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RELATIONS BETWEEN CONVERGENCE OF SERIES AND CONVERGENCE OF SEQUENCES

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Let $A = (a_n)_{n \in \mathbf{N}}$ be a sequence of real numbers. For $\xi \in (0, 1)$ define

$$S_n(\xi, A) := \sum_{k=[n\xi]+1}^{n} a_k, \qquad n \in \mathbf{N}$$

where $[x]$ is the greatest integer less than or equal to x. If no ambiguity can arise we write $S_n(\xi)$ instead of $S_n(\xi, A)$. In the theory of regularly varying sequences the problem arose of concluding from the convergence of the sequence $S_n(\xi)$, $n \in \mathbf{N}$, for all ξ in an appropriate set $K \subset (0, 1)$ of real numbers, that the sequence a_n, $n \in \mathbf{N}$, converges to zero. In this paper we give some positive results for the case that K consists of two elements.

In **[3]** it was shown that such a conclusion is not possible if K consists only of a single rational number and that the conclusion is possible if $K = \{\xi, 1 - \xi\}$ with $\xi \in (0, 1)$ irrational. The question whether such a conclusion is possible if K consists of one irrational or all rational numbers was answered negatively in **[4]**.

DEFINITION 1. If $a_n \in \{0, 1\}$, $n \in \mathbf{N}$, and $a_n = 1$ for infinitely many $n \in \mathbf{N}$, then we call $A := (a_n)_{n \in \mathbf{N}}$ a 0–1 sequence.

Let $A = (a_n)_{n \in \mathbf{N}}$ be a sequence of real numbers such that $S_n(\xi_1, A)$, $n \in \mathbf{N}$, and $S_n(\xi_2, A)$, $n \in \mathbf{N}$, are convergent for different $\xi_1, \xi_2 \in (0, 1)$. Let $\alpha = \liminf_{n \in \mathbf{N}} a_n$ and $\beta = \limsup_{n \in \mathbf{N}} a_n$. Since $\alpha = \beta$ implies $\lim_{n \in \mathbf{N}} a_n = 0$ — as otherwise $\lim_{n \in \mathbf{N}} |S_n(\xi_1, A)| = \infty$ — the Lemma below shows that only the following three cases are possible:

(I) $\lim_{n \in \mathbf{N}} a_n = 0$

(II) $\alpha < \beta$ and each $\gamma \in (\alpha, \beta)$ is an accumulation point of a_n, $n \in \mathbf{N}$

(III) $\alpha < \beta$ and there exists a 0–1 sequence B such that $S_n(\xi_i, B)$, $n \in \mathbf{N}$ converges for $i = 1, 2$.

LEMMA 2. *Let $A = (a_n)_{n \in \mathbf{N}}$ be a sequence of real numbers such that not every point between $\alpha := \liminf_{n \in \mathbf{N}} a_n$ and $\beta := \limsup_{n \in \mathbf{N}} a_n$ is an accumulation point of the sequence a_n, $n \in \mathbf{N}$. If $\xi_i \in (0, 1)$, $i = 1, \cdots, k$, and $S_n(\xi_i, A)$, $n \in \mathbf{N}$, is convergent for $i = 1, \cdots, k$, then there exists a 0–1 sequence $B = (b_n)_{n \in \mathbf{N}}$, such that $S_n(\xi_i, B)$, $n \in \mathbf{N}$, is convergent for $i = 1, \cdots, k$.*

Proof. Since not every point between α and β is an accumulation point of the sequence a_n, $n \in \mathbf{N}$, there exist γ, δ with $\alpha < \gamma < \delta < \beta$ such that $a_n \notin (\gamma, \delta)$ for all $n \in \mathbf{N}$. Since we can consider the sequence $(a_n)_{n\in\mathbf{N}}$ or the sequence $(-a_n)_{n\in\mathbf{N}}$, we may w.l.g. assume that $\gamma \geqq 0$.

Let $b_n = 0$ if $a_n \leqq \gamma$ and $b_n = 1$ if $a_n \geqq \delta$. Then $B = (b_n)_{n\in\mathbf{N}}$ is a 0–1 sequence. According to our assumption there exists $n_0 \in \mathbf{N}$ such that

$$(+) \qquad |S_{n+1}(\xi_i, A) - S_n(\xi_i, A)| < \delta - \gamma \quad \text{if} \quad n \geqq n_0, \quad i = 1, \cdots, k.$$

Since

$$S_{n+1}(\xi_i, A) - S_n(\xi_i, A) = \begin{cases} a_{n+1}, & \text{if} \quad [n\xi_i] = [(n+1)\xi_i] \\ a_{n+1} - a_{[n\xi_i]+1}, & \text{otherwise} \end{cases}$$

we obtain from $(+)$ that $S_{n+1}(\xi_i, B) = S_n(\xi_i, B)$ for all $n \geqq n_0$, $i = 1, \cdots, k$, whence $S_n(\xi_i, B)$, $n \in \mathbf{N}$, converges for $i = 1, \cdots, k$.

