PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 1, 1976

A HAHN DECOMPOSITION FOR LINEAR MAPS

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The question is studied whether every bounded self-adjoint linear map φ be tween two C^*-algebras can be written as the difference of bounded positive linear maps. Such a decomposition is called a *Hahn decomposition* of φ.

THEOREM. *Let X be an infinite compact Hausdorff space. Then there is a bounded, self-adjoint linear map, with domain $C(X)$, that does not admit a Hahn decomposition.*

A bounded linear map φ is said to have finite total variation if

$$\sup\left\{\left\|\sum_{i=1}^{n}|\varphi(a_i)|\right\|: a_i\in\mathscr{A},\ 0\leqq a_i,\ \sum a_i\leqq 1\right\}<\infty\ .$$

THEOREM. *If the domain is commutative, and if the range is a von Neumann algebra, then a sufficient condition for a self-adjoint map to admit a Hahn decomposition is that the map have finite total variation.*

0. Introduction. It is a well-known theorem [4] that every linear functional τ on a C^*-algebra $\mathscr{A}$ can be written $\tau=\tau_1-\tau_2+i(\tau_3-\tau_4)$, where the τ_j are positive linear functionals. It is, therefore, natural to ask whether every bounded linear map φ between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ admits a decomposition $\varphi=\varphi_1-\varphi_2+i(\varphi_3-\varphi_4)$, where the φ_j are positive linear maps.

Given any bounded linear map φ, if we define the linear map $\tilde{\varphi}$ by $\tilde{\varphi}(a)=\varphi(a^*)^*$, it is easy to see that $\|\tilde{\varphi}\|=\|\varphi\|$, and that $\tilde{\varphi}$ is the natural "adjoint" map to φ. Hence, the map $\varphi_1=(\varphi+\tilde{\varphi})/2$ is self-adjoint, i.e., $\varphi_1(a^*)=\varphi_1(a)^*$, as is $\varphi_2=(\varphi-\tilde{\varphi})/2i$, and therefore φ can be written (uniquely) as $\varphi=\varphi_1+i\varphi_2$, the usual combination of self-adjoint elements.

We are now reduced to the following problem: Given a bounded, self-adjoint linear map φ between two C^*-algebras, when can we write $\varphi=\varphi_1-\varphi_2$ where φ_1, φ_2 are bounded, positive linear maps?

DEFINITION 0.1. We shall call such a form a *Hahn decomposition* of φ.

In general, a Hahn decomposition is not always possible. Even in the commutative case, pathology can occur [see Theorem 2.2 below].

For future references, we state here Grothendieck's result for functionals on a C^*-algebra [4].

THEOREM 0.2. *Let φ be a bounded self-adjoint functional on a C^*-algebra. Then we can write $\varphi = \varphi^+ - \varphi^-$, with φ^+, φ^- positive, and $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\|$.*

In our terminology, bounded self-adjoint functionals admit a Hahn decomposition.

Most of this material appeared in my Ph. D. thesis, Berkeley, 1973, written under the direction of Professor William B. Arveson. During the course of this research, I benefitted greatly from conversations with Professors Arveson, Oscar Lanford, and Donald Sarason.

1. Preliminaries. In this section, we shall study some simple cases where a Hahn decomposition exists, and some consequences of the decomposition. As usual, M_n denotes the C^*-algebra of $n \times n$ complex matrices.

LEMMA 1.1. *Let $\mathscr{A}$ be a C^*-algebra, let $\varphi: \mathscr{A} \to M_n$ be a bounded linear self-adjoint map. Then we can write $\varphi(x) = \sum_{j=1}^N \rho_j(x)A_j$, where the ρ_j are bounded self-adjoint functionals, $\|\rho_j\| \leqq \|\varphi\|$, and $A_j \in M_n$ satisfy $A_j = A_j^*$, $\|A_j\| \leqq 1$, and A_j^2 is a projection.*

Proof. Let $\{E_{jk}\}$ denote the usual basis of M_n, let $z_1, \cdots, z_n$ be the canonical basis for C^n. Define the functionals φ_{jk} by $\varphi_{jk}(x) = (\varphi(x)z_k, z_j)$. Then $\|\varphi_{jk}\| \leqq \|\varphi\|$, and we have $\varphi = \sum \varphi_{jk}E_{jk}$. Since φ is self-adjoint, it is easy to see that the functionals φ_{jk} satisfy $\varphi_{jk} = \tilde{\varphi}_{kj}$. In particular the φ_{jj} are self-adjoint. Hence

$$\varphi = \sum_{j=1}^n \varphi_{jj}E_{jj} + \sum_{j\neq k} \varphi_{jk}E_{jk} = \sum \varphi_{jj}E_{jj} + \sum_{j\neq k}\Big(\frac{\varphi_{jk}+\varphi_{kj}}{2}\Big)(E_{jk}+E_{kj}) + \sum_{j\neq k}\Big(\frac{\varphi_{jk}-\varphi_{kj}}{2i}\Big)\Big(\frac{E_{kj}-E_{jk}}{i}\Big)$$

which writes φ in the promised manner.

PROPOSITION 1.2. *Let $\mathscr{A}$ be a C^*-algebra, let $\varphi: \mathscr{A} \to M_n$ be a bounded, linear, self-adjoint map. Then φ admits a Hahn decomposition.*

Proof. Write $\varphi = \sum \rho_j A_j$ as in 1.1. Then we can apply Theorem 0.2 to say $\rho_j = \rho_j^+ - \rho_j^-$ and $\|\rho_j\| = \|\rho_j^+\| + \|\rho_j^-\|$. We also write $A_j = A_j^+ - A_j^-$, with $\|A_j\| = \max(\|A_j^+\|, \|A_j^-\|)$, where the A_j^+, A_j^- are positive. Then $\rho_j A_j = (\rho_j^+ A_j^+ + \rho_j^- A_j^-) - (\rho_j^+ A_j^- + \rho_j^- A_j^+)$, is the difference of positive maps, and hence $\varphi = \sum \rho_j A_j$ is the difference of positive maps.

REMARKS 1.3.

(i) We do not get a good bound on $\|\varphi^+\|$, $\|\varphi^-\|$ from this proof. However, we will see later [see Remark 2.3(ii)] that good bounds cannot be obtained without more detailed knowledge of φ.

(ii) We say that a linear map φ between two Banach spaces X and Y is *nuclear* if we can write $\varphi = \sum \rho_i \otimes y_i$, where $\rho_i \in X'$ and $y_i \in Y$, with $\sum \|\rho_i\| \|y_i\| < \infty$, and $(\rho_i \otimes y_j)(x) = \rho_i(x) y_i$. Let $|||\varphi||| = \inf\{\sum \|\rho_i\| \|y_i\| : \varphi = \sum \rho_i \otimes y_i\}$, then if $|||\varphi||| < \infty$, where φ is a bounded linear map of C^*-algebras, the proof of Proposition 1.2 shows that φ admits a Hahn decomposition. Unfortunately, most linear maps are not nuclear.

