

## FUNDAMENTAL UNITS AND CYCLES IN THE PERIOD OF REAL QUADRATIC NUMBER FIELDS

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### PART II

1. Fundamental unit in  $Q(\sqrt{M})$  from the expansion of  $\sqrt{M}$ . In the first part of this paper we succeeded to state explicitly the periodic expansion of  $\sqrt{M}$ ,  $M$  a squarefree natural number, for infinitely many classes  $\sqrt{M}$ , each containing infinite many numbers. There are 14 types of these infinitely many classes, and they will all be enumerated here for the calculation of the fundamental unit  $e_f, |e_f| > 1$ , of the quadratic field  $Q(\sqrt{M})$ . There are many ways to calculate  $e_f$ . Many an elaborate mathematician like G. Degert [4] and H. Yokoi [7] have done so by finding the smallest solution of Pell's equation  $x^2 - My^2 = 1$ , or  $x^2 - My^2 = \pm 4$ , if the latter is solvable which necessitates  $M \equiv 1 \pmod{4}$ . Now to solve Pell's equation, poses another problem. If the expansion of  $\sqrt{M}$  as a periodic continued fraction is known, the problem is solved. For numerical values of  $M$ , this causes arithmetic difficulties only. If  $M$  is just a symbol standing for any natural number, the challenge of stating the periodic expansion of  $\sqrt{M}$  explicitly as a function of  $\sqrt{M}$ , has yet not been taken by mathematicians, except in a few cases enumerated by O. Perron [5]. These few cases have recently been enriched by a brilliant paper by Yamamoto [6], and by the author in [3]. Of course,  $M = D^2 + d, 1 \leq d \leq 2D$ , and the author conjectures that if we know a functional relationship  $D = D(d)$ , the periodic expansion can be stated explicitly, as was indeed demonstrated by the author in the first part of this paper for certain arithmetic functions  $D(d)$ . But if the expansion of  $\sqrt{M}$  as a periodic continued fraction is stated explicitly, the fundamental unit  $e_f$  of  $Q(\sqrt{M})$  can be also stated explicitly by methods which are generally known, and will be briefly reviewed here. We shall also restate the notations and formulas of the first part of this paper of which we shall frequently make use here.

$$(15.1) \quad \left\{ \begin{array}{l} \text{(i)} \quad \sqrt{M} = w = x = \frac{w + P_0}{Q_0} = b_0 + \frac{1}{x_1}; P_0 = 0; Q_0 = 1; \\ \quad \quad b_0 = [w]; \\ \text{(ii)} \quad x_k = \frac{w + P_k}{Q_k} = b_k + \frac{1}{x_{k+1}}; P_k = b_{k-1}Q_{k-1} - P_{k-1}; \end{array} \right.$$

$$(iii) \quad \begin{cases} Q_{k-1}Q_k = M - P_k^2; b_k = [x_k]; k = 1, 2, \dots, \\ (w + P_{k-1})(w + P_k) = Q_{k-1}[Q_k + b_{k-1}(w + P_k)]; \\ x_{k-1}x_k = \frac{Q_{k-1}b_{k-1}(w + P_k)}{Q_k}; k = 1, 2, \dots. \end{cases}$$

It is especially formula (15.1), (iii) we shall use most frequently here. We shall also use the notation we have introduced in previous papers [1] for the  $n-1$ -dimensional Jacobi-Perron algorithm and which should be retained for the Euclidean algorithm when  $n = 2$ , viz.,

$$(15.2) \quad \begin{cases} (i) & A_i^{(v)} = \delta_{iv}, (i, v = 0, 1; \delta_{iv} \text{ is Kronecker's symbol}). \\ (ii) & A_i^{(v+2)} = A_i^{(v)} + b_v A_i^{(v+1)}, (i = 0, 1; v = 0, 1, \dots). \\ (iii) & A_1^{(v+1)^2} - M A_0^{(v+1)^2} = (-1)^v Q_v, (v = 0, 1, \dots). \end{cases}$$

If  $m$  is the length of the primitive period of the expansion of  $\sqrt{M}$  as a periodic continued fraction, we obtain from (15.2), (iii), since  $Q_{tm} = 1, (t = 1, 2, \dots)$

$$(15.3) \quad \begin{cases} \text{All solution vectors of Pell's equation } x^2 - My^2 = (-1)^{mt}, \\ \text{are given by } (A_1^{(mt+1)}, A_0^{(mt+1)}); e_f = A_1^{(m+1)} + A_0^{(m+1)}w \text{ is a} \\ \text{unit in } Q(w), w = \sqrt{M}. \end{cases}$$

From

$$w = \frac{A_1^{(m)} + x_m A_1^{(m+1)}}{A_0^{(m)} + x_m A_0^{(m+1)}} = \frac{A_1^{(m)} + (w + b_0) A_1^{(m+1)}}{A_0^{(m)} + (w + b_0) A_0^{(m+1)}}$$

we obtain  $A_1^{(m+1)} = A_0^{(m)} + b_0 A_0^{(m+1)}$ ; hence

$$(15.4) \quad e_f = A_0^{(m)} + (w + b_0) A_0^{(m+1)} = A_0^{(m)} + x_m A_0^{(m+1)}.$$

$$(15.5) \quad \begin{cases} \text{If } Q_{v_0} = 4, \text{ then the smallest solution vector of Pell's} \\ \text{equation } x^2 - My^2 = (-1)^{v_0} 4 \text{ is given by } (A_1^{(v_0+1)}, A_0^{(v_0+1)}); \\ \text{if } M \equiv 1 \pmod{4}, e'_f = \frac{1}{2}(A_1^{(v_0+1)} + w A_0^{(v_0+1)}) \text{ is a unit in} \\ Q(w), w = \sqrt{M}. \end{cases}$$

As above, we obtain

$$\frac{1}{2} e'_f = A_0^{(v_0)} + x_{v_0} A_0^{(v_0+1)}.$$

In [2] the author proved (for the Jacobi-Perron, hence for the Euclidean algorithm)

$$(15.6) \quad \prod_{i=1}^v x_i = A_0^{(v)} + x_v A_0^{(v+1)}.$$

Since  $(A_1^{(m+1)}, A_0^{(m+1)}), (A_1^{(v_0+1)}, A_0^{(v_0+1)})$  are the smallest solution vectors

of the corresponding Pellian equations.

