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LATTICE ORDERINGS ON THE REAL FIELD

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Since every total order is a lattice order, and the real field $\boldsymbol{R}$ is a totally ordered field, it is a lattice-ordered field. In 1956 Birkhoff and Pierce raised the question of whether $\boldsymbol{R}$ can be made into a lattice-ordered field in any other way. In this paper we answer their question affirmatively by showing that there are, in fact, 2^c such orderings, where c is the cardinal of $\boldsymbol{R}$.

Introduction. We answer the question of the existence of such orderings, raised by Birkhoff and Pierce in [2, p. 68], in Theorem 1, and find the number of orders in Corollary 1.2. We denote the rational field by $\boldsymbol{Q}$, the *positive cone* of $\boldsymbol{R}$ (i.e., the set of reals $\geqq 0$) in the usual order by $\boldsymbol{R}^+$, and the positive cone of $\boldsymbol{Q}$ by $\boldsymbol{Q}^+$.

THEOREM 1. *Let L be any subfield of $\boldsymbol{R}$ except $\boldsymbol{Q}$. Let K be any proper subfield of L, such that L is algebraic over K. Then there is a relation $\leqq$ on L, with positive cone P_L, such that $\langle L, \leqq\rangle$ is a lattice-ordered field which is not totally ordered. Moreover*:

(1) *The order $\leqq$ restricted to K is the usual total order ($K \cap P_L = K \cap \boldsymbol{R}^+$).*

(2) *K is the largest totally ordered subfield of L under $\leqq$.*

(3) *The order $\leqq$ is a distributive lattice order.*

(4) *The order $\leqq$ is* ***R*-compatible** *($P_L \subseteqq \boldsymbol{R}^+$).*

(5) *$L \cap \boldsymbol{R}^+$ is* **quotient-represented** *by P_L, in the sense that for each $l \in L \cap \boldsymbol{R}^+$, there exist $p, q \in P_L$ with $q \neq 0$, such that $l = p/q$.*

We will give the proof in Section 2, where we state the main lemma (see 2.2). We will use the assertion (2) in counting the number of such orders, and we will need the technical feature (5) in the construction process.

COROLLARY 1.1. *Let L be a subfield of $\boldsymbol{R}$ containing κ distinct subfields K such that L is algebraic over K. Then L admits at least κ distinct lattice orders.*

Proof. By (2), these distinct subfields give distinct orders.

COROLLARY 1.2. *$\boldsymbol{R}$ admits exactly 2^c and the algebraic numbers $\boldsymbol{A}$ admit exactly $2^{\aleph_0}$ lattice orders.*

Proof. $\boldsymbol{R}$ is known to be algebraic over 2^c distinct subfields and

A over $2^{\aleph_0}$.

In fact, $\boldsymbol{R}$ may be replaced by any uncountable subfield in Corollary 1.2. Similarly, A may be replaced by any other countable subfield which is algebraic over $2^{\aleph_0}$ subfields.

We observe that $\boldsymbol{R}$-compatibility excludes from consideration many orders on proper subfields L. For example, for every $\boldsymbol{R}$-compatible order on $\boldsymbol{Q}(\sqrt{2})$ the non-trivial field automorphism produces another order which is not $\boldsymbol{R}$-compatible. Even though it can be shown that $\boldsymbol{R}$-compatibility follows from quotient-representability, which plays an important role in the construction process, we require $\boldsymbol{R}$-compatibility during the inductive step to show that quotient-representability extends. Thus we cannot dispense with $\boldsymbol{R}$-compatibility and, indeed, must prove it independently.

When P_M is the positive cone for an order $\leqq$ on some subfield M of $\boldsymbol{R}$, we will refer order expressions to P_M by (wrt P_M) meaning with respect to P_M.