(iii) For the case of a bounded self-adjoint linear map from $C(X) \rightarrow M_n$, we are able to sketch another proof of the Hahn decomposition which is very tempting to generalize to $\mathscr{L}(H)$. Note that using Lemma 1.5, we see that every bounded linear map of $C(X) \rightarrow M_n$ is completely bounded [Definition 1.4].

Sketch of proof. Let $\varphi: C(X) \rightarrow M_n$ be bounded, linear, self-adjoint. Then $\varphi_{ij}(f) = (\varphi(f)z_j, z_i)$ is a bounded linear functional on $C(X)$, hence there is a bounded Borel measure μ_{ij} such that $\varphi_{ij}(f) = \int f d\mu_{ij}$. Since φ is self-adjoint, we have μ_{ii} is real, and $\mu_{ji} = \overline{\mu_{ij}}$. Let $\mu = 1/n^2 \sum_{i,j=1}^n |\mu_{ij}|$, then μ is positive, $\|\mu\| \leqq \|\varphi\|$, and each μ_{ij} is absolutely continuous with respect to μ. Hence, there exist functions $h_{ij} \in L^1(\mu)$, such that $d\mu_{ij} = h_{ij} d\mu$. Let $H(x) = (h_{ij}(x))$, then $H(x)$ is self-adjoint a.e. $(d\mu)$. By the finite-dimensional spectral theorem, $H(x) = P(x) - Q(x)$, where $P(x)$, $Q(x)$ are positive a.e. $(d\mu)$. Then $\varphi(x) = \int f(x)H(x)d\mu = \int f(x)P(x)d\mu - \int f(x)Q(x)d\mu$ is the difference of positive maps. In fact, $K(x) = \|H(x)\| \in L^1(\mu)$.

DEFINITIONS 1.4. Let $\varphi: \mathscr{A} \rightarrow \mathscr{B}$ be a linear map of C^*-algebras. Let M_n be the $n \times n$ matrices, and let φ_n be the natural map from $\mathscr{A} \otimes M_n \rightarrow \mathscr{B} \otimes M_n$. Then φ is *completely positive* if all φ_n are positive ([7]), and *completely bounded* if $\sup_n \|\varphi_n\| < \infty$ ([1]).

A completely positive map is completely bounded ([1]), and if either $\mathscr{A}$ or $\mathscr{B}$ is commutative, a positive map is completely positive ([1, 7]).

LEMMA 1.5. *Let $\varphi: \mathscr{A} \rightarrow \mathscr{B}$ be a bounded self-adjoint map of*

C^*-algebras. In case either $\mathscr{A}$ or $\mathscr{B}$ is abelian, a necessary condition for φ to admit a Hahn decomposition is that φ be completely bounded.

Proof. Suppose $\varphi = \varphi^+ - \varphi^-$ is a Hahn decomposition. Then both φ^+ and φ^- are completely positive, hence completely bounded. Hence, $\|\varphi_N\| = \|\varphi_N^+ - \varphi_N^-\| \leqq \|\varphi_N^+\| + \|\varphi_N^-\| \leqq M_1 + M_2$, $\forall N$, so φ is completely bounded.

REMARK. It is not essential in Lemma 1.5 that φ be self-adjoint.

2. A counterexample. We now proceed with a modification of an example due to O. E. Lanford, showing that a Hahn decomposition is *not* always possible.

LEMMA 2.1. *Let $n \geqq 1$. Then there exist $A_1, \cdots, A_n \in M_{2^n}$ such that*

(1) $A_i = A_i^*$

(2) $A_iA_j + A_jA_i = 2\delta_{ij}\cdot I_{2^n}$

(3) $\mathrm{Tr}\,(A_i) = 0$.

Proof. Let $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $H^* = H$, $I^* = I$, $J^* = J$, $H^2 = I$, $J^2 = I$, and $HJ + JH = 0$. Now, let

$$A_1 = H \otimes \overbrace{I \otimes \cdots \otimes I}^{n-1}$$

$$A_2 = J \otimes H \otimes \overbrace{I \otimes \cdots \otimes I}^{n-2}$$

$$\vdots$$

$$A_k = \overbrace{J \otimes \cdots \otimes J}^{k-1} \otimes H \otimes \overbrace{I \otimes \cdots \otimes I}^{n-k} \quad \text{for} \quad 3 \leqq k \leqq n\,.$$

Clearly $A_i = A_i^*$, $A_i^2 = I$, and $\mathrm{Tr}\,(A_i) = 0$. For $j > 1$, $A_1A_j + A_jA_1 = 0$ by looking at the first factor. If $1 < i < j \leqq n$, then the i^{th} factor of A_i is H, and the i^{th} factor of A_j is J, so we have $A_iA_j + A_jA_i = 0$.

We remark that the A_i generate a Clifford algebra [6].

Note that if $A \in M_{2^n}$, and $A = \sum \alpha_i A_i$, where A_i are from Lemma 2.1, then $A^*A + AA^* = \sum \bar{\alpha}_j\alpha_i A_jA_i + \sum \alpha_i\bar{\alpha}_j A_iA_j = \sum \alpha_i\alpha_j[A_iA_j + A_jA_i] = \sum \alpha_i\bar{\alpha}_j(2\delta_{ij}I) = 2(\sum |\alpha_i|^2)I$. Thus $\|A\| \leqq \sqrt{2}\sqrt{\sum |\alpha_i|^2}$; that is, the A_i are almost an orthonormal basis for their span.

Now, let X be an infinite compact Hausdorff space, and let $\rho_1, \cdots, \rho_n$ be positive linear functionals on $C(X)$ such that the ρ_i have disjoint closed supports and $\|\rho_i\| = n^{-3/4}$. Let $\varphi^{(n)}: C(X) \to M_{2^n}$ by $\varphi^{(n)}(f) = \sum_{i=1}^n \rho_i(f)A_i$. Then $\|\varphi^{(n)}(f)\| \leqq \sqrt{2}\sqrt{\sum |\rho_i(f)|^2} \leqq$

$\sqrt{2}\sqrt{\sum(n^{-3/4}\|f\|)^2} = \sqrt{2}\,\|f\|\sqrt{\sum n^{-3/2}} = \sqrt{2}\|f\|\,n^{-1/4} \leqq \sqrt{2}\,\|f\|$. Note also that $\varphi^{(n)}$ is a self-adjoint linear map.

THEOREM 2.2. *Let X be a compact Hausdorff space such that* $\operatorname{card}(X) = \infty$. *Then there is a bounded, self-adjoint linear map φ from $C(X)$ into the compact operators on a separable Hilbert space such that φ does not admit a Hahn decomposition.*

Proof. Since $\operatorname{card}(X) = \infty$, for every integer $n \geqq 1$, we can find positive linear functionals $\rho_1^{(n)}, \cdots, \rho_n^{(n)}$ such that $\|\rho_i^{(n)}\| = n^{-3/4}$, and all the $\rho_i^{(n)}$ have disjoint closed supports, for $1 \leqq i \leqq n$, $1 \leqq n$. Let $\mathscr{H}_n = \boldsymbol{C}^{2^n}$, and let $\mathscr{H} = \bigoplus_n \mathscr{H}_n$. Let $\varphi\colon C(X) \to \mathscr{L}(H)$ by $\varphi(f) = \bigoplus \varphi^{(n)}(f)$, where $\varphi^{(n)}(f)$ is as above.

