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ON THE CONSTRUCTION OF ONE-PARAMETER SEMIGROUPS IN TOPOLOGICAL SEMIGROUPS

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Let S be a topological Hausdorff semigroup and $s \in S$ be a strongly root compact element. Then there are an algebraic morphism $f: Q_+ \cup \{0\} \to S$ with $f(0) = e$, $f(1) = s$, and a one-parameter semigroup $\phi: H \to S$ which satisfy the following properties: If $K = \cap \{f(\,]0, \varepsilon[_Q): 0 < \varepsilon < 1\}$, then K is a compact connected abelian subgroup of $\mathscr{H}(e)$, $\phi(0) = e$, $\phi(H)$ is in the centralizer $Z = \{x \in eSe : xk = kx$ for all $k \in K\}$ of K in eSe, and $\phi(t) \in f(t)K$ for each $t \in Q_+$. Furthermore, if $\mathscr{U}$ is any neighborhood of s in S, then ϕ may be chosen so that $\phi(1) \in \mathscr{U}$: and, in fact, if K is arcwise connected, then ϕ may be chosen so that $\phi(1) = s$. The above statements also hold for strongly pth root compact elements almost everywhere.

1. Introduction. We are concerned with the question of when a divisible element in a topological semigroup can be embedded in a one-parameter semigroup which has many applications in Probability theory (cf. [4], [8]).

The first result about the existence of one-parameter semigroups in a compact semigroup which we call the One-Parameter Semigroup Theorem is due to Mostert and Shields [7], 1957. In 1960, an independent proof based on the local nature of the compact semigroup was given by Hoffmann (cf. [5], [6]). In 1970, a global proof was presented by Carruth and Lawson [1]. The first result of a generalized one-parameter semigroup theorem dealing with the embedding problems which we will call the Embedding and Density Theorem is indicated by Hofmann in [4] and later proved by Siebert [8]. Siebert's proof is based on the notion of a local semigroup called ducleus (cf. [6]). We will present in this paper a global proof of this theorem by applying the One-Parameter Semigroup Theorem.

Throughout this paper, we maintain that R_+, Q_+ and Z_+ are the totalities of strictly positive real numbers, rational numbers and integers, respectively, $H = R_+ \cup \{0\}$ and $Q_+^p = \{n/p^m : n \in Z_+, m \in Z_+ \cup \{0\}\}$ for a prime p. For convenience, we will use $]a, b]_Q$ (resp. $]a, b[_Q$, etc.) and $]a, b]_{Q^p}$ (resp. $]a, b[_{Q^p}$) to denote $]a, b] \cap Q_+$ (resp. $]a, b[\, \cap Q_+$, etc.) and $]a, b] \cap Q_+^p$ (resp. $]a, b[\, \cap Q_+^p$) respectively. We also maintain that S is a topological (Hausdorff) semigroup and $\mathscr{H}(e)$ is the maximal group of units in the closed subsemigroup eSe for an idempotent $e \in S$.

2. On the existence of a one-parameter semigroup in $\overline{f(A)}$ where $f: A \to S$ is an algebraic morphism with $A = Q_+$, Q_+^p. Throughout this section, we will always assume that $f: Q_+$ (resp. Q_+^p) $\to S$ is an algebraic morphism so that $\overline{f(]0, d]_Q)}$ (resp. $\overline{f(]0, d]_{Q^p})}$) is compact for some $d > 0$ unless mentioned otherwise. As the discussions for Q_+ and for Q_+^p would be almost the same, we will concentrate on Q_+ only.

DEFINITION. For each $s \in S$ and each $n \geqq 1$, let $W_n(s) = \{t \in S: t^n = s\}$, $W(n; s) = \{t^m: 1 \leqq m \leqq n, t^n = s\}$. s is said to be divisible (resp. p-divisible) if $W_n(s) \neq \varnothing$ (resp. $W_{p^n}(s) \neq \varnothing$) for all $n \geqq 1$; root compact (resp. pth root compact) if $W_n(s)$ (resp. $W_{p^n}(s)$) is in addition compact for each $n \geqq 1$; strongly root compact (resp. strongly pth root compact) if $W_\infty(s) = \cup \{W(n; s): n \geqq 1\}$ (resp. $W_{p^\infty}(s) = \cup \{W(p^n; s): n \geqq 1\}$) is in addition relatively compact.

