_p \leqq 16 \| W \|_\Delta^2 p^{1/2} \| \mu \|$,*

(iv) *for all* $\phi \in l^2(\Delta)$, *there is* $f \in C(G)$ *such that* $\hat{f} \mid \Delta = W\phi$ *and* $\|f\|_\infty \leqq 8 \| W \|_\Delta \| \phi \|_2$, *and*

(v) *if* $1 < p \leqq 2$ *and* $f \in L^p(G)$ *then*

$$\Big(\sum_{\chi \in \Delta} | W(\chi)\hat{f}(\chi) |^2\Big)^{1/2} \leqq 2 \| W \|_\Delta p^{-1/2} \| f \|_p .$$

Proof. All are obvious except possibly (iii). If $\mu \in M_\Delta(G)$ and $2 < p < \infty$, by (i) and (ii),

$$\| T_{W^2}\mu \|_p = \| T_W(T_W\mu) \|_p \leqq 2 \| W \|_\Delta p^{1/2} \| T_W\mu \|_2 \leqq 16 \| W \|_\Delta^2 p^{1/2} \| \mu \| .$$

4.8. For which W can 1.10(vi) be tightened to

$$WL^p_\Delta(G)^\wedge \mid \Delta \subseteq l^1(\Delta) \tag{4.8.1}$$

for some $p \in [1, \infty)$? We show that when $1 \leqq p \leqq 2$, (4.8.1) holds iff Δ is a trivial W-Sidon set, and we give a partial answer when $2 < p < \infty$.

THEOREM. *If* $\Delta \subseteq X$ *then*

(i) $1 \leqq p < \infty$ *implies* $(L^p_\Delta(G), A_\Delta(G)) \subseteq L^{p'}(G)^\wedge \mid \Delta$,

(ii) $1 \leqq p \leqq 2$ *implies* $l^p(\Delta) \subseteq (L^p_\Delta(G), A_\Delta(G))$,

(iii) $2 < p < \infty$ *implies* $l^2(\Delta) \subseteq (L^p_\Delta(G), A_\Delta(G))$, *and*

(iv) $2 < p < \infty$ *implies* $(L^p_\Delta(G), A_\Delta(G)) \cap (L^2_\Delta(G), L^{p'}_\Delta(G)) \subseteq l^4(\Delta)$.

Proof. (i) This follows by 4.4(i) but may be proved quickly as follows. If $W \in (L^p_\Delta(G), A_\Delta(G))$ then letting K denote the composition of the isomorphism of 1.8(i) with T^*_W, we have $K: l^\infty(\Delta) \to L^{p'}(G)^\wedge \mid \Delta$ and whenever $\phi \in l^\infty(\Delta)$ and $\chi \in \Delta$, $(K\phi)(\chi) = W(\chi)\phi(\chi)$. Taking $\phi = 1$ this gives

$$f \in L^{p'}(G) \text{ with } \hat{f} \mid \Delta = W \tag{4.8.2}$$

as required.

(ii) If $1 \leqq p \leqq 2$ and $f \in L^p_\Delta(G)$ then by the Hausdorff-Young theorem and Hölder's inequality, whenever $W \in l^p(\Delta)$,

$$\sum_{\chi \in \Delta} | W(\chi)\hat{f}(\chi) | \leqq \| W \|_p \| f \|_p < \infty .$$

(iii) If $2 < p < \infty$ and $f \in L^p_\Delta(G)$ then $\hat{f} \mid \Delta \in l^2(\Delta)$ hence when $W \in l^2(\Delta)$,

$$\sum_{\chi \in \Delta} | W(\chi)\hat{f}(\chi) | \leqq \| W \|_2 \| \hat{f} \|_2 < \infty .$$

(iv) Continuing from (4.8.2), if $2 < p < \infty$, 4.6 shows

$$\left(\sum_{\chi \in \Delta} | W(\chi) |^4\right)^{1/2} \leqq 2 \| W \|_\Delta p^{1/2} \| f \|_{p'} ,$$

so $W \in l^4(\Delta)$.

