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SOME RESULTS ON NORMALITY OF A GRADED RING

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Let $R = \bigoplus_{i \geq 0} R_i$ be a graded domain and let p be a homogeneous prime ideal in R. Let $R_{\mathfrak{p}}$ be be the localization of R at p and $R_{(\mathfrak{p})} = \{r_i/s_i \mid r_i s_i \in R_i$ and $s_i \notin \mathfrak{p}\}$. If $R_1 \cap (R - \mathfrak{p}) \neq \emptyset$, then $R_{\mathfrak{p}}$ is a localization of a transcendental extension of $R_{(\mathfrak{p})}$. Thus $R_{\mathfrak{p}}$ is normal (regular) if and only if $R_{(\mathfrak{p})}$ is normal (regular). Let $\mathrm{Proj}(R) = \{\mathfrak{p} \mid \mathfrak{p}$ is a homogeneous prime ideal and $\mathfrak{p} \subsetneqq \bigoplus_{i>0} R_i\}$. Under certain conditions a Noetherian graded domain R is normal if $R_{(\mathfrak{p})}$, is normal for each $\mathfrak{p} \in \mathrm{Proj}(R)$. If $R = \bigoplus_{i \geq 0} R_i$ is reduced and $F_0 = \{r_i/u_i \mid r_i, u_i \in R_i$ and $u_i \in U$ where U is the set of all nonzero divisors$\}$ is Noetherian, then the integral closure of R in the total quotient ring of R is also graded.

1. Introduction. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded integral domain. Let $\mathrm{Spec}(R)$ be the set of all prime ideals in R. Let $R_+ = \bigoplus_{i>0} R_i$. R_+ is an ideal in R. An ideal $\mathfrak{A}$ in R is said to be irrelevant if $R_+ \subset \sqrt{\mathfrak{A}}$, the radical of $\mathfrak{A}$. Let $\mathrm{Proj}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subset R_+$ is homogeneous and nonirrelevant$\}$. For each $\mathfrak{p} \in \mathrm{Spec}(R)$, let $R_{\mathfrak{p}} = \{r/s \mid, s \in R$ and $s \notin \mathfrak{p}\}$, and for each homogeneous prime ideal $\mathfrak{p}$, let $R_{(\mathfrak{p})} = \{r_i/s_i \mid r_i, s_i \in R_i$ and $s_i \notin \mathfrak{p}\}$. (Note: $R_{(\mathfrak{p})}$ in **[1]** is defined for $\mathfrak{p} \in \mathrm{Proj}(R)$ only.) According to the terminology of Seidenberg **[9]**, $R_{\mathfrak{p}}$ is called the arithmetical local ring of R at $\mathfrak{p}$ and $R_{(\mathfrak{p})}$ the geometrical local ring of R at $\mathfrak{p}$. I prove that if $R_1 \cap (R - \mathfrak{p}) \neq \emptyset$ then $R_{\mathfrak{p}}$ is the ring of quotients of a transcendental extension of $R_{(\mathfrak{p})}$ relative to a multiplicative set, $R_{\mathfrak{p}}$ is normal (regular) if and only if $R_{(\mathfrak{p})}$ is normal (regular); see Theorem 2. In the case of an irreducible projective variety V over a field k in a projective n-space P_k^n, V/k is normal if the geometrical local ring of V at each $\mathfrak{p} \in V$, $\mathfrak{O}_k^v(\mathfrak{p})$ is integrally closed. V is arithmetically normal if the ring of strictly homogeneous coordinates $k[V]$ is integrally closed. The latter implies the former. For the converse, various cohomological criteria are developed; see **[3]**, **[8]**, **[9]**. I attempt to study the normality of a graded domain R if $R_{(\mathfrak{p})}$ is normal for every $\mathfrak{p} \in \mathrm{Proj}(R)$. In this paper, I also obtain the following theorem: Let R be a Noetherian graded domain, say $R = R_0[x_1, \cdots, x_n]$ and $x_1, \cdots, x_n$ are of homogeneous degree 1. Assume that R_0 contains a field k over which R_0 and $k(x_1, \cdots, x_n)$ are linearly disjoint and separable. Let $\mathfrak{B}$ be the kernel of the canonical map from the polynomial ring $R_0[X_1, \cdots, X_n]$. Then R is normal if R_0 is normal, $R_{(\mathfrak{p})}$ is normal for every $\mathfrak{p} \in \mathrm{Proj}(R)$ and $\mathrm{coh.d.}\mathfrak{B} \cdot K[X_1, \cdots, X_n] < n - 1$, where K is the quotient field of R_0.

In the §4, we prove that under certain conditions on a graded ring R (not necessarily integral domain) the integral closure $\bar{R}$ of R in the total quotient ring of R is also graded; see Theorem 6.

Our references on the elementary well known facts about graded rings can be found in **[1]** and **[10]**.