Then $\|\varphi\| = \sup_n \|\varphi^{(n)}\| \leqq \sqrt{2}$. Let $\tilde{\varphi}^{(N)} = \bigoplus_{n=1}^{N} \varphi^{(n)}$, then $\tilde{\varphi}^{(N)}(f)$ has finite rank for all $f \in C(X)$, and

$$\|\varphi - \tilde{\varphi}^{(N)}\| = \sup_{n>N} \|\varphi^{(n)}\| \leqq \sup_{n>N} \sqrt{2}\,n^{-1/4} = \sqrt{2}\,(N+1)^{-1/4} \to 0$$

as $N \to \infty$. Hence $\varphi(f)$ is compact for all $f \in C(X)$.

We shall give two proofs that φ does not admit a Hahn decomposition. In the first proof, we shall show φ is not completely bounded, so that Lemma 1.5 implies φ does not admit a Hahn decomposition.

Let $\varphi_k = \varphi \otimes \mathrm{id}_k\colon C(X) \otimes M_k \to \mathscr{L}(H) \otimes M_k$; then it is easy to see that $\varphi_k = \bigoplus_n (\varphi^{(n)} \otimes \mathrm{id}_k)$, so $\|\varphi_k\| = \sup_n \|\varphi_k^{(n)}\|$: We will show that $\|\varphi_{2^n}^{(n)}\| \geqq n^{1/4}$, hence $\|\varphi_{2^n}\| \geqq n^{1/4}$, and thus φ is not completely bounded.

For convenience of notation, let $\psi = \varphi^{(n)} = \sum \rho_i A_i$. In $C(X) \otimes M_{2^n} = C(X, M_{2^n})$, consider the matrix $F(x) = (f_{ij}(x))$ such that on the support K_i of ρ_i, $F(x) \equiv A_i$, and such that otherwise $F(x)$ is a convex combination of the A_i. Such an F can be constructed in the following manner: Since the supports K_i of the ρ_i are disjoint, a slight variation of the usual partition of unity argument yields continuous functions $g_1, \cdots, g_n$ such that $0 \leqq g_i \leqq 1$, $\sum g_i \equiv 1$, $g_i \equiv 1$ on K_i; we then let $F(x) = \sum g_i(x) A_i$. Note that $F(x) = \overline{F(x)} \forall x$, and $\|F\| = \sup_x \|F(x)\| = 1$.

We have that $\psi_{2^n}(F) = (\psi \otimes \mathrm{id}_{2^n})(F) = (\sum (\rho_i \otimes A_i) \otimes \mathrm{id}_{2^n})(F) = \sum A_i \otimes (\rho_i \otimes \mathrm{id}_{2^n}(F)) = \sum A_i \otimes \|\rho_i\|\,A_i = \sum \|\rho_i\|\,A_i \otimes A_i$. Since $A_i A_j + A_j A_i = 0$ for $i \neq j$, we see that $A_i A_j = -A_j A_i$ for $i \neq j$, and hence $A_i \otimes A_i$ commutes with $A_j \otimes A_j$! It is clear that $\|\psi_{2^n}(F)\| \leqq \sum \|\rho_i\|\,\|A_i \otimes A_i\| = \sum \|\rho_i\|$. We claim: there is a unit vector $z \in \boldsymbol{C}^{2^n} \otimes \boldsymbol{C}^{2^n}$ such that $(A_i \otimes A_i)(z) = z$, $\forall i$. If so, $\|\psi_2^n(F)\| \geqq |\langle \psi_{2^n}(F)(z), z\rangle| = \sum \|\rho_i\|$, so we actually have $\|\psi_{2^n}(F)\| = \sum \|\rho_i\|$. Thus $\|\psi_{2^n}\| \geqq \|\psi_{2^n}(F)\| = \sum \|\rho_i\| = \sum_{i=1}^{n} n^{-3/4} = n^{1/4}$, as desired. (We showed that $\|\psi_{2^n}\| \geqq \sum \|\rho_i(F)\|$ in order that the role of the actual values of $\|\rho_i\|$ can be seen. In fact, we have really shown that $\|\psi_{2^n}\| = \sum \|\rho_i\|$.)

Now, to prove the claim, we need some observations.

(1) There is a unitary operator $U \in M_{2^n}$ so that $U^*A_iU = -A_i$ for all i. We can use $U = \overbrace{J \otimes \cdots \otimes J}^{n}$, so then $U^* = U$ and $U^*U = UU^* = \overbrace{J^2 \otimes \cdots \otimes J^2}^{n} = \overbrace{I \otimes \cdots \otimes I}^{n} = I$. We then have

$$\begin{aligned} U^*A_iU &= [J \otimes \cdots \otimes J]\Big[\overbrace{J \otimes \cdots \otimes J}^{i-1} \otimes H \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i}\Big][J \otimes \cdots \otimes J] \\ &= \overbrace{J \otimes \cdots \otimes J}^{i-1} \otimes JHJ \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i}, \end{aligned}$$

but $HJ + JH = 0$ means $JHJ = -H$, hence $U^*A_iU = -A_i$.

(2) Let $i_1, \cdots, i_k$ be distinct; then $\operatorname{Tr}(A_{i_1} \cdot A_{i_2} \cdots A_{i_k}) = 0$. For if k is odd,

$$\begin{aligned} \operatorname{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) &= \operatorname{Tr}(U^*A_{i_1}A_{i_2} \cdots A_{i_k}U) \\ &= \operatorname{Tr}(U^*A_{i_1}U \cdot U^*A_{i_2}U \cdots U^*A_{i_k}U) \\ &= \operatorname{Tr}(-A_{i_1} \cdot -A_{i_2} \cdots -A_{i_k}) \\ &= (-1)^k \operatorname{Tr}(A_{i_1} \cdots A_{i_k}), \end{aligned}$$

so $\operatorname{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) = 0$.

If k is even, then

$$\operatorname{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) = \operatorname{Tr}(A_{i_k}A_{i_1}A_{i_2} \cdots A_{i_{k-1}}) = (-1)^{k-1} \operatorname{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k})$$

by 2.1, so again $\operatorname{Tr}(A_{i_1}A_{i_2} \cdots A_{i_k}) = 0$.

(3) Let $S_i = A_i \otimes A_i$, $1 \leqq i \leqq n$. Then the S_i are mutually commuting, self-adjoint, and have square $= I$. This follows easily from Lemma 2.1. Further, the S_i can be simultaneously diagonalized, and it follows that the eigenvalues of the S_i are all $+1$ or -1. Also, $S_i = 2P_i - I$, where the P_i are commuting projections.