**THEOREM 15.** *If  $M \not\equiv 1 \pmod{4}$ , or if no  $Q_v$  in the expansion of  $\sqrt{M}$  as a periodic continued fraction equals 4, the fundamental unit of the field  $Q(w)$ ,  $w^2 = M$ , is given by*

$$(15.7) \quad e_f = \prod_{i=1}^m x_i ,$$

where  $m$  is the length of the primitive period of the expansion of  $w$ . If  $M \equiv 1 \pmod{4}$  and  $Q_{v_0} = 4$ , the fundamental unit of the field  $Q(w)$ ,  $w^2 = M$ , is given by

$$(15.8) \quad e_f = \prod_{i=1}^{v_0} x_i .$$

As H. Zassenhaus [8] has pointed out, this is the most effective method to calculate  $e_f$ .

**16. Fundamental unit of  $Q(w)$ ,  $w^2 = [(2a + 1)^k + a]^2 + 2a + 1$ ;  $a, k \geq 1$ .** From Theorem 1, Part I, we recall that the length of the primitive period of the expansion of  $w$  equals  $6k$ . Since, as can be easily verified by the reader,  $w^2 \equiv 1 \pmod{4}$ , the fundamental unit of  $Q(w)$ , according to (15.7), is given by

$$(16.1) \quad \frac{1}{2} e_f = \prod_{i=1}^{6k} x_i .$$

Taking into account formulas (0.9), (0.11), (0.13) and (15.1) and the structures of the  $x_i$  from §1, we obtain

$$(16.2) \quad e_f = x_1^2 \left( \prod_{s=1}^{k-1} x_{3s-1} x_{3s} x_{3s+1} \right)^2 Q_{3k}(x_{3k-1} x_{3k})^2 .$$

From (1.8), (i), (ii) we obtain, with formulas (15.1), (iii),

$$x_{3s-1} x_{3s} = \frac{(w + P_{3s-1})(w + P_{3s})}{Q_{3s-1} Q_{3s}} = \frac{Q_{3s} + b_{3s-1}(w + P_{3s})}{Q_{3s}} ;$$

but  $b_{3s-1} = 1$ ;  $Q_{3s} = 2A^{k-s}$ ;  $P_{3s} = A^k - 2A^{k-s} + (a + 1)$ ; hence

$$(16.3) \quad x_{3s-1} x_{3s} = \frac{w + A^k + (a + 1)}{2A^{k-s}} . \quad (A = 2a + 1)$$

Since  $P_{3s+1} = A^k - (a + 1)$ ;  $Q_{3s+1} = A^{s+1}$ , we obtain from (16.3)

$$\begin{aligned} & x_{3s-1} x_{3s} x_{3s+1} \\ &= \frac{w + A^k + (a + 1)}{2A^{k-s}} \cdot \frac{w + A^k - (a + 1)}{A^{s+1}} \end{aligned}$$

$$\begin{aligned}
&= \frac{w^2 + 2A^k w + A^{2k} - (a+1)^2}{2A^{k+1}} \\
&= \frac{A^{2k} + 2aA^k + (a+1)^2 + 2A^k w + A^{2k} - (a+1)^2}{2A^{k+1}} \\
&= \frac{2A^{2k} + 2aA^k + 2A^k w}{2A^{k+1}},
\end{aligned}$$

$$(16.4) \quad x_{3s-1} x_{3s} x_{3s+1} = \frac{w + A^k + a}{A}.$$

We further obtain  $x_{3k-1} x_{3k} = (Q_{3k} + b_{3k-1}(w + P_{3k}))/Q_{3k}$ , and since  $Q_{3k} = 2$ ,  $b_{3k-1} = 1$ ,  $P_{3k} = A^k - 2 + a + 1$ ,

$$(16.5) \quad x_{3k-1} x_{3k} = \frac{w + A^k + a + 1}{2}.$$

We further have, from (1.3)

$$(16.6) \quad x_1 = \frac{w + A^k + a}{A}.$$

Substituting the values of the  $x_i$  from (16.3)-(16.6) in (16.2), we obtain

$$\begin{aligned}
(16.7) \quad e_f &= \left(\frac{w + A^k + a}{A}\right)^2 \cdot \left(\frac{w + A^k + a}{A}\right)^{2(k-1)} \cdot 2 \cdot \left(\frac{w + A^k + a + 1}{2}\right)^2, \\
e_f &= \frac{\left(\frac{w + A^k + a}{A}\right)^2 \cdot (w + A^k + a + 1)^2}{2}.
\end{aligned}$$

Since the conjugate  $e'_f$  of  $e_f$  equals

$$\left(\frac{-w + A^k + a}{A}\right)^{2k} \cdot \frac{(-w + A^k + a + 1)^2}{2},$$

we verify that the norm of  $e_f$  is 1 by

$$\begin{aligned}
N(e_f) &= \left[\frac{(A^k + a)^2 - w^2}{A^2}\right]^{2k} \cdot \frac{[(A^k + a + 1)^2 - w^2]^2}{4} \\
&= \left[\frac{(A^k + a)^2 - \{(A^k + a)^2 + A\}}{A^2}\right]^{2k} \cdot \frac{[(A^k + a + 1)^2 - \{(A^k + a)^2 + A\}]^2}{4} \\
&= \frac{1}{4A^{2k}} \cdot [(A^k + a)^2 + 2(A^k + a) + 1 - (A^k + a)^2 - A]^2 \\
&= \frac{1}{4A^{2k}} \cdot 4A^{2k} = 1.
\end{aligned}$$