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2. Normality and regularity of local domains. Let R be a commutative ring with identity 1. Let $\mathbf{p}$ be a prime ideal in R. By height of $\mathbf{p}$, we mean the supremum of the length of chains of prime ideals $\mathbf{p}_0 \supsetneqq \mathbf{p}_1 \supsetneqq \mathbf{p}_2 \supsetneqq \cdots \supsetneqq \mathbf{p}_n$ with $\mathbf{p}_0 = \mathbf{p}$ and denote it by $ht(\mathbf{p})$. Let $R = \bigoplus_{i \geqq 0} R_i$ be a graded integral domain. Let K be the quotient field of R. We say that R is integrally closed if R is integrally closed in K. Let $K_q = \{f_i/g_j \mid i - j = q; f_i \in R_i, g_j \in R_j\}$. K_0 is a field, $\Sigma_{q \in Z} K_q$ is a subring of K and the sum is direct, where Z stands for the set of integers. Elements in K_q are known as homogeneous elements of K of degree q. The following theorem was originally proved in **[9]** for projective varieties. We observe that the same holds true for non-Noetherian graded domain also.

THEOREM 1. *Let $R = \bigoplus_{i \geqq 0} R_i$ be a graded domain. Let $\mathbf{p} \in \operatorname{Spec}(R)$ be nonhomogeneous. If $ht(\mathbf{p}) = 1$ then $R_{\mathbf{p}}$ is integrally closed.*

Proof. Let $\mathbf{p}^*$ be the ideal generated by all the homogeneous elements of $\mathbf{p}$. By [**10**, Lemma 3, p. 153] $\mathbf{p}^*$ is a prime ideal and $\mathbf{p} \supsetneqq \mathbf{p}^* \geqq 0$. Since $ht(\mathbf{p}) = 1$, $\mathbf{p}^* = 0$. Therefore $\mathbf{p}$ contains no homogeneous element. Thus every nonzero homogeneous element u is in $R - \mathbf{p}$. It follows therefore $\bigoplus_{q \in Z} K_q \subset R_{\mathbf{p}}$. Let $f \in K$ be integral over $R_{\mathbf{p}}$. Then there exists $h \in R - \mathbf{p}$ such that fh is integral over R. It follows from [**10**, Theorem 11, p. 157] that each of the homogeneous components is integral over R. By the preceeding, each homogeneous component of $f \cdot h$ is in $R_{\mathbf{p}}$. Therefore $f \cdot h \in R_{\mathbf{p}}$ and $f \in R_{\mathbf{p}}$. Thus $R_{\mathbf{p}}$ is integrally closed.

Let $y \in K_1$ be any nonzero element. If $\xi \in K_q$, then $\xi / y^q \in K_0$. Moreover $R \subset K_0[y]$, $K = K_0(y)$, y is transcendental over K_0, $K_q = K_0 y^q$ and $\bigoplus_{q \in Z} K_q = K_0[y, 1/y]$. We have the following theorem.

THEOREM 2.† *Let $R = \bigoplus_{i \geqq 0} R_i$ with that $R_1 \neq 0$. Let $\mathbf{p}$ be a homogeneous prime ideal such that there exists an element $r_1 \in R_1 - \mathbf{p}$. Then*

† Professor A. Seidenberg remarks that the present Theorem 2 strengthens Lemma 2 of [**9**; p. 618] and corrects its proof.

(a) *K_0 is the quotient field of $R_{(\mathbf{p})}$ and $K_0 \cap R_{\mathbf{p}} = R_{(\mathbf{p})}$.*

(b) *$R_{(\mathbf{p})}$ is integrally closed in K_0 implies that $R_{(\mathbf{p})}$ is integrally closed in K.*

(c) *$R_{\mathbf{p}} = (R_{(\mathbf{p})}[r_1])_S$, where $S = R - \mathbf{p}$; r_1 is transcendental over $R_{(\mathbf{p})}$.*

(d) *$R_{\mathbf{p}}$ is integrally closed in K if and only if $R_{(\mathbf{p})}$ is integrally closed in K_0.*

(e) *$R_{(\mathbf{p})}$ is regular if and only if $R_{\mathbf{p}}$ is regular.*

Proof. By definition $R_{(\mathbf{p})} \subset K_0$. Let $x \in K_0$, $x = f_i/g_i$ for some $f_i, g_i \in R_i$ and $g_i \neq 0$. Then $x = f_i/g_i = (f_i/r_1^i)/(g_k/r_1^i)$, since f_k/r_1^i and f_i/r_1^i are both in $R_{(\mathbf{p})}$. Therefore x is in the quotient field of $R_{(\mathbf{p})}$. Thus K_0 is the quotient field of $R_{(\mathbf{p})}$. For the second part of (a) we need only to prove that $K_0 \cap R_{\mathbf{p}} \subset R_{(\mathbf{p})}$. Let $x \in K_0 \cap R_{\mathbf{p}}$. Then $x = f_i/g_i$ for some f_i, $g_i \in R_i$ with $g_i \neq 0$. On the other hand $x = (r_j + r_{j+1} + \cdots + r_{j+m})/(s_l + s_{l+1} + \cdots + s_{l+m})$ with $s_l + s_{l+1} + \cdots + s_{l+m} \notin \mathbf{p}$. Then there exists an index $l+t$ such that $s_{l+t} \notin \mathbf{p}$. $f_i \cdot (s_l + s_{l+1} + \cdots + s_{l+m}) = g_i(r_j + r_{j+1} + \cdots + r_{j+k})$ implies that $l = j$, $m = k$ and $f_i \cdot s_{l+t} = g_i \cdot r_{l+t}$. Thus $x = f_i/g_i = r_{l+t}/s_{l+t}$ i.e. $x \in R_{(\mathbf{p})}$. Therefore $K_0 \cap R_{\mathbf{p}} = R_{(\mathbf{p})}$.