Now, to show that the S_i do indeed have a common $+1$ eigenvalue, it suffices to consider the P_i, for

$$\dim(\Pi P_i) = \dim(\text{common } +1 \text{ eigenspace of the } S_i).$$

But

$$\begin{aligned} \dim(\Pi P_i) = \operatorname{Tr}(\Pi P_i) &= \operatorname{Tr}\Big[\Big(\frac{I + S_1}{2}\Big) \cdots \Big(\frac{I + S_n}{2}\Big)\Big] \\ &= 2^{-n} \operatorname{Tr}[(I + S_1) \cdots (I + S_n)] \\ &= 2^{-n} \operatorname{Tr}[I + \text{products of the } S_i], \text{ which by observation (2) is,} \\ &= 2^{-n}[\operatorname{Tr}(I) + 0 \cdots + 0] = 2^{-n}2^{2n} = 2^n. \end{aligned}$$

Thus, the S_i do indeed have many common $+1$ eigenvalues, proving the claim, and completing the argument.

I am indebted to W. Arveson and O. Lanford for simplifications of my original argument.

The following proof of the impossibility of a Hahn decomposition is essentially Lanford's original proof:

Proof. Suppose θ, τ are bounded positive linear maps such that $\varphi = \theta - \tau$. Then $\theta \geqq \varphi$, and hence the compression $\theta_n = P_{\mathscr{H}_n}\theta \mid_{\mathscr{H}_n}$ of θ to each $\mathscr{H}_n$ satisfies $\theta_n \geqq \varphi^{(n)}$. Also, $\|\theta_n\| \leqq \|\theta\|$, since θ_n is a compression of θ.

From Lemma 2.1, we know that $\mathrm{Tr}\,(A_i) = 0$. Let $g_1, \cdots, g_n$ be the functions we previously constructed. Then $\theta_n(g_i) \geqq \varphi^{(n)}(g_i) = \|\rho_i\| A_i$. Choose a basis for C^{2^n} with respect to A_i. Then the matrix representation of $\theta_n(g_i)$ is as a positive matrix, so all its diagonal elements are nonnegative. But $\theta_n(g_1) \geqq \|\rho_i\| A_i$, so $\mathrm{Tr}\,(\theta_n(g_i)) \geqq 2^{n-1}\|\rho_i\| + 2^{n-1}\cdot 0$, where $\|\rho_i\|$ terms arise from $+1$ eigenvalues of A_i, 0's from -1 eigenvalues of A_i (since diagonal elements of $\theta_n(g_i)$ are nonnegative).

Hence $\theta_n(1) = \sum_i \theta_n(g_i)$, so $\mathrm{Tr}\,(\theta_n(1)) = \sum \mathrm{Tr}\,(\theta_n(g_i)) \geqq 2^{n-1} \sum \|\rho_i\|$. Thus $\|\theta_n(1)\| \geqq 1/2^n\, \mathrm{Tr}\,(\theta_n(1)) \geqq 1/2 \sum_1^n \|\rho_i\| = 1/2 \sum n^{-3/4} = (1/2)n^{1/4}$. But θ_n is positive, so $\|\theta_n\| = \|\theta_n(1)\|$ (see [1]) and so $(1/2)n^{1/4} \leqq \|\theta_n\| \leqq \|\theta\|$. This is true for all n, hence θ is unbounded, so φ does not admit a Hahn decomposition.

REMARKS 2.3.

(i) Basically, all that was needed was that the ρ_i had disjoint supports, that $\sum_i^n \|\rho_i\|^2 \leqq k$ (independent of n), and that $\sum_1^n \|\rho_i\| \to \infty$ as $n \to \infty$. It is interesting to note that this same quantity, $\sum \|\rho_i\|$, appears in both arguments.

(ii) We have shown that if $P \geqq 0$, $P \geqq \varphi^{(n)}$, then $\|P\| \geqq (1/2)n^{1/4}$. Hence, although $\|\varphi^{(n)}\| \leqq 2n^{-1/4}$, we see that the positive part of $\varphi^{(n)}$ (from Proposition 1.2) has norm $\geqq (1/2)n^{1/4}$.

The mappings $\varphi^{(n)}$ we have used in this example, have some other interesting properties. Again, for convenience, we let $\psi = \varphi^{(n)}$, so $\psi = \sum_1^n \rho_i A_i$.

Let $\{f_j\}$, $1 \leqq j \leqq J$, be in $C(X)$ such that $\sum |f_j(x)|^2 \leqq 1$ for all $x \in X$. Let $G \in C(X) \otimes M_J$ be

$$G = \begin{pmatrix} f_1 & \\ \vdots & 0 \\ f_J & \end{pmatrix}.$$

Then $\|G^*G\| = \|G\|^2 = \|\sum |f_j(x)|^2\| \leqq 1$. (For ease of notation, we have put the $\{f_j\}$ into the first column, but the following argument is valid as long as the $\{f_j\}$ all lie in the same row or column.)

Consider $(\psi \otimes \mathrm{id}_J)(G)$. Then $\|(\psi \otimes \mathrm{id}_J)(G)\|^2 = \|(\psi \otimes \mathrm{id}_J(G))^* (\psi \otimes \mathrm{id}_j(G))\| = \|sI + \sum_{1 \leqq j < k \leqq n} iZ_{jk}t_{jk}\|$ where (1) $s = \sum_{j,k} |\rho_k(f_j)|^2$; (2) $iZ_{jk} = A_jA_k$, so $Z_{jk} = Z_{jk}^*$, $Z_{jk}^2 = I$ and (2*) $Z_{jk}Z_{lm} + Z_{lm}Z_{jk} = 0$ if one common index, $Z_{jk}Z_{lm} = Z_{lm}Z_{jk}$ if no common index; (3) $t_{jk} = \sum_i [\overline{\rho_j(f_i)}\rho_k(f_i) - \rho_j(f_i)\overline{\rho_k(f_i)}]$.

We claim: (a) $s \leqq \sum \|\rho_k\|^2$; (b) $|t_{jk}| \leqq 2\|\rho_j\|\|\rho_k\|$.

If so, then $\|sI + \sum_{1 \leqq j < k} iZ_{jk}t_{jk}\| \leqq s + \|\sum_{k=2}^n iZ_{1k}t_{1k}\| + \sum_{k=3}^n iZ_{2k}t_{2k}\| + \cdots + \|iZ_{n-1,n}t_{n-1,n}\|$; but the use of (2*) shows that, e.g.,

$$\left\|\sum_{k=2}^n iZ_{1k}t_{1k}\right\| \leqq 2\sqrt{\sum_{k=2}^n |t_{1k}|^2}\ ,$$

so we obtain

$$\left\|sI + \sum_{1 \leqq j < k} iZ_{jk}t_{jk}\right\| \leqq \sum \|\rho_k\|^2 + 2\sqrt{\sum_{k=2}^n |t_{1k}|^2} + 2\sqrt{\sum_{k=3}^n |t_{2k}|^2} + \cdots + 2\sqrt{|t_{n-1,n}|^2}\ .$$

We now notice that $\|\rho_k\| = 1/n^{1-\varepsilon}$, $0 < \varepsilon < 1/2$, is essentially the weakest estimate needed for $\varphi^{(n)}$ to have the desired properties (see Remark 2.3i).