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2. Main lemma and proof of Theorem 1. Our method of proof employs judiciously chosen algebraic bases to extend orders. Thus, if K is ordered by $\leqq$ with positive cone P_K, if M is an extension field of K, and if B is a basis for M over K, we write $P_K(B)$ for the set of finite sums of the form $\sum k_i b_i$ with $k_i \in P_K$ and $b_i \in B$. For $B = \{b_1, \cdots, b_m\}$, we write $P_K(b_1, \cdots, b_m)$.

REMARK. 2.1. If P_K is the positive cone for a lattice order on K and B is a basis for M over K, then it is immediate that $P_K(B)$ is closed under addition and that $P_K(B)$ induces a lattice order $\leqq$ on M considered as a group (since ordering, like addition, is computed 'coordinatewise'). Moreover, if the order on the 'coordinate' field is total, then $\leqq$ is distributive.

To prove Theorem 1, we start with $P_K = K \cap \boldsymbol{R}^+$ and consider the collection $\mathfrak{m} = \{\langle M, B\rangle\}$ where M is an intermediate field and where $B = B_M$ is a basis for M over K such that $B \subset L \cap \boldsymbol{R}^+$ and such that:

(a) $P_K(B)$ is closed under multiplication (for which it will be sufficient to show that $b \cdot c \in P_K(B)$ for all b and c in B):

(b) $P_K(B)$ is $\boldsymbol{R}$-compatible;

(c) $P_K(B)$ quotient-represents $M \cap \boldsymbol{R}^+$; and

(d) $1 \in B$.

By (a) and (c) above and Remark 2.1 we see that the order $\leqq$ on M with positive cone $P_K(B)$ make $\langle M, \leqq\rangle$ into a lattice-ordered field satisfying (3) and (5). Our original choice of P_K as $K \cap \boldsymbol{R}^+$ and (d) give (1) while the fact that distinct elements of B are incomparable with respect to $\leqq$ gives us (2). Finally, (4) is just (b).

Thus, Theorem 1 will be proven if we show that $\langle L, B_L \rangle$ belongs to $\mathfrak{m}$. In fact, we will employ induction, in the form of Zorn's lemma (though we may also view it as a transfinite induction using successive simple extensions) to choose a maximal $\langle M_0, B_{M_0} \rangle$ from $\mathfrak{m}$ and we will see that $M_0 = L$. For the inductive step we will require the following lemma.

MAIN LEMMA 2.2. *Let M and M' be subfields of $\boldsymbol{R}$ with M' a finite algebraic extension of M. Suppose that P_M is the positive cone of a lattice order $\leqq$ on M which quotient-represents $M \cap \boldsymbol{R}^+$. Then there exists $\alpha \in M'$ such that*

(i) $M' = M[\alpha]$,

(ii) *α satisfies $\alpha^n = a_{n-1}\alpha^{n-1} + \cdots + a_0$ with $a_i \in P_M$* (where n is the degree of M' over M),

(iii) *$P_M(1, \alpha, \cdots, \alpha^{n-1})$ is $\boldsymbol{R}$-compatible, and*

(iv) *$P_M(1, \alpha, \cdots, \alpha^{n-1})$ quotient-represents $M' \cap \boldsymbol{R}^+$.*

The proof will be given in the next section.

For the inductive step we suppose $\langle M, B \rangle$ is a member of $\mathfrak{m}$ so that B is a basis for M over K satisfying (a), (b), (c) and (d). We suppose that $M \neq L$, so that there exists a proper simple extension M' of M with $M' \subset L$. We choose $\alpha \in M'$ using the Main Lemma and consider $B' = \{b\alpha^i \mid b \in B, 0 \leqq i \leqq n-1\}$. Now $B' \supset B$ and is a basis for M' over K satisfying (a), (b), (c) and (d). Thus $\langle M', B' \rangle \in \mathfrak{m}$. Clearly, any maximal member M_0 of $\mathfrak{m}$ must be L.