Now we shall prove that for most pairs of real numbers — more exactly for all $\xi_1, \xi_2 \in (0, 1)$ with $\xi_1^r \neq \xi_2^s$ for all $r, s \in \mathbf{N}$ — case (III) cannot occur.

THEOREM 3. *Let $a_n, n \in \mathbf{N}$, be a sequence of real numbers and $\xi_1, \xi_2 \in (0, 1)$ such that $S_n(\xi_i)$, $n \in \mathbf{N}$, is convergent for $i = 1, 2$.*

Assume that:

$$(*) \qquad \xi_1^r \neq \xi_2^s \quad \textit{for all} \quad r, s \in \mathbf{N}.$$

Then $a_n, n \in \mathbf{N}$, converges to zero or every real number between $\liminf_{n\in\mathbf{N}} a_n$ and $\limsup_{n\in\mathbf{N}} a_n$ is an accumulation point of a_n, $n \in \mathbf{N}$.

Proof. Assume that the assertion is false.

Hence according to Lemma 2 we may assume that a_n, $n \in \mathbf{N}$, is a 0–1 sequence. Let $\xi_1 < \xi_2$ and put $\eta_i := 1/\xi_i$.

Then there exists $z > 1$ with $\eta_1 = \eta_2^z$. We have to prove that z is rational. Since $S_n(\xi_i)$, $n \in \mathbf{N}$, converges for $i = 1, 2$ and a_n, $n \in \mathbf{N}$, is a 0–1 sequence, there exists $n_0 \in \mathbf{N}$ with

$$(1) \qquad S_n(\xi_i) = S_{n_0}(\xi_i) \quad \text{for} \quad n \geqq n_0 \qquad (i = 1, 2)$$

$$(2) \qquad n_0(1 - \eta_2^{-1/3j}) > 2/(\eta_2 - 1)$$

where $j := \lim_{n\in\mathbf{N}} S_n(\xi_2) \in \mathbf{N}$.

Let $N_1 := \{n \in \mathbf{N}: n > n_0 \text{ and } a_n = 1\}$ and let $\langle a \rangle := \min\{n \in \mathbf{N}: a \leqq n\}$ for $n \geqq 1$.

Since $\langle t \cdot \eta \rangle = \inf\{n \in \mathbf{N}\colon [n \cdot 1/\eta] = t\}$, $t \in \mathbf{N}$, $\eta > 1$, we have

$$S_{\langle t \cdot \eta \rangle}\left(\frac{1}{\eta}\right) - S_{\langle t \cdot \eta \rangle - 1}\left(\frac{1}{\eta}\right) = a_{\langle t \cdot \eta \rangle} - a_t \qquad (t \in \mathbf{N},\ \eta > 1)$$

and hence we obtain from (1) that

(3) $$t \in \mathbf{N}_1 \quad \text{implies} \quad \langle t \cdot \eta_i \rangle \in \mathbf{N}_1 \quad \text{for} \quad i = 1, 2.$$

Define inductively for $t \in \mathbf{N}_1$, $\eta > 1$

$$\tau^0(t, \eta) := t$$

and

$$\tau^n(t, \eta) := \langle \tau^{n-1}(t, \eta) \cdot \eta \rangle.$$

According to (3) we directly obtain that

(4) $$t \in \mathbf{N}_1 \quad \text{implies} \quad \tau^n(t, \eta_i) \in \mathbf{N}_1 \quad \text{for} \quad n \in \mathbf{N} \quad \text{and} \quad i = 1, 2.$$

Since $j = S_n(\xi_2) \in \mathbf{N}$ for all $n \geqq n_0$ according to (1), there exist exactly j elements $t_i \in \mathbf{N}_1$, $i = 1, \cdots, j$ with

(5) $$n_0 < t_1 < t_2 < \cdots < t_j \leqq \langle n_0 \cdot \eta_2 \rangle.$$

Since $\eta_2 > 1$, (5) implies

(6) $$\tau^n(n_0, \eta_2) < \tau^n(t_1, \eta_2) < \cdots < \tau^n(t_j, \eta_2) \leqq \tau^{n+1}(n_0, \eta_2)$$

for all $n \in \mathbf{N}$. Now we obtain from relations (1), (4), (5) and (6) that

(7) $$\mathbf{N}_1 = \{\tau^n(t_i, \eta_2)\colon i = 1, \cdots, j, n \in \mathbf{N} \cup \{0\}\}.$$

As by (4) $\tau^n(t_1, \eta_1) \in \mathbf{N}_1$, according to (7) for each $n \in \mathbf{N}$ there exist $k(n) \in \mathbf{N}$, $i(n) \in \{1, \cdots, j\}$ with

(8) $$\tau^n(t_1, \eta_1) = \tau^{k(n)}(t_{i(n)}, \eta_2).$$

By induction it is easily proved that

(9) $$|\tau^n(t, \eta) - t\eta^n| \leqq 1 + \eta + \cdots + \eta^{n-1} = (\eta^n - 1)/(\eta - 1)$$

for $t \in \mathbf{N}$ and $n > 1$.