**17. Fundamental unit of  $Q(w)$ ,  $w = \sqrt{(A^k - a)^2 + A}$ ;  $A = 2a + 1$ ,  $a, k \geq 1$ .** From Theorem 2 we recall that the length of the primi-

tive period of the expansion of  $w$  equals  $6k - 2$ . Since,  $w^2 \equiv 1 \pmod{4}$ , for  $a = 2u$ ,  $Q_v = 4$  is possible. But the reader will verify easily from formulas (2.1) to (2.4), that  $Q$  is either odd or of the form  $2t$ ,  $t = 2s + 1$ . Thus the fundamental unit of  $Q(w)$ , according to (15.7) is given by the formula

$$(17.1) \quad e_f = \prod_{i=1}^{6k-2} x_i .$$

Though the cycle in the expansion of  $w$  is  $(x_{3s-2}, x_{3s-1}, x_{3s})$ , we shall arrange it in the form  $(x_{3s-1}, x_{3s}, x_{3s+1})$ ,  $s = 1, 2, \dots, k-1$ , leaving  $x_1$  outside the cycles, but including  $x_{3k-2}$ . This is done in order to have  $b_{3s} = 1$  not at the end of the cycle while making use of formula (15.1), (iii). We then obtain from (17.1)

$$(17.2) \quad e_f = x_1^2 \left( \prod_{s=1}^{k-1} x_{3s-1} x_{3s} x_{3s+1} \right)^2 x_{3k-1} x_{3k} .$$

From (2.3) we obtain, with formula (15.1), (iii),

$$(17.3) \quad x_{3s} x_{3s+1} = \frac{Q_{3s+1} + b_{3s}(w + P_{3s+1})}{Q_{3s+1}} = \frac{w + A^k + (a + 1)}{A^{s+1}} .$$

We further obtain in virtue of (17.3)

$$(17.4) \quad \left\{ \begin{array}{l} x_{3s-1} x_{3s} x_{3s+1} \\ = \frac{w + A^k + (a + 1)}{A^{s+1}} \cdot \frac{w + A^k - (a + 1)}{2A^{k-s}} = \frac{(w + A^k)^2 - (a + 1)^2}{2A^{k+1}} \\ = \frac{A^{2k} - 2aA^k + (a + 1)^2 + 2A^k w + A^{2k} - (a + 1)^2}{2A^{k+1}} \\ = \frac{2A^k w + 2A^{2k} - 2aA^k}{2A^{k+1}} , \\ x_{3s-1} x_{3s} x_{3s+1} = \frac{w + A^k - a}{A} . \end{array} \right.$$

We further have from (2.3) and since  $Q_{3k-1} = 2$ ,  $P_{3k-1} = P_{3k} = A^k - (a + 1)$

$$(17.5) \quad x_1 = \frac{w + A^k - a}{A}; \quad x_{3k-1} x_{3k} = \frac{[w + A^k - (a + 1)]^2}{2} .$$

From (17.2), (17.4), (17.5) we now obtain

$$(17.6) \quad e_f = \frac{\left( \frac{w + A^k - a}{A} \right)^{2k} [w + A^k - (a + 1)]^2}{2} .$$

We also have, as before,

$$(17.7) \quad N(e_f) = 1 .$$

18. **Fundamental unit of  $Q(w)$ ,  $w^2 = (A^k + a + 1)^2 - A$ ;  $A = 2a + 1$ ;  $a, k \geq 1$ .** From Theorem 3 we recall that the length of the primitive period of the expansion of  $w$  as a continued fraction equals  $4k + 2$ . The reader will verify easily from §3, that in this expansion  $Q_v$  cannot equal 4 for any  $v$ . Hence, according to (15.7), the fundamental unit of  $Q(w)$  is given by

$$(18.1) \quad e_f = \prod_{i=1}^{4k+2} x_i.$$

From (3.3), we obtain

$$P_{2s-1} = A^k + a, Q_{2s-1} = 2A^{k-s+1}; P_{2s} = A^k - a, Q_{2s} = A^s; \\ s = 1, \dots, k-1; Q_{2k+1} = 2.$$

Thus

$$(18.2) \quad e_f = \left( \prod_{s=1}^k x_{2s-1} x_{2s} \right)^2 2x_{2k+1}^2.$$

Now

$$(18.3) \quad \begin{cases} x_{2s-1} x_{2s} = \frac{w + A^k + a}{2A^{k-s+1}} \cdot \frac{w + A^k - a}{2A^s} = \frac{(w + A^k)^2 - a^2}{2A^{k+1}} \\ \quad = \frac{A^{2k} + 2(a+1)A^k + a^2 + 2A^k w + A^{2k} - a^2}{2A^{k+1}}, \\ x_{2s-1} x_{2s} = \frac{w + A^k + (a+1)}{A} \quad (s = 1, \dots, k). \end{cases}$$

Also

$$(18.4) \quad x_{2k+1} = \frac{A^k + a}{2}.$$

From (18.2), (18.3), (18.4) we now obtain

$$(18.5) \quad e_f = \frac{\left[ \frac{w + A^k + (a+1)}{A} \right]^{2k} (w + A^k + a)^2}{2}; N(e_f) = 1.$$

19. **Fundamental unit of  $Q(w)$ ,  $w^2 = [A^k - (a+1)]^2 - A$ ;  $A = 2a + 1$ ,  $a, k \geq 2$ .** From Theorem 4 we recall that the length of the primitive period of the expansion of  $w$  as a periodic continued fraction equals  $4(2k - 1)$ . The reader will verify easily from §4 that in the expansion of no  $Q_v$  can equal 4 for any  $v$ . Hence, according to (15.7), the fundamental unit of  $Q(w)$  is given by

$$(19.1) \quad e_f = \prod_{i=1}^{8k-4} x_i.$$

This case is more complicated than the previous one. The cycle is of length four and starts from the third member of the primitive period. We thus obtain for  $e_f$ , with  $Q_{4k-2} = 2$

$$(19.2) \quad e_f = (x_1 x_2)^2 \left( \prod_{s=1}^{k-2} x_{4s-1} x_{4s} x_{4s+1} x_{4s+2} \right)^2 2(x_{4k-5} x_{4k-4} x_{4k-3} x_{4k-2})^2.$$

