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MATRIX TRANSFORMATIONS AND ABSOLUTE SUMMABILITY

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The main results of this paper are two theorems which give necessary conditions for a matrix to map into ℓ the set of all subsequences (rearrangements) of a null sequence not in ℓ. These results provide affirmative answers to the following questions proposed by J. A. Fridy. Is a null sequence x necessarily in ℓ if there exists a sum-preserving $\ell-\ell$ matrix A that maps all subsequences (rearrangements) of x into ℓ?

1. Introduction. Let s, m, c, c_0 and cs denote, respectively, the set of all complex sequences, the set of all bounded sequences in s, the set of all convergent sequences in s, the set of all null sequences in c, and the set of all sequences in s with sequence of partial sums in c. Let

$$\ell = \{x \in s: \Sigma \mid x_p \mid < \infty\} \text{ and } \ell^2 = \{x \in s: \Sigma \mid x_p \mid^2 < \infty\} .$$

A matrix A which maps each element of ℓ into ℓ is called an $\ell-\ell$ matrix and may be characterized [3] and [6] by the property: $\{\sum_{p=1}^{\infty} | a_{pq} |\}_{q=1}^{\infty} \in m$. If, in addition, $\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq}x_q = \sum_{q=1}^{\infty} x_q$, whenever $x \in \ell$, then A is a sum-preserving $\ell-\ell$ matrix; this is characterized by $\sum_{p=1}^{\infty} a_{pq} = 1$, for each q.

In 1943, R. C. Buck [1] showed that a sequence x is convergent if some regular matrix sums every subsequence of x. J. A. Fridy [5] has obtained an analog to Buck's theorem in which "subsequence" is replaced by "rearrangement." In addition, he has characterized ℓ by showing that $x \in \ell$ if there is a sum-preserving $\ell-\ell$ matrix that transforms every rearrangement of x into ℓ. In § 2 of the present paper, necessary conditions are obtained for a matrix to map into ℓ the set of all subsequences of a null sequence not in ℓ. This result yields as a corollary the affirmative answer to the following question proposed by J. A. Fridy [5]. Is a null sequence x necessarily in ℓ if there exists a sum-preserving $\ell-\ell$ matrix that maps all subsequences of x into ℓ? In § 3, necessary conditions are obtained for a matrix to map into ℓ all rearrangements of a null sequence not in ℓ. This yields as a corollary Fridy's characterization of ℓ mentioned above. Finally, § 4 contains examples of matrix mappings involving both subsequences and rearrangements.

2. Subsequences. The following two lemmas will be instru-

mental in the proof of Theorem 1.

LEMMA 1. *Suppose x and a are sequences such that $\sum_{q=1}^{\infty} a_q y_q$ converges for every subsequence y of x. If $\varepsilon > 0$, then there exist $M > 0$ and a strictly increasing function $\delta: I^+ \to I^+$ such that if $t > M$, then $|\sum_{q=t}^{\infty} a_q y_q| \leqq \varepsilon$ for every subsequence $(y_q)_{q=t}^{\infty}$ of $(x_q)_{q=\delta(t)}^{\infty}$.*

LEMMA 2. *If x is a null sequence not in ℓ and a is a nonnull convergent sequence, then there exists a subsequence y of x such that $\lim_t |\sum_{q=1}^{t} y_q| = \infty$ and $(\sum_{q=1}^{n} a_q y_q)_{n=1}^{\infty}$ is not bounded.*

THEOREM 1. *Let x be a null sequence not in ℓ, and suppose A is a matrix such that $Ay \in \ell$ for every subsequence y of x. Then*
(i) *$\sum_{p=1}^{\infty} |a_{pq}| < \infty$ for $q = 1, 2, 3, \cdots$; and*
(ii) *if $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$, then $L = 0$.*

Proof. To show (i), let k be fixed and $j > i > k$ such that $x_i \neq x_j$. Let y be the subsequence of x such that $y_q = x_q$ for $q = 1, 2, \cdots, k-1$; $y_k = x_i$; and $y_{k+t} = x_{j+t}$ for $t = 1, 2, 3, \cdots$. Let z be the subsequence of x such that $z_k = x_j$ and $z_q = y_q$ otherwise. Then

$$\infty > \sum_{p=1}^{\infty} \left| \sum_{q=1}^{\infty} a_{pq} y_q - \sum_{q=1}^{\infty} a_{pq} z_q \right| = |x_i - x_j| \sum_{p=1}^{\infty} |a_{pk}| .$$

Therefore $\sum_{p=1}^{\infty} |a_{pk}| < \infty$.