Using Claim (a), we get $s \leqq n(1/n^{2-2\varepsilon}) = n^{2\varepsilon}/n$, and by (b), $|t_{jk}| \leqq 2(n^{2\varepsilon}/n^2)$. Hence

$$\begin{aligned}
&\left\|sI + \sum_{1 \leqq j < k} + iZ_{jk}t_{jn}\right\| \\
&\leqq \frac{n^{2\varepsilon}}{n} + 2\sqrt{\sum_{k=2}^n\left(\frac{2n^{2\varepsilon}}{n^2}\right)^2} + 2\sqrt{\sum_{k=3}^n\left(\frac{2n^{2\varepsilon}}{n^2}\right)^2} + \cdots + 2\sqrt{\left(\frac{2n^{2\varepsilon}}{n^2}\right)^2} \\
&= \frac{n^{2\varepsilon}}{n} + 4\frac{n^{2\varepsilon}}{n^2}(\sqrt{n-1} + \cdots + \sqrt{1}) \\
&\leqq \frac{n^{2\varepsilon}}{n} + 4\frac{n^{2\varepsilon}}{n^2}(n^{3/2}) \leqq 5 \quad \text{for} \quad 0 < \varepsilon \leqq 1/4\ .
\end{aligned}$$

This means that we can choose the maps $\varphi^{(n)}$ so that $\varphi = \bigoplus \varphi^{(n)}$ has range in the compact operators, and such that φ is not completely bounded, hence does not admit a Hahn decomposition, but such that φ is completely row or column bounded, in the sense of the above calculation.

We now return to the proofs of Claims (a) and (b).

Proof of Claim (a). $s = \sum_{i,k} |\rho_k(f_i)|^2 = \sum_k (\sum_i |\rho_k(f_i)|^2)$, which by the generalized Schwarz inequality [3], is

$$\leqq \sum_k \left(\sum_i \| \rho_k \| \rho_k(|f_i|^2)\right) = \sum_k \| \rho_k \| \left(\sum_i \rho_k(|f_i|^2)\right) \leqq \sum_k \| \rho_k \| \rho_k(1) ,$$

since $\sum |f_i|^2 \leqq 1$, but $\| \rho_k \| = \rho_k(1)$.

Proof of Claim (b). $|t_{jk}| \leqq 2 \sum_i |\rho_j(f_i)| \, |\rho_k(f_i)|$; by the Schwarz inequality, $|\rho_j(f_i)| \leqq \sqrt{\| \rho_j \|} \sqrt{\rho_j(|f_i|^2)}$; similarly, $|\rho_k(f_i)| \leqq \sqrt{\| \rho_k \|} \sqrt{\rho_k(|f_i|^2)}$. Hence,

$$\begin{aligned} |t_{jk}| &\leqq 2 \sum_i \sqrt{\| \rho_j \| \, \| \rho_k \|} \sqrt{\rho_j(|f_i|^2) \rho_k(|f_i|^2)} \\ &= 2\sqrt{\| \rho_j \| \, \| \rho_k \|} \sum_i \sqrt{\rho_j(|f_i|^2)} \sqrt{\rho_k(|f_i|)^2} \, . \end{aligned}$$

Let

$$x_i = \sqrt{\sigma_j(|f_i|^2)} \, , \qquad y_i = \sqrt{\rho_k(|f_i|^2)} \, ,$$

then the usual Schwarz inequality shows $\sum_i x_i y_i \leqq (\sum x_i^2)^{1/2} (\sum y_i^2)^{1/2} = (\sum_i \rho_j(|f_i|^2))^{1/2} (\sum_i \rho_k(|f_i|^2))^{1/2} \leqq (\rho_j(1))^{1/2} (\rho_k(1))^{1/2} = \| \rho_j \|^{1/2} \| \rho_k \|^{1/2}$.

3. Finite total variation.

As we have seen, self-adjoint linear mappings of C^*-algebras do not necessarily admit a Hahn decomposition, even in extremely nice cases. However, for the case of self-adjoint linear mappings φ from $C(X)$ into a W^*-algebra, we have been able to obtain a sufficient condition for φ to admit a Hahn decomposition.

DEFINITION 3.1. Let $\varphi: \mathscr{A} \rightarrow \mathscr{B}$ be a bounded linear mapping of C^*-algebras. We say that φ has *finite total variation* if

$$V(\varphi) = \sup \left\{ \left\| \sum_{k=1}^{n} |\varphi(f_k)| \right\| : 0 \leqq f_k \in \mathscr{A}, \sum f_k \leqq 1 \right\} < \infty \, .$$

When $\mathscr{A} = C(X)$, and $\mathscr{B} = \boldsymbol{C}$, then $\varphi(f) = \int f d\mu$, for some bounded Borel measure μ, and in that case $V(\varphi)$ is nothing but the total variation of the measure μ.

A positive map ψ has finite total variation, in fact $V(\psi) = \| \psi \|$, and a scalar multiple of a map with finite total variation also has finite total variation.

We say that a map φ between two (e.g.) C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is nuclear if φ can be written as $\varphi = \sum \rho_i \otimes B_i$, where $\rho_i \in \mathscr{A}'$ (the dual of $\mathscr{A}$), $B_i \in \mathscr{B}$, and $\sum \| \rho_i \| \, \| B_i \| < \infty$. The nuclear norm of φ, $|||\varphi|||$, is then defined by $|||\varphi||| = \inf \{ \sum \| \rho_i \| \, \| B_i \| : \varphi = \sum \rho_i \otimes B_i \}$. It was noted in Remark 1.3ii that if φ is nuclear, then φ admits a Hahn decomposition. The proof of Proposition 1.2 shows that every bounded (self-adjoint) map from $\mathscr{A} \to M_n$ is nuclear.

PROPOSITION 3.2. *Let* $\varphi\colon \mathscr{A} \to \mathscr{B}$ *be nuclear. Then* φ *has finite total variation, and* $V(\varphi) \leqq \inf \{\sum \|\tau_i\|\colon \varphi = \sum \tau_i \otimes C_i,\ 0 \leqq \tau_i \in \mathscr{A}',\ \|C_i\| = 1,\ C_i \in \mathscr{B}\}$.