(b) If $R_{(\mathbf{p})}$ is integrally closed in K_0, then, since $K = K_0(r_1)$ and r_1 is transcendental over K_0 as noted in the preceeding, K_0 is algebraically closed in K and $R_{(\mathbf{p})}$ is thus integrally closed in K.

(c) As noted in (b), r_1 is transcendental over $R_{(\mathbf{p})}$. Let $f \in R$ be an element. Then $f = f_r + f_{r+1} + \cdots + f_n$ where $f_i \in R_i$ for some nonnegative integers r and n. But $f = (f_r/r_1^r)r_1^r + (f_{r+1}/r_1^{r+1})r_1^{r+1} + \cdots + (f_n/r_1^n)r_1^n \in R_{(\mathbf{p})}[r_1]$. Therefore $R \subset R_{(\mathbf{p})}[r_1]$. Thus $S = R - \mathbf{p}$ is a multiplicative set in $R_{(\mathbf{p})}[r_1]$. Now let $f/g \in R_{\mathbf{p}}$, $g \in R - \mathbf{p}$. Then for some nonnegative integer t and m,

$$\frac{f}{g} = \frac{f_t}{g} + \cdots + \frac{f_m}{g} = \frac{1}{g}\left(\left(\frac{f_t}{r^t}\right) r_1^t + \left(\frac{f_{t+1}}{r^{t+1}}\right) r_1^{t+1} + \cdots + \left(\frac{f_m}{r_1^m}\right) r_1^m\right).$$

Therefore $f/g \in (R_{(\mathbf{p})}[r_1])_S$ i.e. $R_{\mathbf{p}} \subset [R_{(\mathbf{p})}[r_1])_S$. The other inclusion is obvious. Thus $R_{\mathbf{p}} = (R_{(\mathbf{p})}[r_1])_S$.

(d) Now, if $R_{(\mathbf{p})}$ is integrally closed in K, then clearly $R_{\mathbf{p}} = (R_{(\mathbf{p})}[r_1])_S$, being a localization of transcendental extension of an integrally closed domain, is integrally closed. Conversely if $R_{\mathbf{p}}$ is integrally closed in K, let $f \in K_0$ be an integral element over $R_{(\mathbf{p})}$. Then $f \in R_{\mathbf{p}}$. Thus $f \in R_{\mathbf{p}} \cap K_0 = R_{(\mathbf{p})}$, and $R_{(\mathbf{p})}$ is integrally closed.

(e) Recall that a ring A is said to be regular if A_m is a regular local ring for each maximal ideal m in A. It follows from Serre's theorem [**5**; p. 139] that A is regular if and only if $A_{\mathbf{p}}$ is regular for every $\mathbf{p} \in \operatorname{Spec}(A)$.

If $R_{(\mathbf{p})}$ is a regular local ring, then by [**5**; Theorem 40, p. 126] the polynomial ring $R_{(\mathbf{p})}[r_1]$ is regular. Since localization of a regular ring is regular therefore $R_{\mathbf{p}} = (R_{(\mathbf{p})}[r_1])_S$ is a regular local ring.

Conversely assume that $R_{\mathfrak{p}} = (R_{(\mathfrak{p})}[r_1])_S$ is a regular local ring. Since $R_{(\mathfrak{p})}[r_1]$ is a polynomial ring over $R_{(\mathfrak{p})}$ therefore $R_{(\mathfrak{p})}[r_1]$ is $R_{(\mathfrak{p})}$-flat. $(R_{(\mathfrak{p})}[r_1])_S$ is $R_{(\mathfrak{p})}[r_1]$-flat therefore $R_{\mathfrak{p}}$ is $R_{(\mathfrak{p})}$-flat. Thus $R_{(\mathfrak{p})}$ is Noetherian. The inclusion map $R_{(\mathfrak{p})} \to R_{\mathfrak{p}}$ is obviously a local homomorphism. Therefore it follows from [**1**; IV, 17.3.3 (i), p. 48] that $R_{(\mathfrak{p})}$ is a regular local ring.

There are graded rings in which there are homogeneous prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \cap R_1 \neq R_1$. For example: (1) graded rings which are homogeneous coordinate rings of projective varieties. In this case $\mathfrak{p} \cap R_1 \neq R_1$ for $\mathfrak{p} \in \mathrm{Proj}(R)$. (2) $R = R_0[R_1]$, a graded ring generated over R_0 by R_1; (3) Let $k[X, Y]$ be a polynomial ring in two indeterminantes over a field k. Let $R = k[Y] + (X \cdot Y) \cdot k[X, Y]$. R has a graded structure $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ with $R_0 = k$, $R_1 = k \cdot Y$; $R_2 = kY^2 + k(X \cdot Y)$, $R_3 = kY^3 + kX^2Y + kXY^2$, etc. It follows from the observation that $(X^i \cdot Y^j)^2 \in Ry$ if $j \geqq 1$ that $\mathfrak{p} \cap R_1 = 0$ for every $\mathfrak{p} \in \mathrm{Proj}(R)$.

3. Normality of a graded domain. In this section, a graded domain R is normal if it is integrally closed in its field of fractions.

Recall [**6**; Theorem 8, p. 400]: Let $\mathfrak{O}$ and $\mathfrak{O}'$ be two normal rings which contain a field k. If $\mathfrak{O}$ and $\mathfrak{O}'$ are separably generated over k and if $\mathfrak{O} \otimes_k \mathfrak{O}'$ is an integral domain, then $\mathfrak{O} \otimes_k \mathfrak{O}'$ is a normal ring.