REMARKS. (i) Taking W constant, (4.8.2) shows there can be no infinite Sidon sets Δ with $L^p_\Delta(G)^\wedge | \Delta \subseteqq l^1(\Delta)$ when $1 \leqq p < \infty$.

(ii) Results (i) and (ii) above combine to show that trivial W-Sidon sets are precisely the W-Sidon sets for which (4.8.1) holds when $p \in [1, 2]$.

Results (iii) and (iv) do not interlock in this way but show, thanks to 4.7(v), that when $p \in (2, \infty)$, (4.8.1) cannot hold when Δ is W-Sidon and $W \notin l^4(\Delta)$.

(iii) For comparison, $(L^p_\Delta(G), A_\Delta(G))$ is identified when Δ is a subgroup of X in [6], (36·20) via the method of 1.3.

4.9. When $W = 1$ the inclusions implied by 4.7 for Sidon sets are, by Parseval's identity, equalities. In fact these are the only W-Sidon sets with equality:

THEOREM. *Δ is Sidon whenever it is W-Sidon and one of these holds.*

(i) $l^2(\Delta) \subseteqq WM_\Delta(G)^\wedge | \Delta$,
(ii) $L^\infty(G)^\wedge | \Delta \subseteqq Wl^2(\Delta)$,
(iii) $C(G)^\wedge | \Delta \subseteqq Wl^2(\Delta)$,
(iv) $L^p_\Delta(G)^\wedge | \Delta \subseteqq Wl^2(\Delta)$, *for some* $p \in (2, \infty)$ *and*
(v) $l^2(\Delta) \subseteqq WL^p(G)^\wedge | \Delta$, *for some* $p \in (1, 2)$.

Proof. Theorem 1.12 as used in 1.14 makes short work of these.

4.10. So far we have discussed the behaviour of $T_W\pi$ when π is a Δ-spectral measure of L^p-function and Δ is W-Sidon. Immediately from 1.10(viii) we have: Δ is W-Sidon iff $WPM_\Delta(G)^\wedge | \Delta \subseteqq M(G)^\wedge | \Delta$. From 1.14(i) this inclusion is proper whenever Δ is not Sidon.

Evidently $T_W(PM_\Delta(G)) \subseteqq L^2_\Delta(G)$ iff Δ is a trivial W-Sidon set and if $T_W(PM_\Delta(G)) \subseteqq M_\Delta(G)$ then $W \in l^4(\Delta)$.

4.11. We now deduce more about those W in $\mathfrak{W}(\Delta)$. Specialising to $\mathfrak{T}$ (though (4.11.1) holds in general) we use:

THEOREM. *Let* $F \in \mathfrak{C}^3$. *If* $\phi F \in \bigcap \{L^p(\mathfrak{T})^\wedge : 1 \leqq p < \infty\}$ *for all* $\phi \in c_0(\mathfrak{Z})$ *then for all* $\alpha > 0$, $\sum_{n \neq 0} | n^{-\alpha} F(n) | < \infty$.

Proof. Successive applications of 1.11 and 1.8 show that if

$1<p<\infty$, then $\phi F \in L^p(\mathfrak{T})^\wedge$ for all $\phi \in c_0(\mathfrak{Z})$ implies $WL^{p'}_\Delta(G)^\wedge | \Delta \subseteq l^1(\Delta)$. So the hypothesis entails

(4.11.1) for all $p \in (1, \infty)$ and all $g \in L^p(\mathfrak{T})$, $\sum_{n \in \mathfrak{Z}} | F(n)\hat{g}(n) | < \infty$.

Now if $0<\alpha<1$ then by [2], Exercise 7.8, there exist $p \in (1,(1-\alpha)^{-1})$ and $g \in L^p(\mathfrak{T})$ such that $\hat{g}(n) = n^{-\alpha}$ for $n \neq 0$. If $\alpha \geqq 1$ then the map $n \mapsto n^{-\alpha}$ belongs to $l^2(\mathfrak{Z}\backslash\{0\})$ hence there is $g \in L^2(\mathfrak{T})$ with $\hat{g}(n) = n^{-\alpha}$ whenever $n \neq 0$.

