PACIFIC JOURNAL OF MATHEMATICS
Vol 64, No 1, 1976

A GENERALIZATION OF A THEOREM OF CHACON

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A generalization of a theorem of Chacon is proved simply by an application of a maximal inequality. A pointwise convergence theorem and the submartingale convergence theorem are immediate consequences.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{X_n\}$ be a sequence of integrable random variables adapted to the increasing sequence $\{\mathcal{F}_n\}$ of sub σ-fields of $\mathcal{F}$, B be the collection of all bounded stopping times (with respect to $\{\mathcal{F}_n\}$), and D be the collection of random variables Y which are measurable with respect to $\mathcal{F}_\infty = \sigma(\{\mathcal{F}_n\})$ and, for each w in Ω, $Y(w)$ is a cluster value of the sequence $\{X_n(w)\}$.

The main purpose of this note is to generalize (in Theorem 1) the result stated as Corollary 1, due to Chacon ([3]). The result is a reformulation of a result due to Baxter ([2]) but our method of proof is much simpler than that in ([2]) and ([3]), and is just a simple application of a maximal inequality due to Chacon and Sucheston ([4]). A pointwise convergence theorem and the submartingale convergence theorem are immediate consequences ([1] and [5]).

THEOREM 1. *Suppose that* $\sup_{t\in B} E(|X_t|) < \infty$ *and* Y_1, Y_2 *are any two random variables in D. Then there exist* τ_n^*, t_n^* *in B such that* $\tau_n^* \geqq n$, $t_n^* \geqq n$, *and*

$$\lim_{n\to\infty} E\{|(X_{\tau_n^*} - X_{t_n^*}) - (Y_1 - Y_2)|\} = 0. \tag{1}$$

Proof. By Lemma 1 of [1] and the Borel-Cantelli lemma, for any two random variables Y_1, Y_2 in D, there exist two strictly increasing sequences $\{\tau_n\}$ and $\{t_n\}$ in B such that $\lim_{n\to\infty} X_{\tau_n} = Y_1$ almost surely and $\lim_{n\to\infty} X_{t_n} = Y_2$ almost surely. By the condition that $\sup_{t\in B} E(|X_t|) < \infty$ and the Fatou lemma, Y_1 and Y_2 are integrable.

To prove (1), we need a maximal inequality, which I learned from Chacon and Sucheston.

$$\lambda P\left(\left[\sup_n |X_n| \geqq \lambda\right]\right) \leqq \sup_{t\in B} E(|X_t|) \text{ for each positive constant } \lambda. \tag{2}$$

To see (2), let M be a fixed positive integer and define a bounded stopping time τ by $\tau(w) = \inf\{n \mid 1 \leqq n \leqq M, |X_n(w)| \geqq \lambda\}$, $\tau(w) = M + 1$ if no such n exists, $w \in \Omega$. Then

$$\lambda P\left(\left[\sup_{1 \leqq n \leqq M} |X_n| \geqq \lambda\right]\right) \leqq E(|X_\tau|) \leqq \sup_{t \in B} E(|X_t|).$$

(2) follows immediately on letting $M \to \infty$.