PROPOSITION 2.1. *Let s be a root compact* (resp. pth root compact) *element in S. Then there is an algebraic morphism $f: Q_+$* (resp. *Q_+^p*) *$\to S$ so that $f(1) = s$. If s is strongly root compact* (resp. strongly pth root compact), *then f may be chosen so that $\overline{f(]0, 1]_Q)}$* (resp. $\overline{f(]0, 1]_{Q^p})}$) *is compact.*

Proof. For each $n \geqq 1$ and $i \geqq 0$, pick an $s_{n+i} \in W_{(n+i)!}(s)$ (resp. $s_{n+i} \in W_{p^{(n+i)}}(s)$) and let

$$a_n = (s_n^{n!}, s_n^{n!/2!}, \cdots, s_n, s_{n+1}, \cdots)$$

$$(\text{resp. } a_n = (s_n^{p^n}, s^{p^{n-1}}, \cdots, s_n, s_{n+1}, \cdots)) .$$

Then $\{a_n\}$ is a sequence in the compact set $\prod_{n \geqq 1} W_{n!}(s)$ (resp. $\prod_{n \geqq 1} W_{p^n}(s)$). Hence there is a convergent subnet $\{a_{n(k)}\}$ converging to $a = (t_1, t_2, \cdots) \in \prod_{n \geqq 1} W_{n!}(s)$ (resp. $\prod_{n \geqq 1} W_{p^n}(s)$).

Then

$$\begin{aligned} t_{q+1}^{q+1} &= (\lim s_{n(k)}^{n(k)!/(q+1)!})^{q+1} \\ &= \lim s_{n(k)}^{n(k)!/q!} = t_q \end{aligned}$$

$$\begin{aligned} (\text{resp. } t_{q+1}^{p} &= (\lim s_{n(k)}^{p^{n(k)-q}})^p \\ &= \lim s_{n(k)}^{p^{n(k)-q+1}} = t_q) \end{aligned}$$

for all $q \geqq 1$, and $t_1 = s$. If $n/m! = b/a!$ (resp. $n/p^m = b/p^a$), then

$$t_m^n = (t_m^{m!/a!})^b = t_b^a$$

$$(\text{resp. } t_m^n = (t_m^{p^{m-a}})^b = t_a^b) .$$

Hence $f: Q_+$ (resp. Q_+^p) $\to S$ given by $f(n/m!) = t_m^n$ (resp. $f(n/p^m) = t_m^n$)

is well-defined. If $n/m!$, $b/a! \in Q_+$ (resp. n/p^m, $b/p^a \in Q_+^p$), assuming $a \geqq m$, then

$$f(n/m! + b/a!) = f\left(\frac{n(a!/m!) + b}{a!}\right) = t_a^{n(a!/m!)} t_a^b = t_m^n t_a^b$$

$$\text{resp. } f(n/p^m + b/p^a) = f\left(\frac{np^{a-m} + b}{p^a}\right) = t_a^{np^{a-m}} t_a^b = t_m^n t_a^b) \; ,$$

whence f is an algebraic morphism so that $f(1) = s$. The rest is simple.

LEMMA 2.2. *for each $x > 0$, let $S(x) = \overline{f(]0, x[_Q)}$. Then*

(1) *$S(x + y) = S(x)S(y)$ for all $x, y > 0$. In particular, $S(x)$ is compact for each $x > 0$*

(2) *$\overline{f(Q_+)}$ has the identity e so that $K = \cap \{S(x): x \in Q_+\}$ is a divisible compact abelian subgroup of $\mathscr{H}(e)$. In particular, we may extend f to $Q_+ \cup \{0\}$ so that $f(0) = e$*

(3) *$\overline{Kf([x, y[_Q)} = \overline{f([x, y[_Q)}$ for all $x < y \in Q_+$.*

Proof. Straightforward (cf. § 3, Chapter B, [6]).

LEMMA 2.3. *The following statements are equivalent:*

(1) *$K = \{f(0)\}$*

(2) *f is continuous at 0*

(3) *f is continuous.*

Proof. (cf. 3.9, p. 102, [6].)

LEMMA 2.4. *If f is continuous, then there is a unique one-parameter semigroup ϕ so that $\phi \mid (Q_+ \cup \{0\}) = f$.*

Proof. Given a $d > 0$, there is a net $\{x_\alpha\}$ in $]0, d + 1[_Q$ with $\lim x_\alpha = d$. Since $\{(f(x_\alpha)\}$ is a net in $S(d + 1)$, there is a convergent subnet $\{fx_\beta)\}$. Define $F(d) = \lim f(x_\beta)$. It is straightforward to check that $F: \boldsymbol{H} \to S$ is a well defined morphism so that $\cup \overline{\{F(]0, x[)}: x > 0\} = \{f(0)\}$, whence F is continuous (cf. 3.9, p. 102, [6]).