3. Proof of Main Lemma. In outline, the proof proceeds as follows:

Step 1. We find a β such that $M' = M[\beta]$, $\beta > 1$(wrt $\boldsymbol{R}^+$), and β satisfies $\beta^n = b_{n-1}\beta^{n-1} + \cdots + b_0$ with $b_i \in M \cap \boldsymbol{R}^+$. That is, (ii) holds except that $M \cap \boldsymbol{R}^+$ replaces P_M. (This step depends only on the usual topology of $\boldsymbol{R}$ and $\boldsymbol{C}$ and the usual order structure of $\boldsymbol{R}$.)

Step 2. We use quotient-representability to replace β by $\alpha \in M'$ so that (i) $M' = M[\alpha]$, $\alpha > 1$ (wrt $\boldsymbol{R}^+$), and α satisfies (ii). We write P'_M for $P_M(1, \alpha, \cdots, \alpha^{n-1})$. It is clear that $P'_M \subset M \cap \boldsymbol{R}^+$ (i.e., that P'_M satisfies (iii)).

For use in the remaining steps we define

$$Q'_M = \{p/q \mid p, q \in P'_M, q \neq 0\}\ ,$$

which is the positive cone of an $\boldsymbol{R}$-compatible partial order on M'.

Step 3. To show P'_M quotient-represents $M' \cap \boldsymbol{R}^+$ it is clearly sufficient to show $M' \cap \boldsymbol{R}^+ \subset Q'_M$. To this end, we show that $\boldsymbol{Q}^+ \subset Q'_M$ and, after defining the concept of ***Q**-approximability*, we show how $\boldsymbol{Q}$-approximability of M' implies $M' \cap \boldsymbol{R}^+ \subset Q'_M$.

Step 4. We show that α is $\boldsymbol{Q}$-approximable, that every element of M is $\boldsymbol{Q}$-approximable and that the $\boldsymbol{Q}$-approximable elements of M' constitute a subring and therefore must be $M[\alpha] = M'$ itself.

Details of Step 1. We let γ be such that $M' = M[\gamma]$ and its minimal polynomial is $h(x)$. We suppose $\gamma = \gamma_1, \gamma_2, \cdots, \gamma_n$ are all the (necessarily distinct) roots of h in $\boldsymbol{C}$. We show below how to construct a non-singular linear fractional transform T with rational coefficients so that $\beta = T(\gamma) > 1(\text{wrt}\, \boldsymbol{R}^+)$ and for $2 \leqq i \leqq n$ the $\beta_i = T(\gamma_i)$ are "sufficiently close" to $-1/n$. Since the coefficients are continuous in the roots, a comparison with $(x + 1/n)^{n-1}$ shows that $(x - \beta_2) \cdots (x - \beta_n) = x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_0$ satisfies $1 = c_{n-1} > c_{n-2} > \cdots > c_0 > 0$. (See [3, L. 6.2, p. 40] for details, including proof that "sufficiently close" means "within $\varepsilon = 1/n^2$".) Therefore $g(x) = (x - \beta)\cdot(x - \beta_2) \cdots (x - \beta_n) = x^n - b_{n-1}x^{n-1} - \cdots - b_0$ where $b_i = c_i\beta - c_{i-1} > c_i - c_{i-1} > 0$ for $1 \leqq i \leqq n - 1$ and $b_0 = c_0\beta > 0$. We note that $g(x)$ is the minimal polynomial of β over M and is computed by clearing the denominators of $h(T^{-1}(x))$ and scaling.

To construct T we let $\varepsilon = 1/n^2$ as above and choose rationals t and s such that $0 < t(1/\varepsilon + 1/2) < \min|\beta - \beta_i|$ for $i \geqq 2$ and $0 < \beta/t - s < 1/2$. Then T is the composition of the following maps: $x \to x/t$; $x \to x - s$; $x \to 1/x$; $x \to x - 1/n$. (After the first two the image of γ is in the interval $(0, 1/2)$ and the rest of the roots are outside a circle of radius $1/\varepsilon$ centered at 0, and after the last $\beta = t/(\gamma - ts) - 1/n > 1$ and the other β_i are within ε of $-1/n$.)

