ON VALUE DISTRIBUTION OF FUNCTIONS MEROMORPHIC IN THE WHOLE PLANE

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We make two kinds of decompositions of functions meromorphic in the whole plane into two classes: One is the class of meromorphic functions of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov) and the other is the class of meromorphic functions of the second kind in the sense of Yosida (resp. of the second kind in the sense of Gavrilov). Using these decompositions, we prove a result about the growth of the characteristic functions and some results about value distribution, of meromorphic functions of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov).

1. Introduction. Noshiro [6] decomposed the class of normal meromorphic functions in the unit disc into two categories: Normal meromorphic functions of the first category and those of the second category. Then, Tse [8] generalized this idea to general meromorphic functions in the unit disc and decomposed all functions meromorphic in the unit disc into two classes: One is the class of meromorphic functions of the first kind and the other is the class of meromorphic functions of the second kind. Using this decomposition, he proved some interesting results about value distibution of meromorphic functions in the unit disc.

On the other hand, Yosida [10] defined a class of meromorphic functions in the whole plane, called it the class (A) and divided this class into two categories: Meromorphic functions of the first category and those of the second category. Then, Gavrilov [2] divided the class W_1 of Julia's exceptional functions into two subclasses: The class W_1^0 and the class of functions not belonging to W_1^0. Using each of these notions, they proved some interesting results.

In this paper, by making a consideration parallel to Tse's, we generalize both Yosida's decomposition and Gavrilov's decomposition to general meromorphic functions in the whole plane and according to each generalization we make two kinds of decompositions of functions meromorphic in the whole plane into two classes. Using these decompositions, we prove some results about value distribution of meromorphic functions in the whole plane.

I wish to express my thanks to the referee of this paper for his kind remarks.

2. Preliminaries. We denote the whole z-plane by Z and the w-sphere by W. Let z_0 be a point in Z and ϵ be a positive number. By $D(z_0, \epsilon)$ (resp. $D^*(z_0, \epsilon)$, where $z_0 \neq 0$), we denote the set

$$\{z; |z - z_0| < \epsilon\} \qquad (\text{resp. } \{z; |z - z_0| < \epsilon \cdot |z_0|\}).$$

A meromorphic function $f(z)$ in Z is said to be of *the second kind in the sense of Yosida* (resp. of *the second kind in the sense of Gavrilov*), if there exists a sequence $\{z_n\}$ of points in Z, $|z_n| \to \infty$, such that the sequence $\{f_n(z)\}$ of functions $f_n(z) = f(z + z_n)$ (resp. $f_n(z) = f(|z_n| \cdot z + z_n)$), tends uniformly to a constant on W in some neighborhood of $z = 0$. A meromorphic function $f(z)$ in Z is said to be of *the first kind in the sense of Yosida* (resp. of *the first kind in the sense of Gavrilov*), if it is not of the second kind in the sense of Yosida (resp. of the second kind in the sense of Gavrilov).

REMARK 2.1. A meromorphic function $f(z)$ in Z is of the second kind in the sense of Yosida (resp. of the second kind in the sense of Gavrilov) if, and only if there exist a sequence $\{z_n\}$ of points in Z, $|z_n| \to \infty$, and a positive constant ϵ such that $f(z)$ tends uniformly to a constant on the set

$$\bigcup_n D(z_n, \epsilon) \qquad \left(\text{resp. } \bigcup_n D^*(z_n, \epsilon)\right)$$

as $|z| \to \infty$. From this fact, we can speak roughly: Meromorphic functions of the second kind in the sense of Yosida (resp. of the second kind in the sense of Gavrilov) behave mildly, while those of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov) are comparatively wild in the sense that the values which they assume do not tend to any constant limit on sequences of discs $D(z, \epsilon)$ (resp. $D^*(z, \epsilon)$) for any fixed positive number ϵ.

REMARK 2.2. As "the class (A)", Yosida [10] called the class of meromorphic functions having the following property: For any sequence $\{z_n\}$ of complex numbers, the family $\{f_n(z)\}$ of meromorphic functions $f_n(z) = f(z + z_n)$ is a normal family in Z. And he divided this class into two categories: $f(z)$ is a function of the first category if there is no sequence $\{z_n\}$ of complex numbers such that the normal family $\{f_n(z)\}$ admits a constant limit, and any function which is not of the first category is of the second category. It is evident that a meromorphic function belonging to the class (A) is of the second category if and only if it is of the second kind in the sense of Yosida.