LEMMA 2.5. *Let $\phi: \boldsymbol{H} \to S$ be a nontrivial one-parameter semigroup. Then there is a $d \in]0, 1]$ so that $\phi \mid [0, d]$ is injective. Moreover, if $c > 0$, one may reparameterize ϕ so that $\phi \mid [0, c]$ is injective* (cf. 3.9, p. 102, [6]).

Since K acts on $\overline{f(Q_+)}$ and $\overline{f([x, y[_Q)}$, one has the orbit spaces $\overline{f(Q_+)}/K$ and $\overline{f([v, y[_Q)}/K$. We will use the same letter π to denote the orbit maps.

LEMMA 2.6. *$\overline{f(Q_+)}/K$ is a topological monoid under the multiplication $xK \cdot yK = xyK$.*

LEMMA 2.7. *If $f(Q_+) \not\subset K$, then $\pi \circ f \colon Q_+ \cup \{0\} \to \overline{f(Q_+)}/K$ is nontrivial continuous morphism so that $\pi(\overline{f([x, y[_Q)}) = \overline{f([x, y[_Q)}/K$ for all $x < y \in Q_+ \cup \{0\}$.*

Proof. The continuity of $\pi \circ f$ follows from 2.3. The rest follows from the closedness of π.

In the remainder of this section, we maintain that $f(1) \notin K$ and so $\pi \circ f$ extends to a unique one-parameter semigroup $g \colon \boldsymbol{H} \to \overline{f(Q_+)}/K$ that $g \mid [0, 2]$ is injective by a suitable reparameterization of g or f, i.e. the following diagram commutes:

$$\begin{array}{ccc}]0, 2[_Q & \xrightarrow{\ f\ } & S(2) \\ \cap\downarrow & & \downarrow \pi \\ [0, 2] & \xrightarrow[g]{} & S(2)/K\,. \end{array}$$

Let $\rho = g^{-1} \circ \pi \colon S(2) \to [0, 2]$. Then ρ is a continuous map such that

$$\rho(f(r)) = (g^{-1} \circ \pi)(f(r)) = r \quad \text{for all} \quad r \in [0, 2]_Q$$

and that the following condition is satisfies:

$$\rho(xy) = \rho(x) + \rho(y) \quad \text{for all} \quad x, y \in S(1)\,.$$

LEMMA 2.8. *The following statements hold:*

(1) $x \in Kf(r)$ *iff* $x \in \pi^{-1}(g(r))$ *for each* $r \in Q_+ \cup \{0\}$

(2) $x \in S(2)$ *iff there is a unique* $t \in [0, 2]$ *so that* $x \in \pi^{-1}(g(t))$

(3) $\pi^{-1}(g([x, y])) = K\overline{f([x, y[_Q)} = \overline{f(]x, y[_Q)}$ *for all* $x, y \in Q_+ \cup \{0\}$

(4) $S(1)Kf(1) \subset K\overline{f([1, 2[_Q)}$

(5) $S(1) \backslash Kf(1) = S(2) \backslash K\overline{f([1, 2[_Q)}$.

Proof. Straightforward.

Define a multiplication on the space X obtained from $S(1)$ by collapsing $Kf(1)$ to a point as follows:

$$m_R(x, y) = \begin{cases} xy & \text{if} \quad x, y, xy \in S(1) \backslash Kf(1) \\ Kf(1) & \text{otherwise.} \end{cases}$$

Let $\pi' : S(2) \to X$ be defined via

$$\pi' \mid S(1) \backslash Kf(1) = \pi \mid S(2) \backslash \overline{Kf([1, 2[_Q)} \quad \text{and}$$
$$\pi'(\overline{Kf([1, 2[_Q)}) = \{Kf(1)\} \; ;$$

then

$$\begin{array}{ccc} S(1) \times S(1) & \xrightarrow{m} & S(2) \\ {\scriptstyle \pi' \times \pi'} \downarrow & & \downarrow {\scriptstyle \pi'} \\ X \times X & \xrightarrow[m_R]{} & X \end{array}$$

commutes, hence m_R is a global multiplication on X.

LEMMA 2.9. *X is a compact abelian monoid in the quotient topology.*

Proof. Since π' is a closed map, m_R is continuous.

Let $[0, 1]_*$ denote the space $[0, 1]$ equipped with the multiplication $x + y = \min\{1, x + y\}$. Then $[0, 1]_*$ is a compact monoid in the usual topology. In particular, we have the following factorization:

$$\begin{array}{ccc} S(2) & \xrightarrow{\rho} & [0, 2] \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \tau} \\ X & \xrightarrow[\rho_R]{} & [0, 1]_* = H/[1, \infty] \, , \end{array}$$

where $\tau : \boldsymbol{H} \to [0, 1]_*$ is the canonical map and $\rho_R : X \to [0, 1]_*$ is the unique continuous morphism making the diagram commute.