Suppose $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$ and $L \neq 0$. Let $(y_1, \cdots, y_{M-1})$ be a subsequence of x with $y_{M-1} = x_r$. Since $x \notin \ell$ there exists a subsequence $(w_q)_{q=M}^{\infty}$ of $(x_q)_{q=r+1}^{\infty}$ such that $\lim_t |\sum_{q=M}^{t} w_q| = \infty$. By Lemma 2 there exists a subsequence $(z_q)_{q=M}^{\infty}$ of $(w_q)_{q=M}^{\infty}$ such that $\lim_t |\sum_{q=M}^{t} z_q| = \infty$ and $\limsup_t |\sum_{q=M}^{t} z_q \sum_{p=1}^{\infty} a_{pq}| = \infty$. Choose $k > M$ such that

$$\left| \sum_{q=M}^{k} z_q \sum_{p=1}^{\infty} a_{pq} \right| > M + \sum_{q=1}^{M-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| + 3 .$$

Let $K > 0$ such that $|\sum_{p=K+1}^{\infty} a_{pq}| < 1/(k(|z_q| + 1))$ for $q = M, \cdots, k$. By Lemma 1, letting $\varepsilon = 1/K$, there exist N'_p and δ'_p for $1 \leqq p \leqq K$, such that if $N = \max\{N'_1, \cdots, N'_K, k+2\}$ and $\delta(i) = \max\{\delta'_p(i): p = 1, \cdots, K\}$, then $\sum_{p=1}^{K} |\sum_{q=N}^{\infty} a_{pq} v_q| < 1$ for every subsequence $(v_q)_{q=N}^{\infty}$ of $(x_q)_{q=\delta(N)}^{\infty}$. Let $y_q = z_q$ for $M \leqq q \leqq k$, and choose $(y_{k+1}, \cdots, y_{N-1})$ a subsequence of $(w_q)_{q=\delta(N)}^{\infty}$ such that $\sum_{q=k+1}^{N-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| < 1$. Note that the first $N-1$ terms of a fixed sequence y have now been determined. If y^* is any subsequence of x that agrees with y for these first $N-1$ terms, then $\sum_{p=1}^{K} |\sum_{q=1}^{\infty} a_{pq} y_q^*| > M$.

This process for defining terms of y may be continued so that if $T > 0$, then there exist $M \geqq T$ and $K > 0$ such that

$$\sum_{p=1}^{K}\left|\sum_{q=1}^{\infty} a_{pq}y_q\right| > M .$$

Thus a subsequence y of x can be constructed such that $Ay \notin \ell$, a contradiction.

COROLLARY 1. *A null sequence x is in ℓ if and only if there exists a sum-preserving $\ell - \ell$ matrix A such that $Ay \in \ell$ for every subsequence y of x.*

3. **Rearrangements.** Following J. A. Fridy [5], the sequence y is called a rearrangement of the sequence x provided that there is a $1-1$ function π from the positive integers onto themselves such that for each k, $x_k = y_{\pi(k)}$. The word "permutation" will be reserved to indicate the reordering of a finite sequence.

THEOREM. *If x is a null sequence not in ℓ and A is a matrix such that $Ay \in \ell$ for every rearrangement y of x, then* $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.