According to Gavrilov [2], a Julia's exceptional function $f(z)$

belongs to W_1^0 if there is no sequence $\{z_n\}$ of points in Z, $|z_n| \to \infty$, such that the normal family $\{g_n(z)\}$, $g_n(z) = f(z_n \cdot z)$, in $\{z; 0 < |z| < \infty\}$, admits a constant limit. Now, if we consider the sequence $\{h_n(t)\}$, $h_n(t) = f(z_n(1+t)) = f(z_n + z_n \cdot t)$, in a neighborhood of $t = 0$, which is the same as $\{g_n(z)\}$ in a neighborhood of 1, it is evident that a Julia's exceptional function belongs to W_1^0 if and only if it is a meromorphic function of the first kind in the sense of Gavrilov.

REMARK 2.3. There exists an interesting example of integral functions of the first kind in the sense of Yosida (resp. in the sense of Gavrilov). From Yosida [10, Theroem 6] (resp. Lehto and Virtanen [3, p. 7–8]), we see that this function does not belong to the class (A) (resp. the class of Julia's exceptional functions).

So, in order to obtain this example, we shall define the following terms, in the manner analogous to the case of the unit disc. Let $\zeta(t)$ be a continuous, complex-valued function for $0 \le t < \infty$ with the properties:

$$\lim_{t \to \infty} |\zeta(t)| = +\infty, \qquad \lim_{t \to \infty} \arg \zeta(t) = +\infty,$$

and let the equation $z = \zeta(t)$ define a simple curve. We denote by S the set $\{z; z = \zeta(t), 0 \le t < \infty\}$ and we call it a *spiral* in Z. Then, from Mergelyan [4, Theorem 1.2., p. 326], we see that S is a Carleman continuum, and hence, we can find an integral function $f(z)$ such that $f(z)$ is bounded on S. We shall refer to integral functions with the property that it remains bounded on some spiral S in Z, as *integral functions of class* (V) *relative to S in Z*. We shall also introduce the following measures for the "tightness" of a spiral in Z whose equation is $z = \zeta(t)$. For any value of t, starting with the point $\zeta(t)$, describe the curve in the sense of increasing t and let t' denote the first value of t for which $\arg \zeta(t') = \arg \zeta(t) + 2\pi$. Then, we put

$$\bar{\mu}(S) = \overline{\lim_{t \to \infty}} |\zeta(t) - \zeta(t')|, \qquad \bar{\mu}^*(S) = \overline{\lim_{t \to \infty}} \frac{|\zeta(t) - \zeta(t')|}{|\zeta(t)|}.$$

Now, take an integral function $f(z)$ of class (V) relative to a spiral S in Z, for which $\bar{\mu}(S) = 0$ (resp. $\bar{\mu}^*(S) = 0$). Then, since any integral function has an asymptotic value ∞, there is no sequence $\{z_n\}$ of points in Z, $|z_n| \to \infty$, such that the sequence $f_n(z) = f(z + z_n)$ (resp. $f_n(z) = f(|z_n| \cdot z + z_n)$), for this $f(z)$, converges uniformly to a constant in a neighborhood of $z = 0$. Therefore, this $f(z)$ is of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov).

3. The characteristic functions of meromorphic functions of the first kind.

First of all, we shall prove a characteristic property of meromorphic functions of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov) (Theorem 1).

LEMMA 1. (See Ostrowski [7, Satz 1 and p. 234].) *Let $\{z_n\}$ be a sequence of points in Z satisfying $|z_n| \to \infty$ and $f(z)$ be a meromorphic function in Z. Then, if the family $\{f_n(z)\}$ of functions*

$$f_n(z) = f(z + z_n) \qquad (\text{resp. } f_n(z) = f(|z_n| \cdot z + z_n))$$

is not normal at $z = 0$, we can choose a subfamily $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ such that for any ϵ, $\epsilon > 0$, and any δ, $\delta > 0$, each of functions $f_{n_k}(z)$ for $k > k_0$ assumes every values on W in $D(0, \epsilon)$ with the possible exception of two sets of values E and G on W whose spherical diameters do not exceed δ, where k_0 is a positive integer dependent on ϵ and δ, and both E and G are dependent on ϵ, δ and k.