In either case, substitution into (4.11.1) yields

$$\sum_{n\neq 0} | F(n)n^{-\alpha} | < \infty$$

as required.

NOTES. (i) In [12] we show the converse of this theorem to be false.

(ii) The sum $\sum_{n\neq 0} | n^{-\alpha}F(n) |$ was first considered by Hardy and Littlewood in [5]. Their results imply that it is finite whenever $\alpha > 1/2$ and may be infinite otherwise, when $F \in \bigcap \{L^p(\mathfrak{T})^\wedge : 1 \leqq p < \infty\}$.

4.12. The information this gives about W is:

COROLLARY. *If $W \in \mathfrak{W}(\Delta)$ then for all $\mu \in M_\Delta(\mathfrak{T})$, if $\alpha > 0$ then*

$$\sum_{n\neq 0} | n^{-\alpha}\hat{\mu}(n) W^2(n) | < \infty .$$

Proof. In fact if $\phi \in l^\infty(\mathfrak{Z})$ (not merely $c_0(\mathfrak{Z})$) and Δ is W-Sidon then evidently Δ is $W\phi^{1/2}$-Sidon. Hence by 4.7(ii), whenever $\mu \in M_\Delta(G)$,

$$\phi W^2 \hat{\mu} \in \bigcap \{L^p_\Delta(G)^\wedge : 1 \leqq p < \infty\}$$

so the conclusion follows from 4.11.

4.13. Using $l^\infty(\mathfrak{Z})$ rather than $c_0(\mathfrak{Z})$ above seems to be stronger. However in this context they are equivalent.

THEOREM. *Let $F \in \mathfrak{C}^X$. Then ϕF belongs to $\bigcap \{L^p(G)^\wedge : 1 \leqq p < \infty\}$ for all $\phi \in c_0(X)$ iff it does for all $\phi \in l^\infty(X)$.*

Proof. This follows readily upon taking the bidual of the map $K: c_0(X) \to L^p(G)$ given by $(K\phi)^\wedge = \phi F$.

4.14. It might be hoped that a tight necessary condition for W to belong to $\mathfrak{W}(\Delta)$ follows from 4.12 by eliminating μ somehow to give a purely combinatorial property. However the Δ-spectral measures compensate for variations in the thickness of Δ, so we turn to other means for this.

Refer to [3], 3.1 for the definition of a test family of order m.

THEOREM. *If* $W \in (L^p_\Delta(G), L^q_\Delta(G))$ *where* $1 \leqq p \leqq 2$ *and* $1 < q < \infty$, *and* $\mathfrak{F}$ *is a test family of order* m *then for each* $\Phi \in \mathfrak{F}$,

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leqq \kappa^2 m \nu(\Phi)^{2/q}$$

where κ *is the unnamed constant in* 4.4.

Proof. This is a routine modification of [3], 3.2 for which details appear in [11].

COROLLARY 4.15. *If* Δ *is* W*-Sidon and* $\mathfrak{F}$ *is a test family of order* m *then for each* $\Phi \in \mathfrak{F}$ *with* $\nu(\Phi) \geqq 3$,

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leqq 8e \|W\|_\Delta m \log \nu(\Phi) .$$

Proof. By hypothesis and 4.7(ii), $W \in (L^2_\Delta(G), L^q_\Delta(G))$ whenever $q \in (2, \infty)$ and so by 4.14,

$$\sum_{\chi \in \Phi \cap \Delta} |W(\chi)|^2 \leqq 4 \|W\|^2_\Delta q m \nu(\Phi)^{2/q} .$$

Taking $q = 2 \log \nu(\Phi)$ so that $q > 2$ because $\nu(\Phi) \geqq 3$, this entails the result.

NOTES. (i). This means that if $\varepsilon > 0$, the number of elements of Δ in Φ with $|W(\chi)| > \varepsilon$ remains small as Φ enlarges.

(ii). For $q = \infty$ the result above is overshadowed by the note to 4.6.

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