THEOREM 3. *Let R_0 be a normal integral domain containing a field k such that R_0 is separable over k. Let $R = R_0[x] = R_0[x_1, \cdots, x_n]$ be an integral domain finitely generated over R_0 as an R_0-algebra such that the quotient field K of R_0 and the quotient field $k(x)$ of $k[x_1, \cdots, x_n]$ are linearly disjoint over k, and $k(x)$ separable over k. Then $k[x]$ is normal if and only if R is normal.*

Proof. Let $X_1, \cdots, X_n$ be n indeterminantes over R_0. Let $\mathfrak{A}$ be the prime ideal in $k[X] = k[X_1, \cdots, X_n]$ such that $k[x_1, \cdots, x_n] \cong k[X_1, \cdots, X_n]/\mathfrak{A}$ and let $\mathfrak{B}$ be the prime ideal in $R_0[X] = R_0[X, \cdots, X_n]$ such that $R = R_0[X]/\mathfrak{B}$. Then $\mathfrak{B} \cdot K[X] \cap R_0[X] = \mathfrak{B}$ and $\mathfrak{A} = \mathfrak{B} \cap k[X]$. Since K and $k(x)$ are linearly disjoint over k, it is well known that $\mathfrak{A} \cdot K[X] = \mathfrak{B} \cdot K[X]$ and $\mathfrak{A} \cdot R_0[X] = \mathfrak{B}$, [**4**; Corollary 1, p. 67]. We shall use $\mathfrak{B}$ in both $R_0[X]$ and $K[X]$ as the prime ideal determined by $(x) = (x_1, \cdots, x_n)$. Since $R_0 \otimes_k k[X] = R_0[X]$, it follows that $R_0 \otimes_k k[x] = R_0[x]$, i.e. $R_0 \otimes_k k[x]$ is an integral domain. It follows from [**6**; Theorem 8, p. 400] that $R_0[x]$ is normal. Conversely if $R_0[x]$ is normal, then $R_0[x]_{\mathfrak{p}}$ is normal for each $\mathfrak{p} \in \mathrm{Spec}(R_0[x])$. Let $\mathfrak{p}^c = \mathfrak{p} \cap k[x]$ for $\mathfrak{p} \in \mathrm{Spec}(R_0[x])$ and $\mathfrak{p} \cap R_0 = \{0\}$. Then $k[x]_{\mathfrak{p}^c}$ is also normal. Indeed let $\xi \in k(x)$ be integral over $k[x]_{\mathfrak{p}^c}$. Since $k[x]_{\mathfrak{p}^c} \subset R_0[x]_{\mathfrak{p}}$, therefore $\xi \in R_0[x]_{\mathfrak{p}}$. Thus $\xi \in R_0[x]_{\mathfrak{p}} \cap k(x)$. It is sufficient to show that $R_0[x]_{\mathfrak{p}} \cap k(x) \subset k[x]_{\mathfrak{p}^c}$. Let $S = R_0 - \{0\}$. $K[x] = S^{-1}R_0[x]$ and

$S^{-1}\mathbf{p}$ is a prime ideal in $K[x]$. $S^{-1}\mathbf{p} \cap k[x] = \mathbf{p} \cap k[x]$. Since K and $k(x)$ are linearly disjoint over k, it follows from [**4**; Proposition 6, p. 92] that $K[x]_{S^{-1}\mathbf{p}} \cap k(x) = k[x]_{\mathbf{p}^c}$. Thus $k[x]_{\mathbf{p}^c} \supset R_0[x]_{\mathbf{p}} \cap k(x)$, and $k[x]_{\mathbf{p}^c} = R_0[x]_{\mathbf{p}} \cap k(x)$. So $\xi \in k[x]_{\mathbf{p}^c}$ and $k[x]_{\mathbf{p}^c}$ is therefore normal.

We shall finish the proof by showing that $\mathrm{Spec}(k[x]) = \{\mathbf{p} \cap k[x] \mid \mathbf{p} \in \mathrm{Spec}(R_0[x]) \text{ and } \mathbf{p} \cap R_0 = 0\}$. Let $\mathbf{q}_{\mathfrak{X}}$ be a prime ideal. There exists a prime ideal $Q_{\mathfrak{X}}$ in $K[X]$ such that $Q_{\mathfrak{X}} \cap k[X] = \mathbf{q}_{\mathfrak{X}}$. Indeed, using Zariski's terminology [**10**; pp. 21–22 and pp. 161–176], we consider an algebraically closed field Ω containing K and Ω is of infinite transcendence degree over K. Let A_n^{Ω} be the n dimensional affine space, i.e. $A_n^{\Omega} = \{(a_1, \cdots, a_n) \mid a_1, \cdots, a_n \in \Omega\}$. Every prime ideal P in $K[X]$ defines an irreducible algebraic variety V over K in A_n^{Ω}. Every irreducible algebraic variety V over K carries a generic point $(\xi) = (\xi_1, \cdots, \xi_n) \in A_n^{\Omega}$ over K, and $P = \{g(X) \in K[X] \mid g(\xi) = 0\}$. Let $(\eta) = (\eta_1, \cdots, \eta_n) \in A_n^{\Omega}$ be a generic point of $\mathbf{q}_{\mathfrak{X}}$ over k, i.e. $\mathbf{q}_{\mathfrak{X}} = \{f(X) \in k[X] \mid f(\eta) = 0\}$. Let $Q_x = \{F(X) \in K[X] \mid F(\eta) = 0\}$. Then $Q_{\mathfrak{X}}$ is a prime ideal and $Q_{\mathfrak{X}} \cap k[X] = \mathbf{q}_{\mathfrak{X}}$. Let $Q'_{\mathfrak{X}} = Q_{\mathfrak{X}} \cap R_0[X]$, $Q'_{\mathfrak{X}} \cap R_0 = 0$ and $Q'_{\mathfrak{X}} \cap k[X] = \mathbf{q}_{\mathfrak{X}}$. Since $\mathfrak{A} \subset \mathbf{q}_{\mathfrak{X}} \Leftrightarrow \mathfrak{B} \cdot K[X] \subset Q_{\mathfrak{X}} \Leftrightarrow \mathfrak{B} \subset Q'_{\mathfrak{X}}$. Let $Q' = Q'_{\mathfrak{X}}/\mathfrak{B} \subset R_0[x]$. Then $Q' \cap k[x] = \mathbf{q}$. Thus each prime ideal in $k[x]$ is the contraction of a prime ideal in $R_0[x]$ intersecting R_0 at 0.