THEOREM 1. *In order that a meromorphic function $f(z)$ in Z is of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov), it is necessary and sufficient that for each positive number ϵ*

(3.1)
$$\operatorname*{infimum}_{|z_0| < \infty} \iint_{D(z_0, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z > 0$$
$$\left(\text{resp. } \operatorname*{infimum}_{1 \leqq |z_0| < \infty} \iint_{D^*(z_0, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z > 0 \right),$$

where $d\sigma_z$ denotes the area element.

Proof. The condition is sufficient. Suppose that there exist a sequence $\{z_n\}$ of points, $|z_n| \to \infty$ (if necessary, we may suppose $|z_n| \geqq 1$), and a neighborhood $D(0, \epsilon)$ such that the sequence

$$f_n(z) = f(z + z_n) \qquad (\text{resp. } f_n(z) = f(|z_n| \cdot z + z_n))$$

tends uniformly to a constant in $D(0, \epsilon)$. Then, we have that

$$\iint_{D(0, \epsilon)} \left(\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \right)^2 d\sigma_z = \iint_{D(z_n, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z \to 0 \qquad (n \to \infty)$$

$$\left(\text{resp. } \iint_{D(0, \epsilon)} \left(\frac{|f'_n(z)|}{1 + |f_n(z)|^2} \right)^2 d\sigma_z \right.$$
$$\left. = \iint_{D^*(z_n, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z \to 0 \qquad (n \to \infty) \right).$$

This fact contradicts (3.1).

The condition is necessary. Suppose that there exist a positive constant ϵ and a sequence $\{z_n\}$, $|z_n| \to \infty$ (resp. $|z_n| \to \infty$, $|z_n| \geqq 1$), such that

$$\iint_{D(z_n,\epsilon)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z = \iint_{D(0,\epsilon)} \left(\frac{|f_n'(z)|}{1+|f_n(z)|^2}\right)^2 d\sigma_z \to 0 \qquad (n \to \infty)$$

(3.2)

$$\left(\text{resp.} \iint_{D^*(z_n,\epsilon)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z \right.$$

$$= \iint_{D(0,\epsilon)} \left(\frac{|f_n'(z)|}{1+|f_n(z)|^2}\right)^2 d\sigma_z \to 0 \qquad (n \to \infty).$$

Then, the family $\{f_n(z)\}$,

$$f_n(z) = f(z + z_n) \qquad (\text{resp. } f_n(z) = f(|z_n| \cdot z + z_n)),$$

for this sequence $\{z_n\}$ is normal at $z = 0$. Because, if we suppose that this family $\{f_n(z)\}$ is not normal at $z = 0$, we see from Lemma 1 that (3.2) can not hold. Thus, we can obtain a subfamily $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ and a neighborhood $D(0, \epsilon)$ such that $f_{n_k}(z)$ converges uniformly to a function $p(z)$ in $D(0, \epsilon)$. And we have that

$$\iint_{D(0,\epsilon)} \left(\frac{|f_{n_k}'(z)|}{1+|f_{n_k}(z)|^2}\right)^2 d\sigma_z \to \iint_{D(0,\epsilon)} \left(\frac{|p'(z)|}{1+|p(z)|^2}\right)^2 d\sigma_z$$

as $n \to \infty$. Hence, we can conclude from (3.2) that

$$(3.3) \qquad \iint_{D(0,\epsilon)} \left(\frac{|p'(z)|}{1+|p(z)|^2}\right)^2 d\sigma_z = 0.$$

But, on the other hand, since $f(z)$ is of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov), we see that $p(z) \not\equiv$ constant, and hence,

$$\iint_{D(0,\epsilon)} \left(\frac{|p'(z)|}{1+|p(z)|^2}\right)^2 d\sigma_z > 0.$$

This contradicts (3.3).