Since $t_i < \eta_2 t_1$ for $i = 1, \cdots, j$ (see (5)) there exist $x_i \in [0, 1]$ with

$t_i = t_1\eta_2^{x_i}$. Then $x_1 = 0 < x_2 < \cdots < x_j < 1 =: x_{j+1}$. Hence there exists $l \in \{1, \cdots, j\}$ with

$$x_{l+1} - x_l \geqq \frac{1}{j}\,. \tag{10}$$

Let us now assume that z is irrational. According to ([2], p. 69) there exists an element $m \in \mathbf{N}$ with

$$x_l + \frac{1}{3j} < mz - [mz] < x_l + \frac{2}{3j}\,. \tag{11}$$

Since $\eta_1 = \eta_2^z$ we obtain from (8) and (9) that

$$|t_1\eta_2^{mz} - t_{i(m)}\eta_2^{k(m)}| \leqq \frac{1}{\eta_1 - 1}\,\eta_2^{mz} + \frac{1}{\eta_2 - 1}\,\eta_2^{k(m)}$$

and hence

$$|t_1\eta_2^{mz} - t_{i(m)}\eta_2^{k(m)}| \leqq \frac{2}{\eta_2 - 1}\,\eta_2^{\max(mz,k(m))}. \tag{12}$$

Now we distinguish four cases

(i) If $mz < k(m)$ then $mz - k(m) \leqq -1/3j$ according to (10) and (11). Hence we obtain from (5) and (2) that

$$|t_1\eta_2^{mz-k(m)} - t_{i(m)}| \geqq t_1 - t_1\eta_2^{mz-k(m)} \geqq t_1(1 - \eta_2^{-1/3j}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

In the following three cases we assume that $mz > k(m)$.

(ii) Let $i(m) \leqq l$: As $mz - k(m) \geqq x_l + 1/3j$ by (11) we obtain from (5) and (2) that

$$|t_1 - t_{i(m)}\eta_2^{k(m)-mz}| \geqq t_1 - t_1\eta_2^{x_{i(m)}+k(m)-mz} \geqq t_1(1 - \eta_2^{-1/3j}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

(iii) Let $i(m) > l$ and $[mz] = k(m)$: Then $mz - k(m) \leqq x_{l+1} - 1/3j$ by (10) and (11), and we obtain from (5) and (2) that

$$|t_1 - t_{i(m)}\eta_2^{k(m)-mz}| \geqq t_{l+1}\eta_2^{-(x_{l+1}-1/3j)} - t_1 = t_1(\eta_2^{1/3j} - 1) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

(iv) If $i(m) > l$ and $[mz] > k(m)$, then $mz - k(m) \geqq 1 + 1/3j$ by (11), and we obtain from (5) and (2)

$$|t_1 - t_{i(m)}\eta_2^{k(m)-mz}| \geqq t_1 - t_j\eta_2^{-(1+1/3j)} \geqq t_1 - t_1\eta_2^{-1/3j} = t_1(1-\eta_2^{-1/3j}) > \frac{2}{\eta_2 - 1}$$

which contradicts (12).

Thus we have shown that the assumption of z being irrational leads to a contradiction.

If $r, s \in \mathbf{N}$ denote by (r, s) the greatest common divisor of r and s.

The following remark shows that for two rational numbers condition (*) of Theorem 3 is nearly always fulfilled.

REMARK 4. If $\xi_1, \xi_2 \in (0,1)$ are rational numbers and $\xi_1^r = \xi_2^s$ for $r, s \in \mathbf{N}$ with $(r, s) = 1$ then there exist $t, u \in \mathbf{N}$ such that $\xi_1 = (t/u)^s$ and $\xi_2 = (t/u)^r$.

Proof. Let w.l.g. $\xi_i = l_i/m_i$ where $l_i, m_i \in \mathbf{N}$ and $(l_i, m_i) = 1$ for $i = 1, 2$. If $\xi_1^r = \xi_2^s$ i.e. $l_1^r m_2^s = l_2^s m_1^r$, then $l_1^r = l_2^s$ and $m_1^r = m_2^s$.

We may choose r and s such that $(r, s) = 1$. Then by representation of l_i, m_i as a product of prime numbers we obtain $t, u \in \mathbf{N}$ with

$$t^s = l_1, \qquad t^r = l_2 \quad \text{and} \quad u^s = m_1, \qquad u^r = m_2.$$

According to Theorem 3 Cases I and II can occur. According to Example 2 of **[4]** it is not possible to exclude Case II. Even if $S_n(\xi, A)$, $n \in \mathbf{N}$, converges for each rational number $\xi \in (0,1)$ the sequence a_n, $n \in \mathbf{N}$, need not converge to zero.

We remark that the following questions remain unsolved:

(1) If ξ_1 and ξ_2 are two different irrational numbers, does the convergence of $S_n(\xi_i, A)$, $n \in \mathbf{N}$, (for $i = 1, 2$) imply that a_n, $n \in \mathbf{N}$, converges to zero?

(2) Give an exact characterization of those pairs of rational numbers ξ_1, ξ_2 for which only Case I or II is possible; is for instance the condition (*) of Theorem 3 such an exact characterization?

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