As the assertion in the last part of the proof of the above theorem will be referred later, we would like to state it as a corollary.

COROLLARY. *Let R_0 be an integral domain containing a field k. Let $R = R_0[x_1, \cdots, x_n]$ be an integral domain finitely generated over R_0 as an algebra such that the quotient field K of R_0 and the quotient field $k(x)$ of $k[x] = k[x_1, \cdots, x_n]$ are linearly disjoint over k. Then* $\mathrm{Spec}(k[x]) = \{\mathbf{p} \cap k[x] \mid \mathbf{p} \in \mathrm{Spec}(R_0[x])$ *and* $\mathbf{p} \cap R_0 = 0\}$. *Moreover if R is graded with R_0 as the component of homogeneous degree* 0, *then* $\mathrm{Proj}(k[x]) = \{\mathbf{p} \cap k[x] \mid \mathbf{p} \in \mathrm{Proj}(R_0[x])\} = \{\mathbf{p} \cap k[x] \mid \mathbf{p} \in \mathrm{Proj}\, K[x]\}$.

Proof (of the last part). Let $\mathfrak{A}, \mathfrak{B}, \mathbf{q}, \mathbf{q}_{\mathfrak{X}}$, and $Q_{\mathfrak{X}}$ be the same as those in the proof of Theorem 3. If R is a graded domain, then both $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous ideals. If $\mathbf{q}$ is a nonirrelevant and homogeneous prime ideal in $k[x]$, then so is $\mathbf{q}_{\mathfrak{X}}$. Let $Q^*_{\mathfrak{X}}$ be the ideal in $K[x]$ generated by the homogeneous elements belonging to $Q_{\mathfrak{X}}$. Then, by [**10**; Lemma 3, p. 153], $Q^*_{\mathfrak{X}}$ is a prime ideal and clearly $Q^*_{\mathfrak{X}} \cap k[X] = \mathbf{q}_{\mathfrak{X}}$. Since $\mathbf{q}_{\mathfrak{X}}$ is nonirrelevant, $Q^*_{\mathfrak{X}}$ is also nonirrelevant, and $Q^*_{\mathfrak{X}} \supset \mathfrak{B}$. Let $Q^* = Q^*_{\mathfrak{X}}/\mathfrak{B}$. We have $Q^* \cap k[x] = \mathbf{q}$. Therefore $\mathrm{Proj}(k[x]) = \{\mathbf{p} \cap k[x] \mid \mathbf{p} \in \mathrm{Proj}(R) \text{ and } \mathbf{p} \cap R_0 = 0\}$.

Let us recall some definitions and facts: Let $R = \bigoplus_{i \geq 0} R_i$ be a graded integral domain. R is Noetherian if and only if R_0 is Noetherian and R is an R_0-algebra of finite type. Let $\bar{R}$ be the integral closure of R in its field of quotients K. Let K_i be the homogeneous component of K of degree i as defined in §2. Then $\bar{R}$ is graded with $\bar{R}_i = \bar{R} \cap K_i$. Thus if R is normal then R_0 must be normal.

Corresponding to Krull's characterization of a Noetherian domain being normal [7; (12.9), p. 41], we have the following theorem for normality of a Noetherian graded domain.

THEOREM 4. *Let R be a graded Noetherian domain such that $R_1 - \mathbf{p} \neq \emptyset$ for each homogeneous prime ideal $\mathbf{p}$ of ht 1 in R. If* (1) *$R_{(\mathbf{p})}$ is normal for every homogeneous prime ideal $\mathbf{p}$ of height* 1 *and* (2) *the associated prime ideals of every nonzero homogeneous ideal are of height* 1, *then R is normal.*