Now, for each positive integer k and each positive constant d, define $j(k, d) = \inf\{n \mid k \leqq n, |X_n| \geqq d\}$, $j(k, d) = \infty$ if no such n exists. Let $A(k, d) = [j(k, d) < \infty]$. Since, by (2), for fixed k, $P(A(k, d)) \to 0$ as $d \to \infty$, $E\{|(Y_1 - Y_2)\chi_{A(k,d)}|\} \to 0$ as $d \to \infty$. Therefore, for each positive integer k, there exists a d_k such that $E\{|(Y_1 - Y_2)\chi_{A(k,d_k)}|\} \leqq 1/k$. Next, for each fixed k, let $Z = \max\{|X_1|, |X_2|, \cdots, |X_{k-1}|, d_k\chi_{A^c(k,d_k)} + |X_{j(k,d_k)}\chi_{A(k,d_k)}|\}$, $Z_n = X_{n \wedge j(k,d_k)}$ for all $n \geqq 1$. Then it is easy to see that $|Z_n| \leqq Z$ for all $n \geqq 1$ and $E\{Z\} < \infty$. Since $\lim_{n\to\infty}(X_{\tau_n} - X_{t_n}) = (Y_1 - Y_2)$ almost surely and, on $A(k, d_k)$, $\lim_{n\to\infty}(Z_{\tau_n} - Z_{t_n}) = 0$ (since $\{\tau_n\}$ and $\{t_n\}$ are strictly increasing). $\lim_{n\to\infty}(Z_{\tau_n} - Z_{t_n}) = (Y_1 - Y_2)\chi_{A^c(k,d_k)}$ almost surely. Therefore, by the Lebesgue dominated convergence theorem, $E\{|(Z_{\tau_n} - Z_{t_n}) - (Y_1 - Y_2)\chi_{A^c(k,d_k)}|\} \to 0$ as $n \to \infty$. Since $j(k, d_k) \geqq k$ and $\{\tau_n\}$, $\{t_n\}$ are strictly increasing, we can and do choose, for each positive integer k, two bounded stopping times τ_k^* and t_k^* in B such that $\tau_k^* \geqq k$, $t_k^* \geqq k$, and $E\{|(X_{\tau_k^*} - X_{t_k^*}) - (Y_1 - Y_2)\chi_{A^c(k,d_k)}|\} \leqq 1/k$. Therefore, $\tau_k^* \geqq k$, $t_k^* \geqq k$, and $E\{|(X_{\tau_k^*} - X_{t_k^*}) - (Y_1 - Y_2)|\} \leqq 2/k$ for all $k \geqq 1$. (1) follows on letting $k \to \infty$ and the proof of Theorem 1 now is complete.

COROLLARY 1 (Chacon). *Let $\{X_n\}$ be a sequence of integrable random variables such that* $\liminf_{n\to\infty} E(|X_n|) < \infty$. *Then,*

(3) $\limsup_{\tau, t \in B} E(X_\tau - X_t) \geqq E(X^* - X_*)$, where $X^* = \limsup_{n\to\infty} X_n$, and

$$X_* = \liminf_{n\to\infty} X_n.$$

Further, if $\sup_{t\in B} E(|X_t|) < \infty$, *then X^* and X_* are integrable.*

Proof. If $\sup_{t\in B} E(|X_t|) < \infty$, then, by Theorem 1, X^*, X_* are integrable and $\limsup_{\tau, t\in B} E(X_\tau - X_t) \geqq E(X^* - X_*)$. If $\sup_{t\in B} E(|X_t|) = \infty$, without loss of generality, we can and do assume that $\sup_{t\in B} E(X_t^+) = \infty$. Since $\liminf_{n\to\infty} E(|X_n|) < \infty$, there exists a strictly increasing sequence $\{n_j\}$ of positive integers such that $E(|Xn_j|) \leqq M$ for all $j \geqq 1$ and some constant M. Now, for each bounded stopping time t in B, let $t' = t$ on $\{X_t^+ > 0\}$ and $t' = n$ on $\{X_t^+ = 0\}$ where $n = \inf\{n_j \mid n_j \geqq \sup\{t(w) \mid w \in \{X_t^+ = 0\}\}\}$. We then have $E(X_{t'} - X_n) \geqq E(X_t^+) - M$ and

$\sup_{\tau, t} E(X_\tau - X_t) = \infty \geqq E(X^* - X_*)$ and (3) follows immediately from this fact. The proof of Corollary 1 now is complete.

COROLLARY 2 (Theorem 2 of **[1]**). *Under the conditions of Corollary* 1 *and consider the following two assertions*:

(a) *The generalized sequence* $\{E(X_t) \mid t \in B\}$ *is convergent.*

(b) X_n *converges almost surely to a finite limit.*

Then (a) *implies* (b).

COROLLARY 3 (the submartingale convergence theorem). *Suppose that* $\{X_n\}$ *is a sequence of* L_1*-bounded random variables adapted to the increasing sequence* $\{\mathcal{F}_n\}$ *of* σ*-fields. Suppose that* $E(X_{n+1} \mid \mathcal{F}_n) \geqq X_n$ *almost surely for all* $n \geqq 1$. *Then* X_n *converges almost surely to a finite limit.*

REMARK. Corollaries 1 and 2 also hold under any one of the following two conditions.

(i) $\sup_n E(X_n^+) < \infty$.

(ii) $\sup_n E(X_n^-) < \infty$.

ACKNOWLEDGEMENTS. I would like to thank Professors Chacon and Sucheston for their valuable suggestions and comments.

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Received November 25, 1975.

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