Details of Step 2. Using quotient-representability, we choose $d \in P_M$ so that $db_i \in P_M$ for $0 \leqq i \leqq n - 1$ and so that $d > 1(\text{wrt}\, \boldsymbol{R}^+)$. (The latter Condition may be achieved by positive integer scaling without affecting the former conditions.) Then $\alpha = d\beta$ has $f(x) = d^n g(x/d) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$ as its minimal polynomial over M and $a_i = d^{n-i}b_i \in P_M$ for $0 \leqq i \leqq n - 1$. That is, α satisfies (ii). Clearly $M[\alpha] = M[\beta] = M[\gamma] = M'$.

Details of Step 3. Let $r \in \boldsymbol{Q}^+$ and $p \in P'_M$. We may scale p by positive integers, as above, and by reciprocals of such using [1, Thm. 3, p. 293] with the result rp in P'_M. Since we may write $r = rp/p$ for any non-zero p, we see that $r \in Q'_M$.

We say that $m \in M'$ is $\boldsymbol{Q}$-*approximable* if for each positive rational t there is a rational s such that $m < s < m + t(\text{wrt } Q'_M)$. (Of course, $m - t < s - t < m(\text{wrt } Q'_M)$ also.)

Now if $m > 0(\text{wrt } \boldsymbol{R}^+)$ and m is $\boldsymbol{Q}$-approximable, we choose a positive rational t so that $m - t > 0(\text{wrt } \boldsymbol{R}^+)$ and rational s so that $m < s < m + t(\text{wrt } Q'_M)$. By $\boldsymbol{R}$-compatibility

$$s - t > m - t > 0(\text{wrt } \boldsymbol{R}^+) ,$$

and since $\boldsymbol{Q}^+ \subset Q'_M$, $s - t \in Q'_M$. Thus $m = ((m + t) - s) + (s - t) \in Q'_M$ by the additive closure of $\boldsymbol{Q}'_M$.

Details of Step 4. To show M is $\boldsymbol{Q}$-approximable, we arbitrarily choose m in M and t positive in $\boldsymbol{Q}$. By density of $\boldsymbol{Q}$ in $\boldsymbol{R}$, there is a rational s such that $m < s < m + t(\text{wrt } \boldsymbol{R}^+)$ and by quotient-representability of $M \cap \boldsymbol{R}^+$ by P_M these inequalities hold wrt Q_M. But $Q_M \subset Q'_M$ so $m < s < m + t(\text{wrt } Q'_M)$, which is $\boldsymbol{Q}$-approximability.

To verify that α is $\boldsymbol{Q}$-approximable, we again choose an arbitrary positive t in $\boldsymbol{Q}$. Since $f(x) = 0$ and f is separable, the derivative $f'(\alpha)$ is non-zero. Because $\alpha > 1(\text{wrt } \boldsymbol{R}^+)$ this implies there is a rational $s > 1(\text{wrt } \boldsymbol{R}^+)$ such that $0 < f(s) < t(\text{wrt } \boldsymbol{R}^+)$. For such s we show $\alpha < s < \alpha + t(\text{wrt } Q'_M)$ and hence that α is $\boldsymbol{Q}$-approximable:

First, since s, $f(s)$, and t are in M and M is quotient-representable, we see that $s > 1$ and $0 < f(s) < t(\text{wrt } Q_M)$ and hence $(\text{wrt } Q'_M)$. Next we note that Q'_M is closed under division and $s - \alpha = f(s)/(f(s)/(s - \alpha))$, so, to show $s - \alpha > 0(\text{wrt } Q'_M)$, we need only show $f(s)/(s - \alpha) \in Q'_M$. Now $f(s) = f(s) - f(\alpha) = (s^n - \alpha^n) - a_{n-1}(s^{n-1} - \alpha^{n-1}) - \cdots - a_1(s - \alpha)$. Thus $f(s)/(s - \alpha) = s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_0$, where the $d_i = \alpha^{n-i-1} - a_{n-1}\alpha^{n-i-2} - \cdots - a_{i+1}$ are "scaled truncations" of $f(\alpha)$. In fact, $d_i = (f(\alpha) + a_i\alpha^i + \cdots + a_0)/\alpha^{i+1} = (a_i\alpha^i + \cdots + a_0)/\alpha^{i+1} \in Q'_M$. Since $s^i > 1 > 0(\text{wrt } Q'_M)$, $f(s)/(s - \alpha) > 1 > 0(\text{wrt } Q'_M)$. Thus $0 < s - \alpha < f(s) < t(\text{wrt } Q'_M)$ so that $\alpha < s < \alpha + t(\text{wrt } Q'_M)$.

To finish Step 4 and thus the proof of the Main Lemma, we need to show the set of $\boldsymbol{Q}$-approximable elements of M' is a subring. The proof of closure under subtraction is straightforward, after recalling that we can approximate below also. The proof of closure under multiplication, though resembling the proof of the product rule for derivatives, takes some care. At several points when dealing with the rationals used as "epsilons and deltas" by the approximating process it is necessary to switch from $\boldsymbol{R}^+$ to $\boldsymbol{Q}'_M$ using $\boldsymbol{Q}^+ \subset Q'_M$ or from Q'_M to $\boldsymbol{R}^+$ using $\boldsymbol{R}$-compatibility.

4. Alternate theorem, examples and questions. By a slight modification in the proof of Theorem 1 we can prove Theorem 1*, which differs from Theorem 1 in that (1) and (2) are replaced by their complete opposites (1*) and (2*) and in that (0*), which has no counterpart in Theorem 1, is added.

(0*) *There are no totally ordered subfields of L under $\leqq$.*

(1*) *The order $\leqq$ restricted to K is the trivial partial order.* (In particular, $1 \not> 0$.)

(2*) *K is the largest trivially ordered subfield of L under $\leqq$.*

Before we prove Theorem 1*, we note that Corollaries 1.1 and 1.2 also hold for this type of order. In the proof of Theorem 1* we indicate by * the changes from the proof of Theorem 1.

We again start with $P_K = K \cap \boldsymbol{R}^+$ and seek $B^* \subset L \cap \boldsymbol{R}^+$ satisfying (a), (b) and (c) as before, but

$$(\mathrm{d}^*) 1 \notin P_K(B^*)$$

instead of (d). Now (3), (4) and (5) follow as before and (d*) implies (1*). To see this, we suppose (1*) false and pick $k \in K \cap P_K(B^*)$ with $k \neq 0$. Then $0 < k(\mathrm{wrt}\,\boldsymbol{R}^+)$ by $\boldsymbol{R}$-compatibility and so $k^{-1} \in K \cap \boldsymbol{R}^+$. Thus $0 < (k^{-1})k = 1(\mathrm{wrt}\,P_K(B^*))$, contradicting (d*).

The fact that every $b \in B^*$ satisfies $0 < b(\mathrm{wrt}\,P_K(B^*))$ gives (2*) while (1*) shows that $\boldsymbol{Q}$ must be trivially ordered and this gives (0*).

The Main Lemma is unaltered and applies as before during the inductive step to show that (a), (b), (c) and (d*) are preserved by finite extensions. Thus, in order to achieve (d*), we start the induction so that the *first* nontrivial finite extension has basis $B^* = \{\alpha, \alpha^2, \cdots, \alpha^n\}$ rather than $\{1, \alpha, \cdots, \alpha^{n-1}\}$. Then we note that, in the proof of the Main Lemma, the b_i in step 1 and hence the a_i in step 2 are all nonzero. Thus $1 = (\alpha^n - a_{n-1}\alpha^{n-1} - \cdots - a_1\alpha)/a_0 \notin P_K(B^*)$, which is (d*).