LEMMA 2.10. *The following statements hold:*
(1) *X has exactly two idempotents e and $0 \equiv Kf(1)$*
(2) *K is the maximal group of units in X*
(3) *K is not open in X*
(4) *$X \backslash \{0\}$ is isomorphic to $S(1) \backslash Kf(1)$.*

Proof. (1) and (4) are clear. (2): We have $X \backslash K = \rho_R^{-1}(]0, 1])$ which is an ideal. Thus K is maximal. (3): If K were open, then $X \backslash K$ would be closed, hence compact, and thus $\rho_R(X \backslash K) =]0, 1]$ would be compact which is not the case.

PROPOSITION 2.11. *There is a continuous morphism $\phi_* : [0, 1]_* \to X$ so that $\phi_*(0) = e$ and $\phi_*^{-1}(\{0\}) = \{1\}$.*

Proof. By 2.10 we can apply the One-Parameter Semigroup Theorem (Thm. 1, p. 510, [7]; [1]) to obtain ϕ_*.

PROPOSITION 2.12. *$\rho_R \circ \phi_*$ is the identity map on $[0, 1]_*$.*

Proof. We observe first that $\rho_R \circ \phi_*$ is an endomorphism α of $[0, 1]_*$ with $\alpha^{-1}(\{1\}) = \{1\}$ and is therefore the identity.

PROPOSITION 2.13. *There is a one-parameter semigroup $\phi: \boldsymbol{H} \to S$ such that $\phi(r) \in Kf(r)$ for all $r \in Q_+$.*

Proof. For all $r \in [0, 1[_Q$, $r = \rho_R \circ \phi_*(r) = \rho \circ \phi_*(r)$ and so $\phi_*(r) \in \rho^{-1}(r) = Kf(r)$. Let ϕ be the unique lifting of ϕ_* to $\boldsymbol{H}$. Then $\phi(r) \in Kf(r)$ for all $r \in Q_+$.

3. On the Embedding and Density Theorem.

PROPOSITION 3.1. *Let G be a locally compact abelian group and $LG = \mathrm{Hom}\,(R, G)$ the totality of one-parameter subgroups in G. If $\exp: LG \to G$ denotes the map $\exp(f) = f(1)$, then*

(1) $\overline{\exp(GL)} = G_0$, *where G_0 is the identity component of G*

(2) $\exp(LG) = G_0$ *iff G_0 is arcwise connected.*

Proof. (1) (25.20, p. 410, [3]). (2) (Thm. 1, p. 40, [2]).

EMBEDDING AND DENSITY THEOREM 3.2. *Let s be strongly root compact in S. Then there are an algebraic morphism $f: Q_+ \cup \{0\} \to S$ with $f(0) = e$, $f(1) = s$, and a one-parameter semigroup $\phi: \boldsymbol{H} - S$ which satisfy the following properties: If $K = \cap\{\overline{f(]0, \varepsilon[_Q)}: 0 < \varepsilon < 1\}$, then K is a compact connected abelian subgroup of $\mathscr{H}(e)$, $\phi(0) = e$, $\phi(\boldsymbol{H})$ is in the centralizer $Z = \{x \in eSe: xk = kx$ for all $k \in K\}$ of K in eSe, and $\phi(t) \in Kf(t)$ for each $t \in Q_+$.*

Furthermore, if $\mathscr{U}$ is any neighborhood of s in S, then ϕ may be chosen so that $\phi(1) \in \mathscr{U}$; and, in fact, if K is arcwise connected, then ϕ may be chosen so that $\phi(1) = s$.

Proof. By 2.1, there is an algebraic morphism $f: Q_+ \cup \{0\} \to S$ such that $f(0) = e$, $f(1) = s$, $\overline{f(]0, 1]_Q)}$ is compact, $K \subset \mathscr{H}(e)$ is a compact connected abelian subgroup and $\overline{f(Q_+)} \subset eSe$.