Now, we can prove a result concerning the growth of the characteristic function $T(r, f)$ of $f(z)$ of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov).

THEOREM 2. *Let $f(z)$ be a meromorphic function in Z of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov). Then, the characteristic function $T(r, f)$ of $f(z)$ has the property*

$$(3.4) \qquad \varlimsup_{r \to \infty} \frac{T(r, f)}{r^2} > 0 \qquad \left(\text{resp. } \varlimsup_{r \to \infty} \frac{\mathcal{T}(r, f)}{\log^2 r} > 0 \right).$$

Proof. From Theorem 1, for an ϵ, $\epsilon > 0$ (resp. $0 < \epsilon < 1$), we can find a positive number η such that

$$(3.5) \quad \iint_{D(z_0, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z \left(\text{resp. } \iint_{D^*(z_0, \epsilon)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z \right) \geq \eta$$

for all z_0 (resp. for all z_0, $|z_0| \geq 1$). Using this fact, we shall estimate the evaluation of the characteristic function $T(r, f)$ of $f(z)$

$$T(r, f) = \int_0^r \frac{S(r, f)}{r} \, dr, \qquad S(r, f) = \frac{1}{\pi} \iint_{D(0, r)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma_z.$$

Take the sequence of real numbers $\{x_n\}$ decided by the following recurrence formula

$$x_1 = 2\epsilon, \ x_{n+1} = x_n + 2\epsilon \ (\text{resp. } x_1 = 1, \ x_{n+1} = x_n + \epsilon \cdot x_{n+1} + \epsilon \cdot x_n)$$
$$(n = 1, 2, 3, \cdots)$$

and consider the sequence of discs

$$\Gamma_{n,1} = D(x_n, \epsilon) \qquad (\text{resp. } \Gamma_{n,1} = D^*(x_n, \epsilon)) \qquad (n = 1, 2, 3, \cdots).$$

Then, two discs $\Gamma_{n,1}$ and $\Gamma_{n+1,1}$ intersect only at

$$r_n = x_n + \epsilon \qquad (\text{resp. } r_n = x_n + \epsilon \cdot x_n) \qquad (n = 1, 2, 3, \cdots).$$

Next, for the angle θ_n subtended at $z = 0$ by the disc $\Gamma_{n,1}$, we have

$$\frac{\theta_n}{\pi} < \sin\left(\frac{\theta_n}{2} \right) = \frac{1}{2n} \qquad \left(\text{resp. } \frac{\theta_n}{\pi} < \sin\left(\frac{\theta_n}{2} \right) = \epsilon \right) \qquad (n = 1, 2, 3, \cdots).$$

Hence, in the annulus $\Omega_n = \{z ; r_{n-1} < |z| < r_n\}$ $(n = 1, 2, 3, \cdots)$, where $r_0 = \epsilon$ (resp. $r_0 = 1 - \epsilon$), we can arrange $4n$ (resp. $[2/\epsilon]$, where $[\ ]$ denotes Gauss's symbol) number of discs $\Gamma_{n,i}$ $(i = 1, 2, 3, \cdots, 4n)$ (resp. $i = 1, 2, 3, \cdots, [2/\epsilon])$, each of which has radius ϵ (resp. $\epsilon \cdot x_n$) and its center on $\{z ; |z| = x_n\}$, in the way that any two of these discs not intersect mutually. Thus, we have that for $n = 1, 2, 3, \cdots$

$$(3.6) \qquad D(0, r_n) \supset \bigcup_{\substack{k=1,2,3,\cdots,n \\ i=1,2,3,\cdots,4k}} \Gamma_{k,i} \qquad \left(\text{resp.} \bigcup_{\substack{k=1,2,3,\cdots,n \\ i=1,2,3,\cdots,[2/\epsilon]}} \Gamma_{k,i} \right).$$