Proof. We first note that it follows from condition (1), Theorem 1 and Theorem 2 that $R_{\mathbf{p}}$ is normal for every $\mathbf{p} \in \mathrm{Spec}(R)$ and $ht(\mathbf{p}) = 1$. Let $K, \bar{R}$ and $\bar{R}_i$ be the same as defined in the preceeding. Let $\alpha \in \bar{R}$, $\alpha = \sum_{i=m}^{n} \alpha_i$ for some nonnegative integers m and n and $\alpha_i \in \bar{R}_i$. Let $\alpha_i = b_{ij}/a_{il}$ where $j - l = i$, $b_{ij} \in R_j$ and $a_{il} \in R_l$. If a_{il} is a unit in R then $\alpha_i \in R$. If a_{il} is a nonunit, then the nonzero homogeneous principal ideal $(a_{il})R$ has a primary decomposition $\bigcap_{t=1}^{u} \mathbf{q}_t$ with $\mathbf{p}_1, \cdots, \mathbf{p}_u$ as the associated prime ideals. In view of [**10**; Theorem 9 and Corollary; pp. 153–154] we may assume that $\mathbf{q}_t$'s and $\mathbf{p}_t$'s are homogeneous, (2) implies that $ht(\mathbf{p}_t) = 1$ for $t = 1, 2, \cdots, u$. Thus $R_{\mathbf{p}_t}$ is normal for $t = 1, 2, \cdots, u$. α_i is integral over R implies that α_i is integral over $R_{\mathbf{p}_t}$ for $t = 1, 2, \cdots, u$. Hence $\alpha_i \in R_{\mathbf{p}_t}$ for $t = 1, 2, \cdots, u$. Therefore $b_{ij} \in \bigcap_{t=1}^{u} ((a_{il})R_{\mathbf{p}_t} \cap R) = \bigcap_{t=1}^{u} \mathbf{q}_t = (a_{il})R$. Thus $\alpha_i = b_{ij}/a_{il} \in R$ and $\alpha = \sum_{i=m}^{n} \alpha_i \in R$. R is therefore normal.

Let $A = K[X_1, \cdots, X_n]$ be a polynomial ring over a field K. The smallest integer d such that any chain of syzygies of the A-module M terminates at $(d+1)$th step is called the cohomological dimension of M and is denoted by coh.d.(M). Let $\mathfrak{A} \subset A$ be a homogeneous ideal such that $\mathfrak{A} \neq (0)$, $\neq (1)$. coh.d.$(\mathfrak{A}) \leqq n$ and it is n if and only if $(X_1, \cdots, X_n)A$ is an associated prime ideal of $\mathfrak{A}$. Let l be a form in A, and $l \notin K$. If $\mathfrak{A} : l = \mathfrak{A}$ then coh.d.$(\mathfrak{A}, l) = 1 +$ coh.d.$(\mathfrak{A})$.

THEOREM 5. *Let $R = \bigoplus_{i \geqq 0} R_i$ be a Noetherian graded integral domain generated over R_0 by nonzero homogeneous elements $x_1, \cdots, x_n$ of degree* 1. *Assume that R_0 contains a subfield k over which R_0 and $k(x) = k(x_1, \cdots, x_n)$ are linearly disjoint and R_0 is normal. Assume* $\mathrm{tr.deg}_k k(x) > 0$. *Let $R_0[X] = R_0[X_1, \cdots, X_n]$ be the polynomial ring over R_0 in indeterminantes $X_1, \cdots, X_n$ and let $\mathfrak{B}$ be the ideal such that $R_0[x] \cong R_0[X]/\mathfrak{B}$. Let $\mathfrak{A} = \mathfrak{B} \cap k[X]$, and let $S = R_0 - \{0\}$.*

(1) *If, for each $\mathbf{p} \in \mathrm{Proj}(R_0[x])$, $R_0[x]_{(\mathbf{p})}$ is normal and* coh.d.$S^{-1}\mathfrak{B} < n - 1$, *then $k[x]$ is normal.*

(2) *If R_0 and $k(x)$ are both separable over k, and if $R_0[x]_{(\mathbf{p})}$ is normal*

for all $\mathfrak{p} \in \mathrm{Proj}(R_0[x])$, *and* coh.d.$S^{-1}\mathfrak{B} < n - 1$ *then* $R_0[x]$ *is normal.*

(3) *If* $R_{(\mathfrak{p})}$ *is normal for each* $\mathfrak{p} \in \mathrm{Proj}(R)$ *and if* coh.d.$\mathfrak{B} \cdot S^{-1}R_0[X] = n - 1$ *then* $R_0[x]$ *is not normal.*

Proof. (1) Both $\mathfrak{A}$ and $\mathfrak{B}$ are homogeneous ideals, $k[x]$ is graded. As projective scheme $\mathrm{Proj}(R_0[x]) \cong \mathrm{Proj}((S^{-1}R_0)[x])$ [**1**, Prop. (2.4.7), p. 30]. Therefore $(S^{-1}R_0)[x]$ is locally normal, i.e. $(S^{-1}R_0)[x]_{(\mathfrak{p})}$ is normal for each $\mathfrak{p} \in \mathrm{Proj}(S^{-1}R_0[x])$. Since tr.deg.$S^{-1}R_0[x] > 0$. If coh.d.$S^{-1}\mathfrak{B} < n - 1$, by [**9**, Theorem 3, p. 619], $S^{-1}R_0[x]$ is normal. Therefore $S^{-1}R_0[x]_{\mathfrak{p}}$ is normal for every $\mathfrak{p} \in \mathrm{Spec}(S^{-1}R_0[x])$. Since $(S^{-1}R_0)[x]_{\mathfrak{p}} \cap k(x) = k[x]_{\mathfrak{p}^c}$ as shown in the preceeding, where $\mathfrak{p}^c = \mathfrak{p} \cap k[x]$. $k[x]_{\mathfrak{p}^c}$ is normal. By the Corollary to Theorem 3, $\mathrm{Spec}(k[x]) = \{\mathfrak{p}^c \mid \in \mathrm{Spec}(S^{-1}R_0)[x]\}$, we have that $k[x]_{\mathfrak{q}}$ is normal for every $\mathfrak{q} \in \mathrm{Spec}(k[x])$. Therefore $k[x]$ is normal.