Proof. We can write $\varphi = \sum \rho_i \otimes B_i$, where $\sum \|\rho_i\| \|B_i\| < \infty$. By Theorem 0.2, we can, in fact, write $\varphi = \sum \tau_i \otimes C_i$, with $\sum \|\tau_i\| \|C_i\| < \infty$ and $\tau_i \geqq 0$. We can also assume all $\|C_i\| = 1$. Then for $0 \leqq a \in \mathscr{A}$, we have $\varphi(a) = \sum \tau_i(a) C_i$, which implies

$$\begin{aligned} \varphi(a)^* \varphi(a) &= \sum_{i,j} \tau_i(a) \tau_j(a) C_i^* C_j \\ &= \sum \tau_i^2(a) C_i^* C_i + \sum_{i \neq j} \tau_i(a) \tau_j(a) [C_j^* C_i + C_j^* C_i] \\ &\leqq \sum \tau_i^2(a) I + \sum_{i \neq j} \tau_i(a) \tau_j(a) \cdot 2I = (\sum \tau_i(a))^2 I . \end{aligned}$$

Hence $|\varphi(a)| \leqq (\sum \tau_i(a)) I$. So if $a_1, \cdots, a_n$ are positive, and $\sum a_k \leqq 1$, we see

$$\sum_{k=1}^{n} |\varphi(a_k)| \leqq \sum_k \sum_i \tau_i(a_k) I = \sum_i \sum_k \tau_i(a_k) I \leqq \sum_i \tau_i(1) I = \sum_i \|\tau_i\| I .$$

Hence $V(\varphi) \leqq \sum \|\tau_i\| < \infty$, so φ has finite total variation, and we get the estimate for $V(\varphi)$.

COROLLARY. *If* $\varphi^{(n)}$ *are as in* § 2, *then* $V(\varphi^{(n)}) = \sum_{i=1}^{n} \|\rho_i\|$.

Proof. The ρ_i have disjoint support, and all $|A_i| = I$.

REMARK 3.3. If φ is self-adjoint and has finite total variation, then φ is necessarily bounded. For if not, there exist $\{a_n\}$ such that $\|a_n\| \leqq 1$, $\|\varphi(a_n)\| \to \infty$. But φ self-adjoint implies $\varphi(\mathrm{Re}\,(a_n)) = \varphi((a_n + a_n^*)/2) = \mathrm{Re}\,(\varphi(a_n))$, and similarly $\varphi(\mathrm{Im}\,(a_n)) = \mathrm{Im}\,(\varphi(a_n))$, so that there are self-adjoint a_n with $\|a_n\| \leqq 1$ and $\|\varphi(a_n)\| \to \infty$. But then $a_n = a_n^+ - a_n^-$, where $a_n^+, a_n^- \geqq 0$, and $\|a_n\| = \max(\|a_n^+\|, \|a_n^-\|)$. So there exist c_n with $0 \leqq c_n$, $\|c_n\| \leqq 1$, and $\|\varphi(c_n)\| \to \infty$. But $0 \leqq c_n \leqq 1$, so by finite total variation, $\| |\varphi(c_n)| \| \leqq V(\varphi) < \infty$. However, $\| |\varphi(c_n)| \| = \|\varphi(c_n)\|$, since $\varphi(c_n)^* = \varphi(c_n)$. This contradiction completes the proof.

PROPOSITION 3.4. *Let* $\{P_n\}$ *be finite dimensional projections such that* $P_n \uparrow I$. *Let* $\varphi\colon \mathscr{A} \to \mathscr{L}(H)$. *Suppose* $\varphi_n = P_n \varphi P_n$ *has finite total variation with* $V(\varphi_n) < M < \infty$ *for all* n. *Then* φ *has finite total variation with* $V(\varphi) \leqq M$.

Proof. For any $B \in \mathscr{L}(H)$, $|P_n B P_n| \xrightarrow{w} |B|$ [2]. Hence,

$\sum |P_n\varphi(a_i)P_n| \xrightarrow{w} \sum |\varphi(a_i)|$, where $0 \leqq a_i$, $\sum a_i \leqq 1$. So if $\| \sum |P_n\varphi(a_i)P_n| \| \leqq M$ for all n, we have $\| \sum |\varphi(a_i)| \| \leqq M$ also.

Note. The example φ of § 2 shows the need of a uniform constant, for $V(\varphi^{(n)}) = \sum \|\rho_i\| \to \infty$.

We should say that the difficulty with the notion of finite total variation occurs in attempting to show that the sum of maps with finite total variation also has finite total variation. There is, however, a tractable case.

PROPOSITION 3.5. *Let $\varphi_1, \varphi_2: \mathscr{A} \to C(Y)$ be bounded. Suppose φ_1, φ_2 have finite total variation. Then so does $\varphi = \varphi_1 + \varphi_2$.*

Proof. For $f, g \in C(Y)$, $|f + g| \leqq |f| + |g|$. Thus, $|\varphi_1(a) + \varphi_2(a)| \leqq |\varphi_1(a)| + |\varphi_2(a)|$. The rest follows by addition.

PROPOSITION 3.6. *Let $\varphi: \mathscr{A} \to C(Y)$ be bounded. Then φ has finite total variation.*

Proof. By 3.5, we may assume φ is self-adjoint. Let $y \in Y$, consider $\tau_y = \delta_y \circ \varphi$. Then τ_y is a bounded self-adjoint functional on $\mathscr{A}$, so by Theorem 0.2, $\tau_y = \mu_y^+ - \mu_y^-$, where $\|\mu_y^+\| + \|\mu_y^-\| = \|\tau_y\| \leqq \|\varphi\|$. Then if $a_i \in \mathscr{A}$, $0 \leqq a_i$, $\sum a_i \leqq 1$, we have

$$\begin{aligned}\sum |\varphi(a_i)|(y) = \sum |\tau_y(a_i)| &\leqq \sum (\mu_y^+(a_i) + \mu_y^-(a_i)) \leqq \mu_y^+(1) + \mu_y^-(1) \\ &= \|\mu_y^+\| + \|\mu_y^-\| \leqq \|\varphi\| .\end{aligned}$$

By taking the supremum over y, we obtain $\| \sum |\varphi(a_i)| \| \leqq \|\varphi\|$, which shows that φ has finite total variation.

4. A Hahn decomposition theorem.

We begin with a well-known result, which we state here in the form we need (see [5]).

LEMMA 4.1. *Let X be a compact Hausdorff space, let $U_1, \cdots, U_n$ be an open cover of X. Then there are continuous functions $g_1, \cdots, g_n$ on X so that $0 \leqq g_i \leqq 1$, $\sum g_i \equiv 1$, and support of $g_i \subseteq U_i$. Furthermore, if the cover is nonredundant,* i.e., *$\forall j \bigcup_{i \neq j} U_i \neq X$, then there are $x_1, \cdots, x_n \in X$ such that $g_i(x_j) = \delta_{ij}$.*

From now on, we will assume all partitions of unity are constructed with respect to nonredundant covers, and so the word "cover" will denote a nonredundant cover, with associated points $\{x_i\}$.

Note, therefore, that given a cover $U_1, \cdots, U_n$ and associated partition of unity $g_1, \cdots, g_n$, we have $\| \sum \alpha_i g_i \| = \sup |\alpha_i|$. For, $|\sum \alpha_i g_i(x)| \leqq \sum |\alpha_i| g_i(x) \leqq \sup |\alpha_i| \sum g_i(x) = \sup |\alpha_i|$; but if $|\alpha_{i_0}| = \sup |\alpha_i|$, then by 4.1, $|\sum \alpha_i g_i(x_{i_0})| = |\alpha_{i_0}|$.