From (4.3) we obtain, since  $b_1 = 1$

$$(19.3) \quad \begin{cases} x_1 x_2 = \frac{w + P_1}{Q_1} \cdot \frac{w + P_2}{Q_2} = \frac{Q_2 + b_1(w + P_2)}{Q_2} \\ \quad = \frac{A + w + A^k - (A + a + 1)}{A}, \\ x_1 x_2 = \frac{w + A^k - (a + 1)}{A}. \end{cases}$$

From (4.3), we obtain, since  $b_{4s-1} = 1, b_{4s+1} = 1$ ,

$$\begin{cases} x_{4s-1} x_{4s} = \frac{Q_{4s} + b_{4s-1}(w + P_{4s})}{Q_{4s}} = \frac{2A^{k-s} + w + A^k - 2A^{k-s} - a}{2A^{k-s}} \\ \quad = \frac{w + A^k - a}{2A^{k-s}}; \\ x_{4s+1} x_{4s+2} = \frac{Q_{4s+2} + b_{4s+1}(w + P_{4s+2})}{Q_{4s+2}} = \frac{A^{s+1} + w + A^k - A^{s+1} + a}{A^{s+1}} \\ \quad = \frac{w + A^k + a}{A^{s+1}} \end{cases}$$

hence

$$(19.4) \quad \begin{cases} x_{4s-1} x_{4s} x_{4s+1} x_{4s+2} = \frac{(w + A^k - a)(w + A^k + a)}{2A^{k+1}} \\ \quad = \frac{w^2 + 2A^k w + A^{2k} - a^2}{2A^{k+1}} \\ \quad = \frac{A^{2k} - 2(a + 1)A^k + a^2 + 2A^k w + A^{2k} - a^2}{2A^{k+1}} \\ x_{4s-1} x_{4s} x_{4s+1} x_{4s+2} = \frac{w + A^k - (a + 1)}{A}. \end{cases}$$

We further obtain from (4.3), for  $s = k - 1$  (these formulas indeed hold for  $s = 1, 2, \dots, k - 1$ )

$$\begin{cases} x_{4k-5} \cdot x_{4k-4} = \frac{Q_{4k-4} + b_{4k-5}(w + P_{4k-4})}{Q_{4k-4}} = \frac{2A + w + A^k - 2A - a}{2A} \\ \quad = \frac{w + A^k - a}{2A} \end{cases}$$

$$\left\{ \begin{aligned} x_{4k-3}x_{4k-2} &= \frac{Q_{4k-2} + b_{4k-3}(w + P_{4k-2})}{Q_{4k-2}} = \frac{2 + 2(w + A^k - a - 2)}{2} \\ &= w + A^k - (a + 1). \end{aligned} \right.$$

Thus

$$(19.5) \quad x_{4k-5}x_{4k-4}x_{4k-3}x_{4k-2} = \frac{w + A^k - a}{2} \cdot \frac{w + A^k - (a + 1)}{A}.$$

Substituting the values of (19.3), (19.4), (19.5) into (19.2), we obtain

$$(19.6) \quad \left\{ \begin{aligned} e_f &= \left( \frac{w + A^k - (a + 1)}{A} \right)^2 \left( \frac{w + A^k - (a + 1)}{A} \right)^{2(k-2)} \\ &\cdot 2 \left( \frac{w + A^k - (a + 1)}{A} \right)^2 \cdot \left( \frac{w + A^k - a}{2} \right)^2, \\ e_f &= \left[ \frac{w + A^k - (a + 1)}{A} \right]^{2k} \frac{(w + A^k - a)^2}{2}. \quad N(e_f) = 1. \end{aligned} \right.$$

20. **Fundamental units of  $Q(w)$ ,  $w^2 = [A^k + (A - 1)]^2 + 4A$ ;  $A = 2^d b$ ,  $b$  odd;  $d \geq 1$ , (not both  $d, b = 1$ );  $k \geq 2$ .** From the expansion of  $w$  in §5, the reader will verify easily that no  $Q_v$  equals 4. Thus  $e_f$  has the form  $e_f = a + bw$ ,  $a$  and  $b$  natural numbers. In this chapter, we shall approach the calculation of  $e_f$  by a different approach, since the direct calculation of  $\prod_{i=1}^m x_i$  (in the length of the primitive period) seems to raise insurmountable difficulties here. We shall prove the important formula

$$(20.1) \quad \underline{\underline{e_f = \left( \frac{w + A^k + A - 1}{2A} \right)^k \cdot \frac{w + A^k + A + 1}{2}}}$$

To justify statement (20.1), we have to prove three facts:

$$(20.2) \quad \left\{ \begin{aligned} (i) \quad &e_f \text{ is an algebraic integer of the form } a + bw, a, b \in N; \\ (ii) \quad &N(e_f) = \pm 1; \\ (iii) \quad &e_f \text{ is minimal.} \end{aligned} \right.$$

To prove (20.2), (i), we shall show that

$$(w + A^k + A - 1)^k (w + A^k + A - 1) \equiv 0 \pmod{2^{k+1}A^k}.$$

We obtain

$$(20.3) \quad \begin{aligned} &(a + A^k + A - 1)(w + A^k + A + 1) \\ &= w^2 + 2(A^k + A)w + (A^k + A)^2 - 1 \\ &= A^{2k} + 2(A - 1)A^k + A^2 + 2A + A^{2k} + 2A^{k+1} + A^2 + 2(A^k + A)w \\ &= 2[A^{2k} + 2A^{k+1} + A^2 + A - A^k + (A^k + A)w] \\ &= 2[A^{2k} + A^{k+1} + A^{k+1} + A^2 + A + A^k + (A^k + A)w - 2A^k] \end{aligned}$$

$$\begin{aligned}
 &= 2[A^k(A^k + A) + A(A^k + A) + (A^k + A) + (A^k + A)w - 2A^k] \\
 &= 2[(A^k + A)(w + A^k + A + 1) - 2A^k], \\
 (w + A^k + A - 1)(w + A^k + A + 1) \\
 &= 2(A^k + A)(w + A^k + A + 1) - 4A^k.
 \end{aligned}$$