(2) By (1), $k[x]$ is normal. R_0 is normal. It follows from Theorem 3, $R_0[x]$ is normal.

(3) If coh.d.$\mathfrak{B} \cdot S^{-1}R_0[X] = n - 1$, then it is well known that for a form l in $R_0[X]$ prime to $\mathfrak{B}$ i.e. $\mathfrak{B}: l = \mathfrak{B}$, coh.d.$(\mathfrak{B}, l) \cdot S^{-1}R_0[X] = n$. Therefore $(\mathfrak{B}, l) \cdot S^{-1}R_0[X]$ has $(X) \cdot S^{-1}R_0[X]$ as an associated prime ideal. Since $\dim \mathfrak{B} \cdot S^{-1}R_0[X] > 0$, $(\mathfrak{B}, l)S^{-1}R_0[X]$ has an embedded associated prime. On the other hand, it is easy to see that $(X)S^{-1}R_0[X] \cap R_0[X] = (X)R_0[X]$. Therefore it follows from [**5**, Lemma 7c, p. 50] that $(\mathfrak{B}, l)R_0[X]$ has $(X)R_0[X]$ as an embedded associated prime ideal. Let $(\bar{l})R_0[X] = (\mathfrak{B}, l)R_0[X]/\mathfrak{B}$. Therefore $(\bar{l})R_0[x]$ is a principal homogeneous ideal having $(x) \cdot R_0[x]$ as an embedded associated prime ideal. It follows from Theorem 4 that R is not normal.

4. Integral closure of a graded ring. In this section, we study a general graded ring, $R = \bigoplus_{i \geqq 0} R_i$. Let F be the total quotient ring of R, and let $\bar{R}$ be the integral closure of R in F. In case of a graded domain, the integral closure $\bar{R}$ of R in its quotient field K is again graded and $\bar{R}_i = \bar{R} \cap K_i$ for $i \geqq 0$. We investigate $\bar{R}$ when R is not an integral domain. A ring R is normal if $R_{\mathfrak{p}}$ is an integral domain and integrally closed in its quotient field for each $\mathfrak{p} \in \mathrm{Spec}(R)$.

Let $R = \bigoplus_{i \geqq 0} R_i$. Let U be the set of all nonzero divisors of R. Let F be the total quotient ring and let $F_i = \{r_l/u_j \mid r_l \in R_l, u_j \in R_j \cap U, l - j = i\}$. These are the notations going to be used in the sequel.

THEOREM 6. *Assume* $U \cap R_1 \neq \emptyset$ *and let* $u_1 \in U \cap R_1$. *Then* (1) *the ring* $\Sigma_{i \in Z} F_i$ *is a direct sum, and* $\bigoplus_{i \in Z} F_i = F_0[u_1, 1/u_1]$, $F = F_0[u_1]_U$, u_1 *is algebraically independent over* F_0, *and* $F_i = F_0 \cdot u_1^i$ *for all* $i \in Z$. *If* F_0 *is Noetherian then so is* F. (2) F_0 *is reduced, i.e.* F_0 *has no nonzero nilpotent element, if and only if* R *is reduced.* (3) *If* R *is reduced and* F_0 *is*

Noetherian, then $F_0[u_1]$ *is integrally closed in F.* (4) *If R is reduced and* F_0 *is Noetherian, then* $\bar{R}$ *is a graded subring of* $\bigoplus_{i\in Z} F_i$.

Proof. (1) It follows from the definition of F_i's that each F_i is an additive group and $F_i \cdot F_j \subset F_{i+j}$. $\Sigma_{i\in Z} F_i$ is a ring. Let $f_k + \cdots + f_s \in \Sigma_{i\in Z} F_i$. Suppose $f_k + \cdots + f_s = 0$. Let $f_m = r_{l_m}/u_{j_m}$ where $l_m - j_m = m$ and $m = k, \cdots, s$. Let $u = \Pi_{m=k}^{s} u_{j_m}$. Then $uf_k + \cdots + uf_s = 0$ in R, and $uf_k, \cdots, uf_s$ are homogeneous elements of distinct degrees. Therefore $uf_k = \cdots = uf_s = 0$. Thus $f_k = \cdots = f_s = 0$, and the sum ΣF_i is therefore a direct sum. Let $f_k \in F_k$. Then $f_k/u_1^k \in F_0$. Therefore $f_k \in F_0 \cdot u_1^k$ and $F_k = F_0 \cdot u_1^k$. Hence $\bigoplus_{i\in Z} F_i = F_0[u_1, 1/u_1]$. For any $f \in F$,

$$f = (f_k + \cdots + f_s)/u = \frac{1}{u}\left(\frac{f_k}{u_1^k} u_1^k + \cdots + \frac{f_s}{u_1^s} u_1^s\right).$$

Therefore $F = F_0[u_1, 1/u_1]_U = F_0[u_1]_U$. u_1 is algebraically independent over F_0. Indeed, let $a_0 u_1^n + a_1 u_1^{n-1} + \cdots + a_n = 0$, where $a_i \in F_0$ and $a_0 \neq 0$. Writing $a_i = r_{l_i}/u_{j_i}$ with $l_i - j_i = i$, we have $a_i u_1^{n-1} \in F_{n-i}$. Therefore $a_i u_1^{n-1} = 0$, and $a_i = 0$ for $i = 0, 1, \cdots, n$. Therefore u_1 is algebraically independent over F_0.