LEMMA 4.2. *Let (U_i, g_i) be a cover and associated partition of unity. Then for $F \in C(X)$, the map $Q: F \rightarrow \sum F(x_i) g_i$ is a positive linear projection of norm 1 from $C(X)$ onto the span of the $\{g_i\}$.*

Proof. Let $Q(F) = \sum F(x_i) g_i$. Then $Q(g_j) = g_j$, and if $F \geqq 0$, $Q(F) \geqq 0$. Clearly, $Q(1) = 1$, and the above note shows $\| Q \| \leqq 1$ ([3].).

LEMMA 4.3. *Let $\varphi: C(X) \rightarrow \mathscr{B}$ have finite total variation. Then given (U_i, g_i) a cover and associated partition of unity, there exists a positive linear map $\Pi: \operatorname{span}\{g_i\} \rightarrow \mathscr{B}$ so that $\| \Pi \| \leqq V(\varphi)$, and if φ is self-adjoint, $\Pi \geqq \varphi |_{\operatorname{span}\{g_i\}}$.*

Proof. Let $\Pi(\sum \alpha_i g_i) = \sum \alpha_i |\varphi(g_i)|$. Clearly Π is linear, and $\sum \alpha_i g_i \geqq 0$ iff all $\alpha_i \geqq 0$; it is trivial that all $\alpha_i \geqq 0$ implies $\sum \alpha_i g_i \geqq 0$, and conversely, we need only evaluate at the points $\{x_j\}$. Hence Π is positive, and if φ is self-adjoint, $\Pi \geqq \varphi |_{\operatorname{span}\{g_i\}}$, for if $\sum \alpha_i g_i \geqq 0$, $(\Pi - \varphi)(\sum \alpha_i g_i) = \sum \alpha_i(|\varphi(g_i)| - \varphi(g_i)) \geqq 0$, since all $\alpha_i \geqq 0$, and $g_i \geqq 0 \Rightarrow \varphi(g_i)^* = \varphi(g_i)$.

Now, we can extend Π to $\tilde{\Pi}: C(X) \rightarrow \mathscr{B}$ by defining $\tilde{\Pi} = \Pi \circ Q$, Q as in 4.2. Then $\| \Pi \| \leqq \| \tilde{\Pi} \|$ since $\tilde{\Pi}$ extends Π, and $\| \tilde{\Pi} \| \leqq \| \Pi \|$ since $\| Q \| \leqq 1$. Thus $\| \tilde{\Pi} \| = \| \Pi \|$. But $\tilde{\Pi}$ is positive, since Q and Π are, so by [1], $\| \tilde{\Pi} \| = \| \tilde{\Pi}(1) \|$. However, $1 \in \operatorname{span}\{g_i\}$, so $\tilde{\Pi}(1) = \Pi(1)$, so $\| \Pi \| = \| \Pi(1) \| = \| \sum |\varphi(g_i)| \| \leqq V(\varphi)$.

We now proceed with the main result of this paper.

THEOREM 4.4. *Let $\varphi: C(X) \rightarrow \mathscr{R}$ be a bounded self-adjoint linear map with finite total variation, where $\mathscr{R}$ is any von Neumann algebra. Then φ admits a Hahn decomposition, $\varphi = \varphi^+ - \varphi^-$, where φ^+, φ^- are bounded positive linear maps into $\mathscr{R}$, and $\| \varphi^+ \| \leqq V(\varphi)$.*

Proof. It suffices to find a positive linear map φ^+ into $\mathscr{R}$ such that $\| \varphi^+ \| \leqq V(\varphi)$, and $\varphi^+ \geqq \varphi$, for then we may take $\varphi^- = \varphi^+ - \varphi$.

Let $\mathscr{C}$ be the family of all nonredundant covers of X (and corresponding partitions of unity), partially ordered by inclusion of the covers. For $P \in \mathscr{C}$, let Q_P and Π_P denote the mappings of Lemmas 4.2 and 4.3, respectively. Let $\tilde{\Pi}_P = \Pi_P \circ Q_P$, then the proof of 4.3 shows $\tilde{\Pi}_P: C(X) \rightarrow \mathscr{R}$ is positive, $\| \tilde{\Pi}_P \| = \| \Pi_P \| \leqq V(\varphi)$ and $\Pi_P \geqq \varphi |_P$, so by setting $\tilde{\varphi}_P = \varphi \circ Q_P$, we have $\| \tilde{\varphi}_P \| \leqq \| \varphi \|$ and $\tilde{\Pi}_P \geqq \tilde{\varphi}_P$.

We claim $Q_P \xrightarrow{s} \mathrm{id}$, i.e., $\forall f \in C(X)$, $Q_P(f) \to f$. Since X is compact, f is uniformly continuous, so there is a cover $P = \{U_1, \cdots, U_n\}$ of X such that $x, y \in U_i \Rightarrow |f(x) - f(y)| \leqq \varepsilon$, where $\varepsilon > 0$ is arbitrary. Then let $x \in X$, so

$$\begin{aligned}|f(x) - Q_P(f)(x)| &= |f(x) - \sum f(x_j)g_j(x)| \\ &= |\sum (f(x) - f(x_j))g_j(x)| \leqq \sum |f(x) - f(x_j)|\, g_j(x)\,.\end{aligned}$$

Fix a j_0: if $x \in U_{j_0}$, then $|f(x) - f(x_{j_0})| < \varepsilon$; if $x \notin U_{j_0}$, then $g_{j_0}(x) = 0$. In either case, $|f(x) - f(x_{j_0})|\, g_{j_0}(x) \leqq \varepsilon g_{j_0}(x)$. Summing, we have $|f(x) - Q_P(f)(x) \leqq \sum \varepsilon g_j(x) = \varepsilon$. Thus $\|f - Q_P(f)\| \leqq \varepsilon$, and, by refinement of the cover, this also holds for all $P' \geqq P$, proving our claim.

In particular, since φ is continuous, we see that $\tilde{\varphi}_P(f) \to \varphi(f)$, and since any subnet of a convergent net is convergent, we see that $\tilde{\varphi}_{P_\alpha}(f) \to \varphi(f)$ for any subnet $\{P_\alpha\} \subseteqq \mathscr{C}$.

Now, for any $K > 0$, the set of all positive linear maps from $C(X)$ into $\mathscr{R}$, with norm $\leqq K$, is a compact set in the BW-topology (this is just a variant of Alaoglu's theorem, see [1]). Hence, the partial ordering on $\mathscr{C}$ makes $\{\tilde{\Pi}_P\}$ into a net, and by the above comment, with $K = V(\varphi)$, there is a cluster point $\tilde{\Pi}$ of $\{\tilde{\Pi}_P\}$. Then $\tilde{\Pi}$ is positive, $\|\tilde{\Pi}\| \leqq V(\varphi)$, and there is a cofinal set $\{P_\alpha\}$ such that $\tilde{\Pi}_{P_\alpha} \xrightarrow{BW} \tilde{\Pi}$.