From (20.3) we obtain, multiplying both sides by  $w + A^k + A - 1$ , and using formula (20.3) on the right side, we obtain

$$\begin{aligned}
 (w + A^k + A - 1)^2(w + A^k + A + 1) \\
 &= 2(A^k + A)[2(A^k + A)(w + A^k + A + 1) - 4A^k] \\
 &\quad - 4A^k(w + A^k + A - 1);
 \end{aligned}$$

Now, since  $A^k \equiv O(2^k)$ ,  $4A^{2k} \equiv O(2^{k+1}A^k)$ , hence

$$\begin{aligned}
 (20.4) \quad (w + A^k + A - 1)^2(w + A^k + A + 1) \\
 &\equiv 4(2A^{k+1} + A^2)(w + A^k + A + 1) - 8A^{k+1} \\
 &\quad - 4A^k(w + A^k + A - 1) \pmod{2^{k+1}A^k}.
 \end{aligned}$$

For  $k = 2$ , we obtain

$$\begin{aligned}
 (w + A^k + A - 1)^2(w + A^2 + A + 1) \\
 &\equiv (8A^3 + 4A^2)(w + A^k + A + 1) - 8A^3 - 4A^2(w + A^2 + A - 1) \\
 &\equiv 4A^2(w + A^2 + A + 1) - 4A^2(w + A^2 + A - 1) \\
 &\equiv 8A^2 \equiv O(8A^2).
 \end{aligned}$$

Thus (20.2), (i) is correct for  $k = 2$ .

The reader will have no difficulty to obtain from (20.4), multiplying both sides by  $w + A^k + A - 1$

$$\begin{aligned}
 (20.5) \quad (w + A^k + A - 1)^3(w + A^k + A + 1) \\
 &\equiv 8(3A^{k+2} + A^3)(w + A^k + A + 1) \\
 &\quad - 8^k(4A^2 - A + 1) - 8A^k(2A - 1)w \pmod{2^{k+1}A^k}.
 \end{aligned}$$

For  $k = 3$ , we obtain from (20.5), as can be easily verified

$$(w + A^3 + A - 1)^3(w + A^3 + A + 1) \equiv 16A^3w \equiv O(2^4A^3).$$

Thus (20.2), (i) is correct for  $k = 3$ . It is then easily proved by induction that (20.2), (i) correct for any  $k \geq 2$ .

To prove (ii), we obtain for the conjugate of  $e_f$

$$(20.6) \quad e'_f = \left( \frac{-w + A^k + A - 1}{2A} \right)^k \left( \frac{-w + A^k + A + 1}{2} \right).$$

From (20.6), (20.1) we obtain

$$\begin{aligned}
e_f \cdot e'_f &= \left[ \frac{(A^k + A - 1)^2 - w^2}{4A^2} \right]^k \left[ \frac{(A^k + A + 1)^2 - w^2}{4} \right] \\
&= \left[ \frac{A^{2k} + 2(A-1)A^k + (A-1)^2 - A^{2k} - 2(A-1)A^k - (A+1)^2}{4A^2} \right]^k \\
(20.7) \quad &\cdot \left[ \frac{A^{2k} + 2(A+1)A^k + (A+1)^2 - 2(A-1)A^k - (A+1)^2}{4} \right] \\
&= \left( \frac{-4A}{4A^2} \right)^k \left( \frac{4A^k}{4} \right) = (-1)^k \\
N(e_f) &= (-1)^k .
\end{aligned}$$

To prove (iii), namely that  $e_f$  given by (20.1) is really minimal, we first observe that, since the highest powers of  $A$  in both  $(w + A^k + A - 1)/2A$  and in  $(w + A^k + A - 1)/2$  are respectively  $A^{k-1}$  and  $A^k$ , the highest power of  $A$  in  $e_f$  equals  $(A^{k-1})^k \cdot A^k = A^{k^2}$ . We return now to formula (15.4), viz.,  $e_f = A_0^{(m)} + x_m A_0^{(m+1)}$ , where  $m$  is the length of the primitive period in the expansion of  $w$ . Since, by Theorem 5,  $m = 5k - 6$ , we obtain from formula (15.4)

$$(20.8) \quad e_f = A_0^{(5k-6)} + b_0 A_0^{(5k-5)} + A_0^{(5k-5)} w = A_0^{(5k-4)} + A_0^{(5k-5)} (w - b_0) .$$

We, therefore, have to calculate the power of  $A_0^{(5k-5)}$ . By Theorem 5, different expansions are obtained for odd and for even values of  $k$ . We shall carry out the calculation for odd  $k$ . From the values of the  $b_s - s$  in the expansion of  $w$  in §5, we obtain, by formulas (5.2),

$$\begin{aligned}
(20.9) \quad A_0^{(5s-1)} &= A_0^{(5s-3)} + b_{5s-3} A_0^{(5s-2)} = A_0^{(5s-3)} + 2A_0^{(5s-2)} ; \\
A_0^{(5s)} &= A_0^{(5s-2)} + b_{5s-2} A_0^{(5s-1)} = A_0^{(5s-2)} + \left( \frac{1}{2} A^s - 1 \right) (A_0^{(5s-3)} + 2A_0^{(5s-2)}) , \\
A_0^{(5s)} &= \left( \frac{1}{2} A^s - 1 \right) A_0^{(5s-3)} + (A^s - 1) A_0^{(5s-2)} . \\
A_0^{(5s+1)} &= A_0^{(5s-1)} + b_{5s-1} A_0^{(5s)} = A_0^{(5s-3)} + 2A_0^{(5s-2)} \\
&\quad + \left( \frac{1}{2} A^s - 1 \right) A_0^{(5s-3)} + (A^s - 1) A_0^{(5s-2)} , \\
A_0^{(5s+1)} &= \frac{1}{2} A^s A_0^{(5s-3)} + (A^s + 1) A_0^{(5s-2)} . \\
A_0^{(5s+2)} &= A_0^{(5s)} + b_{(5s)} A_0^{(5s+1)} = \left( \frac{1}{2} A^s - 1 \right) A_0^{(5s-3)} \\
&\quad + (A^s - 1) A_0^{(5s-2)} + \frac{1}{2} A^s A_0^{(5s-3)} + (A^s + 1) A_0^{(5s-2)} , \\
A_0^{(5s+2)} &= (A^s - 1) A_0^{(5s-3)} + 2A^s A_0^{(5s-2)} .
\end{aligned}$$