The following examples illustrate how bases are constructed using the Main Lemma. Of course, all of them satisfy (a), (b), and (c) and either (d) or (d*).

EXAMPLE 4.1. We let $M = \boldsymbol{Q}$, $M' = \boldsymbol{Q}(\gamma)$ where $\gamma^2 = 2$ and $\gamma > 0(\mathrm{wrt}\,\boldsymbol{R}^+)$ and choose $t = 1$, $s = 1$. This gives β satisfying $\beta^2 = 2\beta + 1$ and choosing $d = 1$ gives $\alpha = \beta$. Thus $B = \{1, \alpha\}$ satisfies (d) and $B^* = \{\alpha, \alpha^2\}$ satisfies (d^*). Note that γ is neither in $P_{\boldsymbol{Q}}(B)$ nor in $P_{\boldsymbol{Q}}(B^*)$.

EXAMPLE 4.2. We let $M = \boldsymbol{Q}(\alpha)$, $M' = \boldsymbol{Q}(\gamma')$ where $\gamma'^2 = \gamma$ and $\gamma' > 0(\mathrm{wrt}\,\boldsymbol{R}^+)$, and choose $t = 1/2$, $s = 2$. This gives β' satisfying $\beta'^2 =$

$(\alpha - 1)\beta' + (3\alpha - 1)/4$. The coefficients, while positive wrt $\boldsymbol{R}^+$ are not positive in either order of 4.1. If $P_M = P_{\boldsymbol{Q}}(1, \alpha)$, then $\alpha - 1 = (\alpha + 1)/\alpha$ and $3\alpha - 1 = (5\alpha + 3)/\alpha$. Thus we choose $d = \alpha$ and get α' satisfying $\alpha'^2 = (\alpha + 1)\alpha' + (5\alpha^2 + 3\alpha)/4$. This gives $B = \{1, \alpha, \alpha', \alpha\alpha'\}$. On the other hand, if $P_M = P_{\boldsymbol{Q}}(\alpha, \alpha^2)$, then $\alpha - 1 = (\alpha^2 + \alpha)$, $3\alpha - 1 = (5\alpha^2 + 3\alpha)/\alpha^2$ and we choose $d = \alpha^2$. This gives α^* satisfying $\alpha^{*2} = (\alpha^2 + \alpha)\alpha^* + (5\alpha^4 + 3\alpha^3)/4$ and $B^* = \{\alpha, \alpha^2, \alpha\alpha^*, \alpha^2\alpha^*\}$.

We note that Corollaries 1.1 and 1.2 fail to determine the cardinality of the class of lattice orders for those countable subfields L which are algebraic over only countably many subfields K. For instance, Corollary 1.1 only accords finitely many lattice orders to simple extensions L of $\boldsymbol{Q}$, but:

COROLLARY 4.3. *If L is a simple extension of $\boldsymbol{Q}$ then L admits at least $\aleph_0$ lattice orders.*

Proof. We recall from the proof of Theorem 1* that in $f(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0$, the minimal polynomial of α, the (rational)a_i are *greater than* 0(i.e., α satisfies (ii)). Thus there are $\aleph_0$ distinct sufficiently small rationals r such that the minimal polynomial of $\alpha - r$ still satisfies (ii).

Questions 4.4.

(a) Do any countable subfields L of $\boldsymbol{R}$ which are algebraic over no more than $\aleph_0$ subfields have $2^{\aleph_0}$ orders?

(b) Are there any $\boldsymbol{R}$-compatible lattice orders on subfields L of $\boldsymbol{R}$ which do not quotient-represent $L \cap \boldsymbol{R}^+$?

(c) Besides the $\boldsymbol{R}$-compatible orders constructed here and those on subfields related to $\boldsymbol{R}$-compatible orders by automorphisms, what other lattice orders are there? In particular, are there any non-$\boldsymbol{R}$-compatible orders on $\boldsymbol{R}$ itself?

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