If $s \in K$, then by 3.1 the assertion is true. If $s \notin K$, then by 2.13 there is a one-parameter semigroup $\phi: \boldsymbol{H} \to S$ so that $\phi(\boldsymbol{H}) \subset \overline{f(Q_+)} \subset eSe$ and $\phi(r) \in Kf(r)$ for all $r \in Q_+ \cup \{0\}$. In particular, $\phi(\boldsymbol{H})$ is in the centralizer of K in eSe. Let $\mathscr{U}$ be a neighborhood of s in S; then there is a neighborhood U of e in K so that $sU \subset \mathscr{U}$. Pick a $k \in K$ so that $\phi(1) = sk$, by the fact that $\overline{\exp(LK)} = K$, there is an $\psi \in LK$ so that $\psi(1) \in Uk^{-1}$. Let $\phi_1: \boldsymbol{H} \to S$ be defined via $\phi_1(r) = \phi(r)\psi(r)$. As $\phi(\boldsymbol{H})$ is in the centralizer of K in eSe, then ϕ_1 is a well-defined one-parameter semigroup so that

$$\phi_1(1) = \phi(1)\psi(1) \in skUk^{-1} = sU .$$

It is easy to check that ϕ_1 also satisfies the same properties as stated above. If K is arcwise connected, by 3.1 ψ may be chosen so that $\psi(1) = k^{-1}$ and so $\phi_1(1) = s$.

COROLLARY 3.3. *If K is a Lie group, then there is a one-parameter semigroup ϕ so that $\phi(1) = s$* (cf. Thm. 7, p. 141, [9]).

THEOREM 3.4. *Let s be a strongly pth root compact element in S. Then there are an algebraic morphism $f: Q_+^p \cup \{0\} \to S$ with $f(0) = e$, $f(1) = s$, and a one-parameter semigroup $\phi: \boldsymbol{H} \to S$ which satisfy the following properties: If $K_p = \cap \{\overline{f(]0, \varepsilon[_{Q^p})}: 0 < \varepsilon < 1\}$, then K_p is a p-divisible compact abelian subgroup of $\mathscr{H}(e)$, $\phi(0) = e$, $\phi(\boldsymbol{H})$ is in the centralizer Z of K_p in eSe, and $\phi(r) \in K_p f(r)$ for all $r \in Q_+^p$.*

REMARK. K_p is in general not divisible (cf. p. 265, [5]; p. 117, [6]).

PROPOSITION 3.5. *Let s be a strongly root compact* (resp. strongly pth root compact) *element in S and f and ϕ be as stated in* 3.2 (resp. 3.4). *Then there is an algebraic morphic morphism $h: Q_+ \to K$* (resp. $h: Q_+^p \to K_p$) so that *$\phi(r) = f(r)h(r)$ for all $r \in Q_+$* (resp. Q_+^p).

Proof. For each $n \geqq 1$, let $A_{n!} = \{x \in K: f(1/n!)x = \phi(1/n!)\}$ (resp. $B(p; n) = \{x \in K_p: f(1/p^n)x = \phi(1/p^n)\}$). Clearly, $A_{n!}$ (resp. $B(p; n)$ is a nonempty compact subset for each $n \geqq 1$. The construction of h then follows as in 2.1.

The following example shows that there are elements which are not strongly root compact but which are neverthless embeddable in one-parameter semigroups:

EXAMPLE 3.5. Let $S = SL(2; R)$ and $s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$: then s is divisible and $W_2(s) \supset \left\{\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}: yz = -1\right\}$ is not compact, whence s is not even 2th root compact. But the map $f: R \to S$ defined via

$$f(t) = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}$$

is a one-parameter subgroup so that $f(1) = s$.

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REFERENCES

1. J. H. Carruth and J. D. Lawson, *On the existence of one-parameter semigroups*, Semigroup Forum, **1** (1970), 85-90.

2. J. Dixmier, *Quelques propriétés des groupes abélian localement compacts*, Bull. Sci. Math. 2^e série, **85** (1957), 38-48.

3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin-Heidelberg-New York, (1963/70).

4. H. Heyer, *Infinitely divisible probability measures on compact groups*, Lecture Notes in Mathematics 247, Springer-Verlag, Heidelberg (1973), 55-247.

5. K. H. Hofmann, *Topologisches Halbgruppen mit dichter submonogener Unterhalbgruppe*, Math. Zeitschrift, **74** (1960), 232-276.

6. K. H. Hofmann and P. S. Mostert, *Elements of Compact Semigroups*, Charless E. Merrill, Columbus, Ohio (1966).

7. P. S. Mostert and A. L. Shields, *One-parameter semigroups in a semigroups*, Trans. Amer. Math. Soc., **96** (1960), 510-517.

8. E. Seibert, *Einbettung unendlich teilbarer Wahrscheinlichkeitsmasse auf topologischen Gruppen*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **28** (1974), 227-247.

9. Ph. Tondeur, *Introduction to Lie Groups and Transformation Groups*, Second Edition, Springer-Verlag. Berlin (1969).

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