$$\begin{aligned}
 A_0^{(5s+3)} &= A_0^{(5s+1)} + b_{5s+1} A_0^{(5s+2)} = A_0^{(5s+1)} + \left(\frac{1}{2} A^{k-s-1} - 1\right) A_0^{(5s+2)} \\
 &= \frac{1}{2} A^s A_0^{(5s-3)} + (A^s + 1) A_0^{(5s-2)} + \left(\frac{1}{2} A^{k-s-1} - 1\right) (A^s - 1) A_0^{(5s-3)} \\
 &\quad + \left(\frac{1}{2} A^{k-s-1} - 1\right) \cdot 2A^s A_0^{(5s-2)}, \\
 (20.10) \quad A_0^{(5s+3)} &= \frac{1}{2} (A^{k-1} - A^s - A^{k-s-1} + 2) A_0^{(5s-3)} + (A^{k-1} + 1) A_0^{(5s-2)}.
 \end{aligned}$$

We shall use the symbolic writing  $u \stackrel{A}{=} v$  to express that  $u$  and  $v$  contain the same highest power of  $A$ . Since  $A_0^{(5s-2)} > A_0^{(5s-3)}$ , we obtain from (20.10)

$$(20.11) \quad A_0^{(5s+3)} \stackrel{A}{=} A^{k-1} A_0^{(5s-2)}.$$

Now  $A_0^{(0)} = 1$ ,  $A_0^{(1)} = 0$ ,  $A_0^{(2)} = 1$ ,  $A_0^{(3)} = A_0^{(1)} + b_1 A_0^{(2)}$ , and from (5.17), (i), since  $b_1 \stackrel{A}{=} A^{k-1}$ ,

$$(20.12) \quad A_0^{(3)} \stackrel{A}{=} A^{k-1}.$$

From (20.11), (20.12) we thus obtain the important relation

$$A_0^{(5s+3)} \stackrel{A}{=} A^{(k-1)(s+1)}.$$

But, according to Theorem 5,  $\max s = (1/2)(k-3)$ , hence

$$\begin{aligned}
 (20.13) \quad A_0^{(5/2(k-3)+3)} &\stackrel{A}{=} A^{(k-1)[1/2(k-3)+1]}, \\
 A_0^{1/2(5k-9)} &\stackrel{A}{=} A^{1/2(k-1)^2}.
 \end{aligned}$$

From (5.17), (iii),  $b_{1/2(5k-11)} \stackrel{A}{=} b_{1/2(5k-7)} \stackrel{A}{=} A^0$ ;  $b_{1/2(5k-9)} \stackrel{A}{=} A^{1/2(k-3)}$ . Hence, from (20.13),

$$\begin{aligned}
 (20.14) \quad A_0^{1/2(5k-7)} &\stackrel{A}{=} A_0^{1/2(5k-9)}; \quad A_0^{1/2(5k-5)} \stackrel{A}{=} A^{1/2(k-3)} A_0^{(5k-9)}, \\
 A_0^{1/2(5k-5)} &\stackrel{A}{=} A^{1/2[(k-1)^2+(k-3)]}.
 \end{aligned}$$

We are now at the midterm of the primitive period. If we still remember that the last term of the primitive period is  $2b_0 \stackrel{A}{=} A^k$ , we obtain  $b_0 A_0^{(5k-5)} \stackrel{A}{=} A^k \cdot (A^{1/2[(k-1)^2+(k-3)]})^2 = A^{k^2-2}$ , which proves (iii). The power  $k^2 - 2$  instead of  $k^2$  is the summation error. Thus it is proved that  $e_f$ , given by (20.1) is, indeed the fundamental unit of  $Q(w)$ ,  $w^2$  in the chapter title.

## 21. Fundamental units of various types of quadratic fields.

In this section we state the fundamental units of quadratic fields that have a structure similar to that of the previous chapter. They were treated in §6, 7, 8. We state the corresponding fundamental unit  $e_f$  by an expression  $g(w, A, k)$ , similar to that in (20.1); the proof that  $g(w, A, k)$  is an algebraic integer of the form  $a + bw$ ,  $a, b \in N$ , and that  $g(w, A, k)$  is minimal is completely analogous to the proof of the corresponding statements for  $e_f$  in (20.1), and will therefore not be repeated here; only the norm of  $g(w, A, k)$  will be found.

(i) Let  $w^2 = [A^k - (A - 1)]^2 + 4A$ ,  $A = 2^ab$ ,  $d, b \geq 1$ ;  $b$  odd;  $d, b$  not both 1;  $k \geq 2$ . The fundamental unit of  $Q(w)$  is given by

$$(21.1) \quad e_f = \frac{\left[ \frac{w + A^k - (A - 1)}{2A} \right]^k \cdot \frac{w + A^k - (A + 1)}{2}}{N(e_f) = (-1)^{k+1}};$$

We obtain from (21.1)

$$e'_f = \left[ \frac{-w + A^k - (A - 1)}{2A} \right]^k \cdot \left( \frac{-w + A^k - (A + 1)}{2} \right).$$