Proof. Let $x_i \neq x_j$ be nonzero elements of x. Suppose the kth column of A is not in ℓ. Let $q \neq k$ and y be a rearrangement of x with $y_k = x_i$ and $y_q = x_j$. Let z be the rearrangement of x such that $z_k = x_j$, $z_q = x_i$, and $z_t = y_t$ otherwise. Then

$$|x_i - x_j| \sum_{p=1}^{\infty} |a_{pk} - a_{pq}| = \sum_{p=1}^{\infty}\left|\sum_{q=1}^{\infty} a_{pq}y_q - \sum_{q=1}^{\infty} a_{pq}z_q\right| < \infty .$$

Therefore $\sum_{p=1}^{\infty} |a_{pk} - a_{pq}| < \infty$ for every $q \neq k$. Since $\sum_{p=1}^{\infty} |a_{pk}| = \infty$, it now follows that $\sum_{p=1}^{\infty} |a_{pq}| = \infty$ for $q \geqq 1$. Suppose $N > 0$ and a permutation $(r_1, \cdots, r_M)$ of M terms of x has been chosen such that $\sum_{q=1}^{M} r_q \neq 0$. If $\lambda = \sum_{p=1}^{\infty} |\sum_{q=1}^{M} a_{pq}r_q| < \infty$, then

$$\infty > \lambda + \sum_{q=2}^{M} |r_q| \sum_{p=1}^{\infty} |a_{p1} - a_{pq}| \geqq \left|\sum_{q=1}^{M} r_q\right| \sum_{p=1}^{\infty} |a_{p1}| ,$$

a contradiction. Therefore $\lambda = \infty$ and there exists $K > N$ such that $\sum_{p=N}^{K} |\sum_{q=1}^{M} a_{pq}r_q| > 2$. Let $i = \min\{q: x_q \in x \backslash (r_1, \cdots, r_M)\}$. J. A. Fridy [5] has shown that each row of A is null. Therefore there exists $T > M+1$ such that $|x_i| \sum_{p=1}^{K} |a_{pT}| < 2^{-(M+1)}$. Let $r_T = x_i$ and $(r_{M+1}, \cdots, r_{T-1})$ be a subsequence of $x \backslash (r_1, \cdots, r_M, r_T)$ such that $\sum_{p=1}^{K} \sum_{q=M+1}^{T-1} |a_{pq}| |r_q| < 2^{-(M+2)}$ and $\sum_{q=1}^{T} r_q \neq 0$. Then

$$\sum_{p=N}^{K}\left|\sum_{q=1}^{T} a_{pq}r_q\right| \geqq \sum_{p=N}^{K}\left|\sum_{q=1}^{M} a_{pq}r_q\right| - \sum_{p=N}^{K}\sum_{q=M+1}^{T-1} |a_{pq}r_q| - |r_T| \sum_{p=N}^{K} |a_{pT}| > 1 .$$

But this process may be continued. Therefore there exists a rearrangement r of x such that if $L > 0$, then there exist $K > N \geqq L$ such that $\sum_{p=N}^{K} |\sum_{q=1}^{\infty} a_{pq} r_q| > 1$, a contradiction. Hence each column of A is in ℓ.

Now suppose there exists $\varepsilon > 0$ such that if $N > 0$, then there exists $q > N$ such that $\sum_{p=1}^{\infty} |a_{pq}| > \varepsilon$. Let $z \in \ell$ be a subsequence of x that includes all zero terms of x. Let $j_1 = \min\{q: x_q \neq 0\}$. Let $N_1 > 0$ such that $\sum_{p=1}^{\infty} |a_{pN_1}| > \varepsilon$. Let $r_{N_1} = x_{j_1}$. Also let $(r_1, \cdots, r_{N_1-1})$ be a subsequence of z such that $\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < 1/2$ and $z_t = r_a$ only if for each $s < t$ such that $z_s = 0$ there exists $b < a$ such that $z_s = r_b$. Let $M_1 > 0$ such that

$$\sum_{p=1}^{M_1} |a_{pN_1}| > \frac{\varepsilon}{2} \quad \text{and} \quad |r_{N_1}| \sum_{p=M_1+1}^{\infty} |a_{pN_1}| < \frac{1}{4}.$$