If F_0 is Noetherian, then so is $F_0[u_1]$. Now $F = F_0[u_1]_U$. Therefore F is also Noetherian.

(2) It is obvious that R is reduced implies that F_0 is reduced. Conversely, we note if $(x_m/u_1^m)^n = 0$, then $x_m = 0$. Also if $y_m \in R_m$ such that $y_m^n = 0$ then $(y_m/u_1^m) = 0$. Thus $y_m = 0$. Now let y be a nilpotent element in R. Write $y = y_k + \cdots + y_s$. For some positive integer b, $y^b = (y_k + \cdots + y_s)^b = 0$. Thus $y_k^b = 0$ and then $(y_{k+1} + \cdots + y_s)^b = 0$ and so on we get $y_m^b = y_{m+1}^b = \cdots = y_s^b = 0$, so $y_m = \cdots = y_s = 0$. Therefore $y = 0$ and R is reduced.

(3) F_0 is reduced. It follows from that $F = F_0[u_1]_U$ and that u_1 is transcendental over F_0, the nonzero divisors of F_0 are the same as the nonzero divisors of R in F_0. Let U_0 be the set of all nonzero divisors of F_0. Let $u_0 \in U_0$, then $u_0 = r_m/u_m$ where $u_m \in U$ and $r_m \in R_m$. Moreover $r_m \in U$ also. Thus u_0 is a unit i.e. U_0 is a multiplicative group in F_0. Hence the total quotient ring $(F_0)_{U_0} = F_0$. Since F_0 is Noetherian and reduced, therefore, $F_0 = \bigoplus_{i=1}^{s} G_i$ where G_i's are fields. It follows from [**2**; Proposition (6.5.2), p. 146] that F_0 is normal.

It follows from [**5**; Proposition (1.7.8), p. 116] that $F_0[u_1]$ is normal. Since $F_0[u_1]$ is a polynomial ring in u_1, and F_0 is reduced, therefore $F_0[u_1]$ is also reduced. F_0 is Noetherian implies that F is Noetherian. Then $F = \bigoplus_{i=1}^{n} H_i$ where H_i's are fields. Thus it follows from [**2**; Proposition (6.5.2), p. 146] that $F_0[u_1]$ is integrally closed.

Note: Let $A = Z/(4)[X]$, the polynomial ring in X over $Z/(4)$. $Z/(4)$ is integrally closed, while A is not. Indeed, let $y = (x+1)/(x-1)$, $y^2 - 1 = 0$, $y \notin A$.

(4) Let $x \in \bar{R}$. Since $R \subset R_0[u_1]$, x is integral over $F_0[u_1]$. By (3), $\bar{R} \subset F_0[u_1]$. The rest of the proof is practically the same argument used in the proof of [**10**; Theorem 11, p. 157]. We summarize the proof: Let $x \in \bar{R}$, $x = x_k + \cdots + x_s$, $k \leqq s$, $x_k \neq 0$ is called the initial homogeneous term. We want to show that each x_i, $i = k, \cdots, s$, is integral over R also. Since $x \in \bar{R} \subset \Sigma F_i$, there exists $u_m \in R_m \cap U$ for some positive integer m, such that $u_m x \in R$. Case (a), if R is Noetherian, then $R[x]$ is a finite R-module. There exists an integer $\lambda > 0$ such that $u_m^{\lambda} x^i \in R$ for all integer $i \geqq 0$. Let $d = u_m^{\lambda}$. Then $dR[x] \subset R$. The initial homogeneous term dx^i is dx_k^i. $dx^i \in R$ implies $dx_k^i \in R$. Therefore $x_k^i \in (1/d)R$, a Noetherian R-module. Therefore $R[x_k] \subset R \cdot 1/d$ is a Noetherian R-submodule. Therefore x_k is integral over R. Repeating that argument to $x - x_k = x_{k+1} + \cdots + x_s$, we conclude that $x_i \in \bar{R}$ for $i = k, \cdots, s$. Therefore $\bar{R}$ is graded in this case. Next we look at case (b): R is not Noetherian. Let $x \in \bar{R}$, and $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ where $a_1, \cdots, a_n \in R$. As in case (a), there is a homogeneous nonzero divisor $d \in R$ such that $dx_k^i \in R$. Let $\{y_1, \cdots, y_{\mathcal{N}}\} = \{d, dx_k$, and homogeneous components of a_i's$\}$. Let $A = k[y_1, \cdots, y_{\mathcal{N}}]$, where $k = Z$ or $Z/(n)$ according to whether R is of characteristic 0 or $n > 0$. $A \subset R$. Let $A_q = A \cap R_q$. Then $A = \Sigma A_a$ is a graded subring of R. $U \cap A$ contains d. Therefore $A_{U \cap A}$, the total quotient ring of A, contains x_k, and hence contains x also. Thus the above integral relation takes place in $A_{U \cap A}$. Since A is Noetherian, therefore case (a) is applicable. Therefore x_k is integral over A. hence x_k is integral over R.

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