Hence

$$(21.2) \quad \begin{aligned} e_f \cdot e'_f &= \left[ \frac{[A^k - (A - 1)]^2 - w^2}{4A^2} \right]^k \cdot \left[ \frac{[A^k - (A + 1)]^2 - w^2}{4} \right] \\ &= \left[ \frac{A^{2k} - 2(A - 1)A^k + (A - 1)^2 - A^{2k} + 2(A - 1)A^k - (A + 1)^2}{4A^2} \right]^2 \\ &\cdot \left[ \frac{A^{2k} - 2(A + 1)A^k + (A + 1)^2 - A^{2k} + 2(A - 1)A^k - (A + 1)^2}{4} \right] \\ &= \left( -\frac{1}{A} \right)^k \cdot (-A^k), \end{aligned}$$

$$N(e'_f) = (-1)^{k+1} = N(e_f).$$

(ii) Let  $w^2 = [A^k + (A + 1)]^2 - 4A$ ;  $A = 2^ab$ ;  $d, b \geq 1$ ,  $b$  odd;  $d, b$  not both = 1;  $k \geq 2$ . The fundamental unit of  $Q(w)$  is given by

$$(21.3) \quad e_f = \frac{\left( \frac{w + A^k + A + 1}{2A} \right)^k \cdot \left( \frac{w + A^k + A - 1}{2} \right)}{N(e_f) = -1}.$$

(iii) Let  $w^2 = [A^k - (A + 1)]^2 - 4A$ ;  $A = 2^ab$ ,  $d, b \geq 1$ ,  $b$  odd;  $d, b$  not both = 1;  $k \geq 2$ .

The fundamental unit of  $Q(w)$  is given by

$$(21.4) \quad e_f = \frac{\left( \frac{w + A^k - (A + 1)}{2A} \right)^k \cdot \left( \frac{w + A^k - (A - 1)}{2} \right)}{N(e_f) = 1}.$$

(iv) Let  $w^2 = [2^{(d+2)k} + (2^d - 1)]^2 + 2^{d+2}$ ,  $d \geq 1$ .

The fundamental unit of  $Q(w)$  is given by

$$(21.5) \quad e_f = \frac{\left( \frac{w + 2^{(d+2)k} + (2^d - 1)}{2^{d+2}} \right)^k \cdot \left( \frac{w + 2^{(d+2)k} + (2^d + 1)}{2} \right)}{N(e_f) = (-1)^k} ;$$

(v) Let  $w^2 = [2^{(d+2)k} + (2^d + 1)]^2 - 2^{d+2}$ ,  $d \geq 1$ .

The fundamental unit of  $Q(w)$  is given by

$$(21.6) \quad e_f = \frac{\left[ \frac{w + 2^{(d+2)k} + (2^d + 1)}{2^{d+2}} \right]^k \cdot \left( \frac{w + 2^{(d+2)k} + (2^d - 1)}{2} \right)}{N(e_f) = -1} ;$$

The units in (iv) and (v) are calculated from the expansions considered in §§9 and 10.

22. Fundamental units from  $\prod_{i=1}^m x_i$ . In §21 we stated a unit  $e_f$  of  $Q(w)$  explicitly and then proved that it is fundamental. The explicit statement of  $e_f$  was not just a guess—we were guided by the pattern  $e_f$  achieved by the formula  $\prod x_i$ , as was done in §16. Concluding this paper, we shall return to the calculation of  $e_f$  from  $\prod_{i=1}^m x_i$ , using an expansion with the maximal length of a cycle in the period hitherto known, namely the length 12. This was achieved in Chapter 14. We recapitulate the results obtained there as follows.

$$(22.1) \quad \begin{aligned} w^2 &= [A^k - (A + 1)]^2 - 4A = A^{2k} - 2(A + 1)A^k + (A - 1)^2 ; \\ A &= 2^d b + 1; d, b \geq 1, b \text{ odd} . \end{aligned}$$

$$(22.2) \quad x_1 = \frac{w + A^k + (A + 2)}{2A^k - 3(2A + 1)}, b_1 = 1; x_2 = \frac{w + A^k - (5A + 1)}{4A} .$$

$$(22.3) \quad \left\{ \begin{aligned} b_{12s+3} &= 1, x_{12s+4} = \frac{w + A^k - A^{k-3s-1} - (A - 1)}{A^{k-3s-1}} ; \\ b_{12s+5} &= 1, x_{12s+6} = \frac{w + A^k - 4A^{3s+2} + (A - 1)}{4A^{3s+2}} ; \\ b_{12s+7} &= 2, x_{12s+8} = \frac{w + A^k - 2A^{k-3s-2} - (A - 1)}{4A^{k-3s-2}} ; \\ b_{12s+9} &= 1, x_{12s+10} = \frac{w + A^k - A^{3(s+1)} + (A - 1)}{A^{3(s+1)}} ; \\ b_{12s+11} &= 1, x_{12s+12} = \frac{w + A^k - 4A^{k-3(s+1)} - (A - 1)}{4A^{k-3(s+1)}} ; \\ b_{12s+13} &= 2, x_{12s+14} = \frac{w + A^k - 2A^{3(s+1)+1} + (A - 1)}{4A^{3(s+1)+1}} . \\ s &= 0, 1, \dots, s_0 - 1; s_0 = \frac{1}{6}(k - 4); k \equiv 4 \pmod{6} . \end{aligned} \right.$$

By Theorem 14, the length of the primitive period of the expansion of  $w$  equals  $m = 4k - 2$ , and has the form:

$$w = \overline{[b_0, \overline{b_1, b_2, \dots, b_{12s+3}, \dots, b_{12s+14}, b_{12s_0+3}, \dots, b_{12s_0+7}, 2, b_{12s_0+7}, \dots}, \overline{b_{12s_0+3}, \dots, b_2, b_1, 2b_0}]}$$

Since, as can be easily verified, no  $Q_v$  equals 4, we obtain for the calculation of the fundamental unit  $e_f$  of  $Q(w)$

$$(22.4) \quad e_f = (x_1 x_2)^2 (x_{12s_0+3} x_{12s_0+4} x_{12s_0+5} x_{12s_0+6})^2 \cdot x_{12s_0+7} x_{12s_0+8} Q_{12s_0+8} \left[ \prod_{s=0}^{s_0-1} (x_{12s+3} x_{12s+4} \dots x_{12s+14}) \right]^2.$$