Let $\tau_\alpha = \tilde{\Pi}_{P_\alpha} - \tilde{\varphi}_{P_\alpha}$, then $\tau_\alpha \geqq 0$, and $\|\tau_\alpha\| \leqq \|\tilde{\Pi}_{P_\alpha}\| + \|\tilde{\varphi}_{P_\alpha}\| \leqq V(\varphi) + \|\varphi\|$. By the above comments, we can choose a cluster point τ of $\{\tau_\alpha\}$ which is positive. Since $\tau_\alpha = \tilde{\Pi}_{P_\alpha} - \tilde{\varphi}_{P_\alpha}$, and the $\tilde{\Pi}_{P_\alpha}$ converge to $\tilde{\Pi}$, we have $0 \leqq \tau = \tilde{\Pi} - \tilde{\varphi}$, where $\tilde{\varphi}$ is the corresponding cluster point of the $\tilde{\varphi}_{P_\alpha}$.

But we have that for this (or any) subnet, $\tilde{\varphi}_{P_\alpha} \xrightarrow{s} \varphi$, so we have $\tilde{\Pi}_{P_\alpha}(f) - \tilde{\varphi}_{P_\alpha}(f) \to \tilde{\Pi}(f) - \varphi(f)$. Thus, we see that $0 \leqq \tau = \tilde{\Pi} - \varphi$, i.e., $\tilde{\Pi}$ dominates φ. We may then choose $\varphi^+ = \tilde{\Pi}$, yielding the desired Hahn decomposition.

THEOREM 4.5. *Let $\varphi: C(X) \to C(\Omega)$, where $C(\Omega)$ is a von Neumann algebra. Then the following are equivalent*:

(i) *φ is bounded*; (ii) *φ has finite total variation*; (iii) *φ admits a Hahn decomposition*; (iv) *φ is completely bounded.*

Proof. Theorem 4.4, Propositions 3.5 and 3.6, and Lemma 1.5.

Note. Oscar Lanford has informed me that (ii) $\Rightarrow$ (iii) of Theorem 4.5 fails if $C(\Omega)$ is not a von Neumann algebra.

REMARKS 4.6 (i). For the mappings $\varphi^{(n)}$ of § 2, it follows that we can write $\varphi^{(n)} = \varphi^+ - \varphi^-$, where φ^+, φ^- are positive and $1/2 \sum \| \rho_i \| \leqq \| \varphi^+ \| \leqq \sum \| \rho_i \|$.

(ii) There is another interesting way of ordering the non-redundant partitions of unity. For any $\varepsilon > 0$, we say $(U_i, g_i) \overset{\varepsilon}{\leqq} (V_j, h_j)$ iff (1) the V_j refine the U_i and (2) $x, y \in V_j \Rightarrow | g_i(x) - g_i(y) | < \varepsilon \ \forall i$. It is not difficult to show that $\overset{\varepsilon}{\leqq}$ is, in fact, a partial order. The relation $\overset{\varepsilon}{\leqq}$ has the following two interesting properties:

(a) $(U_i, g_i) \overset{\varepsilon}{\leqq} (V_j, h_j) \Rightarrow (U_i, g_i) \overset{\varepsilon'}{\leqq} (V_j, h_j) \forall \varepsilon' \geqq \varepsilon$;

(b) if $(U_i, g_i) \overset{\varepsilon}{\leqq} (V_j, h_j)$ and we write $G = \operatorname{span}\{g_i\}$, $H = \operatorname{span}\{h_j\}$, then $\sup_{\substack{g \in G \\ \|g\| \leqq 1}} \inf_{\substack{h \in H \\ \|h\| \leqq 1}} \| g - h \| < \varepsilon$, i.e., G is almost a subspace of H. The usual order relation on the partitions of unity does not have a property resembling property (b).

We now order $R^+ \times \mathscr{P}$ by $[\varepsilon, (U_i, g_i)] \leqq [\varepsilon', (V_j, h_j)]$ iff (1) $\varepsilon' \leqq \varepsilon$ and (2) $(U_i, g_i) \overset{\varepsilon}{\leqq} (V_j, h_j)$. Given a partition (U_i, g_i) and $f \in C(X)$, $\|f\| \leqq 1$, such that $\inf_{\substack{g \in \operatorname{span}\{g_i\} \\ \|g\| \leqq 1}} \|f - g\| < \varepsilon$, note that for all $[\varepsilon', (V_j, h_j)] \geqq [\varepsilon, (U_i, g_i)]$, then $\inf_{\substack{h \in \operatorname{span}\{h_j\} \\ \|h\| \leqq 1}} \|f - h\| < 2\varepsilon$.

We then set $S_{(\varepsilon, P)} = \overline{\bigcup_{(\varepsilon', P') \geqq (\varepsilon, P)} \Pi_{P'}}$, where the closure is taken in the BW-topology. Then we take $\tilde{\Pi} \in \bigcap S_{(\varepsilon, P)}$, for each $S_{(\varepsilon, P)}$ is BW-compact, and the finite intersection property applies.

By passing to a subnet, we can show $\tilde{\Pi} \geqq \tilde{\varphi}$, again using $\tau_P = \Pi_P - \varphi_P$, where $\tilde{\varphi}$ is a cluster point of $\{\varphi_P\}$. Then it is not difficult to show $\tilde{\varphi} = \varphi$, i.e., $Q_P \overset{s}{\longrightarrow} \mathrm{id}$ again.

THEOREM 4.7. *Let $\varphi: C(X) \to \mathscr{R}$, a von Neumann subalgebra of $\mathscr{L}(H)$, be self-adjoint, linear with finite total variation. Then $\forall f \in C(X)$, $\varphi(f) = V^* \pi(f) W$, where K is another Hilbert space, $W: H \to K$, π is a *-representation of $C(X)$ on K, and $V^*: K \to H$. In particular, φ is completely bounded.*

Proof. If we know that $\varphi = V^* \pi W$, it is easy to see that φ is completely bounded. For

$$\| \varphi \otimes \mathrm{id}_n \| \leqq \| V^* \otimes I_n \| \, \| \Pi \otimes \mathrm{id}_n \| \, \| W \otimes I_n \| = \| V^* \| \, \| \Pi \| \, \| W \| .$$

By Theorem 4.4, $\varphi = \varphi^+ - \varphi^-$, where φ^+, φ^- are positive. Two fundamental theorems of Stinespring assert that every positive map of $C(X)$ is completely positive, and hence of the form $T^* \sigma T$, where $T: H' \to K'$, and σ is a *-representation on K' ([7]). So $\varphi^+ = V_1^* \pi_1 V_1$ (π_1 on K_1), and $\varphi^- = V_2^* \pi_2 V_2$ (π_2 on K_2). For $z \in H$ let $W(z) = (V_1 z, -V_2 z) \in K_1 \oplus K_2$, $\pi(f) = \pi_1(f) \oplus \pi_2(f)$ on $K_1 \oplus K_2$, and $V^*(\xi, \eta) = V_1^* \xi + V_2^* \eta$ for $(\xi, \eta) \in K_1 \oplus K_2$. Then $\varphi = V^* \pi W$ as desired.

Theorem 4.7 may be regarded as a generalization of Stinespring's theorem [7].

We plan to discuss uniqueness of the Hahn decomposition in another paper.

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Received November 19, 1973.

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