Now, if we note (3.6) and the following fact obtained from (3.5):

$$\iint_{\Gamma_{n,i}} \left(\frac{|f'(z)|}{1+|f(z)|^2} \right)^2 d\sigma_z \geqq \eta$$

$$(n = 1,2,3,\cdots; \ i = 1,2,3,\cdots,4n \ (\text{resp. } i = 1,2,3,\cdots,[2/\epsilon])),$$

we can obtain the estimation

$$S(r_n, f) \geqq \frac{1}{\pi} \sum_{k=1}^{n} 4\eta k \left(\text{resp.} \frac{1}{\pi} \sum_{k=1}^{n} \left[\frac{2}{\epsilon} \right] \eta \right)$$

$$= \frac{2\eta}{\pi} n(n+1) \left(\text{resp.} \frac{\eta}{\pi} \left[\frac{2}{\epsilon} \right] n \right) (n = 1,2,3,\cdots).$$

Hence, from

$$r_i - r_{i-1} = 2\epsilon \qquad \left(\text{resp. } 2\epsilon \left(\frac{1+\epsilon}{1-\epsilon} \right)^{i-1} \right) \qquad (i = 1,2,3,\cdots)$$

we have

$$T(r_n, f) \geqq \sum_{i=2}^{n} \int_{r_{i-1}}^{r_i} \frac{S(r,f)}{r} dr \geqq \sum_{i=2}^{n} S(r_{i-1}, f) \frac{r_i - r_{i-1}}{r_i}$$

$$\geqq \frac{2\eta}{\pi} \sum_{i=2}^{n} (i-1) \left(\text{resp.} \frac{2\epsilon}{1+\epsilon} \left[\frac{2}{\epsilon} \right] \frac{\eta}{\pi} \sum_{i=2}^{n} (i-1) \right)$$
$$= A(n-1)n,$$

where A is a positive constant independent of n. Thus, from

$$r_n = 2\epsilon n \qquad \left(\text{resp. } r_n = (1+\epsilon) \left(\frac{1+\epsilon}{1-\epsilon} \right)^{n-1} \right)$$

we obtain

$$\varlimsup_{n \to \infty} \frac{T(r_n, f)}{r_n^2} \geqq \frac{An(n-1)}{4\epsilon^2 n^2} \geqq \frac{A}{4\epsilon^2}$$

$$> 0 \qquad \left(\text{resp. } \varlimsup_{n \to \infty} \frac{T(r_n, f)}{\log^2 r_n} \geqq A / \log^2 \left(\frac{1+\epsilon}{1-\epsilon} \right) > 0 \right).$$

This proves Theorem 2.

REMARK 3.1. This Theorem 2 is sharp in the following sense: There exists a meromorphic function in Z of the first kind in the sense of Yosida (resp. of the first kind in the sense of Gavrilov) whose characteristic function has the property

$$(3.7) \qquad 0 < \varlimsup_{r \to \infty} \frac{T(r, f)}{r^2} < \infty \qquad \left(\text{resp. } 0 < \varlimsup_{r \to \infty} \frac{T(r, f)}{\log^2 r} < \infty \right).$$

In fact, Yosida [**10**, 2. Corollary] (resp. Lehto and Virtanen [**3**, p. 9]) showed that the characteristic functions of meromorphic functions belonging to the class (A) of the first category (resp. of Julia's exceptional functions belonging to W_1^0) have the property (3.7).

REMARK 3.2. We remark that the inverse of Theorem 2 is not true. For example, consider the integral function

$$\int_0^z e^{-t^p} dt$$

where p, $p \geqq 3$, is a positive integer (see Nevanlinna [**5**, p. 170]).

REMARK 3.3. It is evident that a meromorphic function in Z of the second kind in the sense of Gavrilov is of the second kind in the sense of Yosida. But, there exists a meromorphic function in Z of the second kind in the sense of Yosida and of the first kind in the sense of Gavrilov. In fact, consider Julia's exceptional functions belonging to W_1^0. Then, we see from Remark 2.2 that these functions are of the first kind in the sense of Gavrilov. But since the characteristic functions of Julia's exceptional functions $f(z)$ have the property

$$\varlimsup_{r \to \infty} \frac{T(r, f)}{\log^2 r} < \infty,$$

Theorem 2 shows that these functions are of the second kind in the sense of Yosida.