We first prove

$$(22.5) \quad [w + A^k - (A - 1)][w + A^k + (A - 1)] = 2A^k [w + A^k - (A + 1)].$$

We obtain from (22.1)

$$\begin{aligned} [w + A^k - (A - 1)][w + A^k + (A - 1)] &= (w + A^k)^2 - (A - 1)^2 \\ &= w^2 + 2A^k w + A^{2k} - (A - 1)^2 \\ &= A^{2k} - 2(A + 1)A^k + (A - 1)^2 + 2A^k w + A^{2k} - (A - 1)^2 \\ &= 2A^{2k} - 2(A + 1)A^k + 2A^k w = 2A^k [w + A^k - (A + 1)]. \end{aligned}$$

We shall use repeatedly the formula  $x_i x_{t+1} = (Q_{t+1} + b_t(w + P_{t+1})) / Q_{t+1}$  and obtain from (22.2), (22.3)

$$(22.6) \quad \begin{aligned} x_1 x_2 &= \frac{4A + w + A^k - (5A + 1)}{4A} = \frac{w + A^k - (A + 1)}{4A}, \\ x_1 x_2 &= \frac{w + A^k - (A + 1)}{4A}. \end{aligned}$$

$$(22.7) \quad \left\{ \begin{aligned} x_{12s+3} \cdot x_{12s+4} &= \frac{w + A^k - (A - 1)}{A^{k-3s-1}}; & x_{12s+5} \cdot x_{12s+6} &= \frac{w + A^k + (A - 1)}{4A^{3s+2}}; \\ x_{12s+7} \cdot x_{12s+8} &= \frac{w + A^k - (A - 1)}{2A^{k-3s-2}}; & x_{12s+9} \cdot x_{12s+10} &= \frac{w + A^k + (A - 1)}{A^{3(s+1)}}; \\ x_{12s+11} \cdot x_{12s+12} &= \frac{w + A^k - (A - 1)}{4A^{k-3(s+1)}}; & x_{12s+13} \cdot x_{12s+14} &= \frac{w + A^k + (A - 1)}{2A^{3(s+1)+1}}. \end{aligned} \right.$$

From (22.7) we thus obtain

$$\prod_{i=0}^{11} x_{12s+3+i} = \left( \frac{[w + A^k - (A - 1)][w + A^k + (A - 1)]^3}{4A^{k+1}} \right)^3,$$

hence, by formula (22.5)

$$(22.8) \quad x_{12s+3} x_{12s+4} \dots x_{12s+14} = \left( \frac{w + A^k - (A + 1)}{2A} \right)^3.$$

we further obtain from (22.3), since these formulas also hold for  $s = s_0$

$$(22.9) \quad x_{12s_0+3}x_{12s_0+4}x_{12s_0+5}x_{12s_0+6} = \frac{w + A^k - (A + 1)}{2A},$$

and further

$$(22.10) \quad x_{12s_0+7}x_{12s_0+8} = \frac{w + A^k - (A - 1)}{2A^{k-3s_0-2}}, \quad Q_{12s_0+8} = 4A^{k-3s_0-2}.$$

(22.4) now takes the form, in virtue of (22.6), (22.8), (22.9), (22.10)

$$(22.11) \quad \left\{ \begin{array}{l} e_f = \left( \frac{w + A^k - (A + 1)}{4A} \right)^2 \left( \frac{w + A^k - (A + 1)}{2A} \right) \\ \quad \cdot 2[w + A^k - (A - 1)] \cdot \left[ \frac{w + A^k - (A + 1)}{2A} \right]^{6s_0}, \\ e_f = \left( \frac{w + A^k - (A + 1)}{2A} \right)^{6s_0+4} \cdot \left( \frac{w + A^k - (A - 1)}{2} \right), \\ \text{and, since } 6s_0 + 4 = k, \\ e_f = \underline{\underline{\left[ \frac{w + A^k - (A + 1)}{2A} \right]^k \cdot \left( \frac{w + A^k - (A - 1)}{2} \right)}}. \quad N(e_f) = 1. \end{array} \right.$$

We have investigated the case  $k$  even. By Theorem 14, when  $k \equiv 1 \pmod{6}$  is odd, the length of the primitive period is again  $4k - 2$ , and the reader will have now no difficulty that  $e_f$  has the same structure in this case as for even  $k$ . The pattern of  $e_f$  remains the same as previously. It is also easily verified that  $e_f$  has the following structure for the fields  $Q(w)$ .

(i) Let  $w^2 = [A^k + (A - 1)]^2 + 4A$ ;  $A = 2^d b + 1$ ;  $d, b \geq 1, b$  odd. Then the fundamental unit of  $Q(w)$  is given by

$$(22.12) \quad e_f = \underline{\underline{\left( \frac{w + A^k + A - 1}{2A} \right)^k \cdot \left( \frac{w + A^k + A + 1}{2} \right)}}; \quad N(e_f) = (-1)^k.$$

(ii) Let  $w^2 = [A^k - (A - 1)]^2 + 4A$ ,  $A = 2^d b + 1$ ;  $d, b \geq 1, b$  odd. Then the fundamental unit of  $Q(w)$  is given by

$$(22.13) \quad e_f = \underline{\underline{\left( \frac{w + A^k - (A - 1)}{2A} \right)^k \cdot \left( \frac{w + A^k - (A + 1)}{2} \right)}}; \quad N(e_f) = (-1)^{k+1}.$$

(iii) Let  $w^2 = [A^k + (A + 1)]^2 - 4A$ ,  $A = 2^d b + 1$ ;  $d, b \geq 1, b$  odd. Then the fundamental unit of  $Q(w)$  is given by

$$(22.14) \quad e_f = \underline{\underline{\left( \frac{w + A^k + A + 1}{2A} \right)^k \cdot \left( \frac{w + A^k + A - 1}{2} \right)}}; \quad N(e_f) = -1.$$

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