Let $j_2 = \min\{q: x_q \in x\backslash(r_1, \cdots, r_{N_1}), \text{ and } x_q \neq 0\}$. Since each row of A is null, there exists $N_2 > N_1 + 1$ such that $\sum_{p=M_1+1}^{\infty} |a_{pN_2}| > \varepsilon/2$ and $|x_{j_2}| \sum_{p=1}^{M_1} |a_{pN_2}| < 1/8$. Let $r_{N_2} = x_{j_2}$. Also let $(r_{N_1+1}, \cdots, r_{N_2-1})$ be a subsequence of $z\backslash(r_1, \cdots, r_{N_1}, r_{N_2})$ such that $\sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < 1/16$ and $z_t = r_a$ only if for each $s < t$ such that $z_s = 0$ there exists $b < a$ such that $z_s = r_b$. Let $M_2 > M_1$ such that $\sum_{p=M_1+1}^{M_2} |a_{pN_2}| > \varepsilon/2$ and $|r_{N_2}| \sum_{p=M_2+1}^{\infty} |a_{pN_2}| < 1/32$. This selection process may be continued so that if k is fixed, then

$$\begin{aligned}\sum_{p=1}^{M_k} \Big| \sum_{q=1}^{\infty} a_{pq} r_q \Big| &\geqq \Big(\sum_{p=1}^{M_1} |a_{pN_1} r_{N_1}| + \cdots + \sum_{p=M_{k-1}+1}^{M_k} |a_{pN_k} r_{N_k}| \Big) \\ &\quad - \Big(\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + \sum_{p=M_1+1}^{M_k} |a_{pN_1} r_{N_1}| \\ &\quad + \sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + |r_{N_2}| \sum_{p=1}^{M_1} |a_{pN_2}| \\ &\quad + \sum_{p=M_2+1}^{M_k} |a_{pN_2} r_{N_2}| + \cdots \Big) \geqq \frac{\varepsilon}{2} \sum_{i=1}^{k} |r_{N_i}| - 1.\end{aligned}$$

But r has been selected so that $\lim_k \sum_{i=1}^{k} |r_{N_i}| = \infty$. Therefore $Ar \notin \ell$, a contradiction. Hence $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.

The proof of Theorem 2 is now complete, and Corollary 2, which was first proved by J. A. Fridy [5], follows directly.

COROLLARY 2. *The null sequence x is in ℓ if and only if there exists a sum-preserving $\ell - \ell$ matrix A such that $Ay \in \ell$ for every rearrangement y of x.*

4. **Examples.** By Theorem 2 a matrix A that maps all rearrangements of a sequence $x \in c_0 \backslash \ell$ into ℓ must be an $\ell - \ell$ matrix.

But Theorem 1 gives little insight into the question of whether A must be $\ell-\ell$ if it maps all subsequences of x into ℓ. The following example shows that A need not be $\ell-\ell$ in this case. Let $x_n = 1/n$ for $n = 1, 2, 3, \cdots$; $a_{qq} = q^{1/3}$ for $q = 1, 8, 27, 64, \cdots$; and $a_{pq} = 0$ otherwise. If y is a subsequence of x and $Ay = z$, then $|z_q| \leqq q^{-2/3}$ for $q = 1, 8, 27, \cdots$ and $z_q = 0$ otherwise. Thus $z \in \ell$, but clearly $x \in c_0 \backslash \ell$ and A is not $\ell - \ell$.

I. J. Maddox [7] showed that a matrix A is Schur if it maps all subsequences of some divergent sequence x into c. This might cause one to suspect that if A maps all subsequences (rearrangements) of a sequence $x \in c_0 \backslash \ell$ into ℓ, then $Az \in \ell$ for every $z \in cs$. The following example shows that this is not true. (The author wishes to thank the referee for his comments which aided in the simplification of this example.) Let $x_n = 1/n$ for $n = 1, 2, 3, \cdots$; $a_{1q} = (-1)^q/q$ for $q \geqq 1$ and $a_{pq} = 0$ otherwise. Since $(a_{1q})_{q=1}^{\infty}$ and x are both in ℓ^2, each subsequence (rearrangement) y of x is also in ℓ^2; hence, $Ay \in \ell$. But if z is defined by $z_q = (-1)^q/(\log(q+1))$ for each q, then $z \in cs$ and $(a_{1q}z_q)_{q=1}^{\infty} \notin cs$; thus, $Az \notin \ell$.

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