COROLLARY 1. *Let $f(z)$ be a meromorphic function in Z satisfying*

$$\varlimsup_{r \to \infty} \frac{T(r, f)}{r^2} = 0 \qquad \left(\text{resp. } \varlimsup_{r \to \infty} \frac{T(r, f)}{\log^2 r} = 0 \right).$$

Then, there exist a sequence $\{z_n\}$ of points, $|z_n| \to \infty$, and a positive constant ϵ such that $f(z)$ converges uniformly to a constant as $|z| \to \infty$, in the set

$$\bigcup_{n=1}^{\infty} D(z_n, \epsilon) \qquad \left(\text{resp.} \; \bigcup_{n=1}^{\infty} D^*(z_n, \epsilon) \right).$$

4. The value distribution of meromorphic functions in Z.

Let $f(z)$ be a meromorphic function in Z.   Gavrilov [1] defined the notion "a sequence of $M^{(0)}$-points for $f(z)$".   Now, a sequence $\{z_n\}$ of points, $|z_n| \to \infty$, is called a *sequence of pseudo-$M^{(0)}$-points* for $f(z)$ if for each increasing sequence $\{r_n\}$ of positive numbers, $r_n \uparrow \infty$, $f(z)$ assumes every value on W infinitely often, with the possible exception of two values, in the set $\bigcup_{m=1}^{\infty} D(z_m, r_m)$ for any subsequence $\{z_m\}$ of $\{z_n\}$.

THEOREM 3.   *If $f(z)$ is a meromorphic function in Z of the first kind in the sense of Yosida; then any sequence $\{z_n\}$ of points, $|z_n| \to \infty$, is a sequence of pseudo-$M^{(0)}$-points for $f(z)$.*

Proof. Suppose that there exist a sequence $\{z_n\}_{n=1}^{\infty}$ of points, $|z_n| \to \infty$, a subsequence $\{z_m\}_{m=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers, $r_n \uparrow \infty$, such that $f(z)$ omits three values on W in the union $\bigcup_{m=1}^{\infty} D(z_m, r_m)$. Then, since each function $f_m(z) = f(z_m + z)$ omits three values in $D(0, r_1)$, we can choose a subsequence $\{z_i^{(1)}\}_{i=1}^{\infty}$ of $\{z_m\}_{m=2}^{\infty}$ such that $f_i^{(1)}(z) = f(z_i^{(1)} + z)$ converges uniformly on each compact subset of $D(0, r_1)$.   Next, since each function $f_i^{(1)}(z)$ omits three values in $D(0, r_2)$, we can choose a subsequence $\{z_i^{(2)}\}_{i=1}^{\infty}$ of $\{z_i^{(1)}\}_{i=2}^{\infty}$ such that $f_i^{(2)}(z) = f(z_i^{(2)} + z)$ converges uniformly on each compact subset of $D(0, r_2)$.   In general, for each integer p, $p > 0$, we can choose a subsequence $\{z_i^{(p+1)}\}_{i=1}^{\infty}$ of $\{z_i^{(p)}\}_{i=2}^{\infty}$ such that $f_i^{(p+1)}(z) = f(z_i^{(p+1)} + z)$, converges uniformly on each compact subset of $D(0, r_{p+1})$.   Thus, if we consider the subsequence $\{z_i^{(i)}\}_{i=1}^{\infty}$ of $\{z_m\}_{m=1}^{\infty}$, it follows that

(4.1) $f_i^{(i)}(z) = f(z_i^{(i)} + z)$ converges uniformly to a meromorphic function $g(z)$ in Z on each compact subset of Z.

Here, if we suppose that $g(z)$ is not a constant, we see from a theorem of Hurwitz that $g(z)$ must omit three values on W in Z.   But, since this does not happen, we can conclude that

(4.2) $g(z)$ is a constant.

Thus, from (4.1) and (4.2), we see that $f(z)$ is of the second kind in the sense of Yosida, and we obtain a contradiction.

The following Corollary 2 immediately follows from Theorem 3.

COROLLARY 2.   *If $f(z)$ is a meromorphic function in Z of the first kind in the sense of Yosida, then any ray is a direction of Julia for $f(z)$.*

THEOREM 4.   *Let $f(z)$ be a meromorphic function in Z and $\{z_n\}$, $|z_n| \to \infty$, be a sequence of points which possesses no subsequence of pseudo-$M^{(0)}$-points for $f(z)$.   If*

$$\lim_{n \to \infty} f(z_n) = w_0,$$

where w_0 is an omitted value of $f(z)$, then $f(z)$ converges uniformly to w_0 as $|z| \to \infty$, in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$ for any r, $0 < r < \infty$.

Proof.   Suppose that there exists a positive number r, $0 < r < \infty$, such that $f(z)$ does not converge uniformly to w_0 as $|z| \to \infty$, in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$. Then, there exist a positive number ϵ, a subsequence $\{z_m\}$ of $\{z_n\}$ and a sequence of points $\{z_m^*\}$, $|z_m^*| \to \infty$, $z_m^* \in D(z_m, r)$, such that

(4.3) $\chi(f(z_m^*), w_0) \geqq \epsilon$,

where $\chi(w_1, w_2)$ denotes the spherical distance between w_1 and w_2. Furthermore, by the assumption in Theorem 4, there exist an increasing sequence of positive numbers $\{r_k\}$, $r_k \uparrow \infty$, and a subsequence $\{z_k\}$ of $\{z_m\}$ such that $f(z)$ omits three values on W in the set $\bigcup_{k=1}^{\infty} D(z_k, r_k)$. Hence, by the method analogous to the proof of Theorem 3, we can choose a subsequence $\{z_i^{(i)}\}$ of $\{z_k\}$ such that

(4.4) $f_i^{(i)}(z) = f(z_i^{(i)} + z)$ converges uniformly to a meromorphic

                function $g(z)$ in Z on each compact subset of Z.

Thus, since $f_i^{(i)}(0) \to g(0)$ $(i \to \infty)$ from (4.4) and $f_i^{(i)}(0) = f(z_i^{(i)}) \to w_0$ $(i \to \infty)$ by the assumption in Theorem 4, we have

(4.5) $g(0) = w_0$.

Here, suppose that $g(z)$ is not a constant w_0. Then, we can conclude from (4.4), (4.5) and a theorem of Hurwitz that each function $f_{i_l}^{(i_l)}(z)$ for a subsequence $\{i_l\}$ of $\{i\}$ must take w_0. But, this contradicts the assumption that w_0 is an omitted value.   Hence, we see that

(4.6) $g(z) \equiv w_0$.

Thus, from (4.4) and (4.6), we have that for ϵ and r chosen above and for sufficiently large i,

(4.7) $\chi(f(z), w_0) < \epsilon$ for any $z, z \in D(z_i^{(i)}, r)$.

But, this fact (4.7) contradicts (4.3). Therefore, we obtain the conclusion of Theorem 4.

COROLLARY 3. *Let $f(z)$ be an integral function and let $\{z_n\}$, $|z_n| \to \infty$, be a sequence of points which possesses no subsequence of pseudo-$M^{(0)}$-points for $f(z)$. Then, for each r, $0 < r < \infty$, we have:*
  (1) *If $\lim_{n \to \infty} f(z_n) = \infty$, then $f(z)$ converges uniformly to infinity as $|z| \to \infty$, in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$.*
  (2) *If $|f(z_n)| < K$ $(n = 1, 2, 3, \cdots)$, for some finite constant K which may be dependent on r, then $f(z)$ is bounded in the set $\bigcup_{n=1}^{\infty} D(z_n, r)$.*

Proof. Only (2) is proved. Suppose that there exists a positive number r_0 such that $f(z)$ is unbounded in $\bigcup_{n=1}^{\infty} D(z_n, r_0)$. Then, there exist a subsequence $\{z_k\}$ of $\{z_n\}$ and a sequence $\{z_k^*\}$ of points, satisfying

(4.8) $z_k^* \in D(z_k, r_0)$ $(k = 1, 2, 3, \cdots)$

and

(4.9) $$\lim_{k \to \infty} f(z_k^*) = \infty.$$

Here, if we suppose that $\{z_k^*\}$ is a sequence of pseudo-$M^{(0)}$-points for $f(z)$, we obtain from (4.8) the result that $\{z_k\}$ is also a sequence of pseudo-$M^{(0)}$-points for $f(z)$, which contradicts the assumption in Corollary 3. Hence, $\{z_k^*\}$ is not a sequence of pseudo-$M^{(0)}$-points for $f(z)$. Thus, from (4.9) and (1), we obtain the result which contradicts the assumption in (2).

Let $f(z)$ be a meromorphic function in Z. A path L tending to ∞ is called a *pseudo-$M^{(0)}$-path* for $f(z)$ if there exists a sequence of pseudo-$M^{(0)}$-points for $f(z)$ on L.

THEOREM 5. *Let $f(z)$ be a meromorphic function in Z and L be an asymptotic path having an asymptotic value w_0. If L is not a pseudo-$M^{(0)}$-path for $f(z)$, then $f(z)$ converges uniformly to w_0 in the set $\bigcup_{z \in L}^{\infty} D(z, r)$ for each r, $0 < r < \infty$, where the summation is taken over all $z \in L$.*

Proof. We can proceed by the same method as Theorem 4 is proved. Hence, we remark only the following fact.
  From the result corresponding to (4.4), we have a sequence $\{z_k\}$, $z_k \in L$, $|z_k| \to \infty$, and a subsequence $\{z_i^{(i)}\}$ of $\{z_k\}$ such that

(4.10)
$$f_i^{(i)}(z) = f(z_i^{(i)} + z)$$ converges uniformly to a meromorphic
function $g(z)$ in some neighborhood $\{z; |z| \leqq \epsilon_0\}$ of $z = 0$.

Now, for any ϵ, $0 < \epsilon \leqq \epsilon_0$, and for each i, take a point $z_i(\epsilon)$ in the nonvoid set $\{z; |z - z_i^{(i)}| = \epsilon\} \cap L$, and put $z_i(\epsilon) - z_i^{(i)} = z_i^*(\epsilon)$. Let $z^*(\epsilon)$ be one of limit points of the set $\bigcup_{i=1}^{\infty} z_i^*(\epsilon)$ and $\{z_{i_j}^*(\epsilon)\}_{j=1}^{\infty}$ be the subsequence of $\{z_i^*(\epsilon)\}$ convergent to $z^*(\epsilon)$. Then, from (4.10) we have that

(4.11) $$f(z_{i_j}(\epsilon)) = f_{i_j}^{(i_j)}(z_{i_j}^*(\epsilon)) \rightarrow g(z^*(\epsilon)) \qquad (j \rightarrow \infty).$$

On the other hand, since w_0 is an asymptotic value of $f(z)$ along L, we have that

(4.12) $$f(z_{i_j}(\epsilon)) \rightarrow w_0 \qquad (j \rightarrow \infty).$$

Thus, we obtain from (4.11) and (4.12) that for any ϵ, $0 < \epsilon \leqq \epsilon_0$, $g(z^*(\epsilon)) = w_0$. Hence, we can conclude that $g(z) \equiv w_0$.

Added in Proof. We can easily prove that in Theorem 2

(3.4) $$\varlimsup_{r \rightarrow \infty} \frac{T(r,f)}{r^2} > 0 \qquad \left(\text{resp. } \varlimsup_{r \rightarrow \infty} \frac{T(r,f)}{\log^2 r} > 0 \right)$$

can be replaced by

$$\varliminf_{r \rightarrow \infty} \frac{T(r,f)}{r^2} > 0 \qquad \left(\text{resp. } \varliminf_{r \rightarrow \infty} \frac{T(r,f)}{\log^2 r} > 0 \right).$$

Because, in the proof of Theorem 2, if for any $r \geqq r_1$, we choose a positive integer n satisfying $r_{n+1} > r \geqq r_n$, we can obtain the following estimation:

$$\frac{T(r,f)}{r^2} \geqq \frac{T(r_n, f)}{r_{n+1}^2} \geqq \frac{An(n-1)}{4\epsilon^2(n+1)^2} \rightarrow \frac{A}{4\epsilon^2}$$

$$\left(\text{resp. } \frac{T(r,f)}{\log^2 r} \geqq \frac{T(r_n, f)}{\log^2 r_{n+1}} \rightarrow A/\log^2 \left(\frac{1+\epsilon}{1-\epsilon} \right) \right).$$

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