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THE DUNFORD-PETTIS PROPERTY FOR CERTAIN UNIFORM ALGEBRAS

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A Banach space B has the Dunford-Pettis property if $x_n^*(x_n) \to 0$ whenever $x_n \to 0$ weakly and the sequence x_n^* tends to zero weakly in B^* (i.e. $\sigma(B^*, B^{**})$). Suppose now that A is a uniform algebra on a compact space X. If ϕ is a nonzero multiplicative linear functional on A then M_ϕ is the set of positive representing measures of ϕ. If A is such that a singular measure which is orthogonal to A must necessarily be zero and if all M_ϕ are weakly compact sets then the algebra A as well as its dual have the Dunford-Pettis property.

The idea of the proof is that A^* the dual of A can be decomposed into components for which the results of Chaumat [1] and Cnop-Delbaen [2] can be applied. The fact that an l_1 sum of Dunford-Pettis spaces is also a Dunford-Pettis space then gives the result. In paragraph two some conditions ensuring the weak compactness of M_ϕ are given. These conditions are related to those used in the definition of core and enveloping measures (see [6]).

1. Notation and preliminaries. X will be a compact space, $A \subset \mathscr{C}(X)$ a closed subalgebra of the space of continuous complex-valued functions on X. The algebra A is supposed to contain the constants and to separate the points of X. The spectrum M_A is the set of all nonzero multiplicative linear functionals on A. If $\phi \in M_A$ then M_ϕ is the set of all positive measures on X representing ϕ, i.e.

$$M_\phi = \left\{\mu \in M(X) \mid \mu \geqq 0 \text{ and } \forall f \in A \text{ we have } \phi(f) = \int f d\mu\right\} .$$

As well known M_ϕ is a convex set, compact for the topology $\sigma(M(X), \mathscr{C}(X))$. We say that two multiplicative linear forms ϕ and ψ belong to the same Gleason part if $\|\phi - \psi\| < 2$ in A^*, the dual of A. It is well known that being in the same Gleason part is an equivalence relation and hence $M_A = \bigcup_{\pi \in \Pi} \pi$ where Π is the set of all Gleason equivalence classes. For more details and any unexplained notion on uniform algebras we refer to [6].

If E is a Banach space then E has the Dunford-Pettis property if $e_n^*(e_n) \to 0$ whenever $e_n \to 0$ weakly and $e_n^* \to 0$ weakly (i.e. $\sigma(E^*, E^{**})$).

For more details and properties of such spaces see Grothendieck

[4] or [5], where it is also proved that L^1 spaces and $\mathscr{C}(X)$ spaces have the Dunford-Pettis property.

2. **Weak compactness of M_ϕ.** We investigate under what conditions M_ϕ is weakly compact. First we remark that if ψ and ϕ are in the same Gleason part then there is an affine isomorphism linking M_ϕ and M_ψ, see [6, p. 143]. It follows that M_ϕ is weakly compact (i.e. $\sigma(M(X), M(X)^*)$) if and only if M_ψ is weakly compact. Moreover if m_ϕ is dominant in M_ϕ and m_ψ is dominant in M_ψ then m_ϕ is absolutely continuous with respect to m_ψ. (The existence of a dominant measure in M_ϕ is given by [3, p. 307].)

LEMMA. *If ϕ is an element of M_A then following are equivalent*

1. *M_ϕ is weakly compact.*
2. *If u_n is a sequence of continuous functions on X such that $1 \geqq u_n \geqq 0$ and $u_n \to 0$ pointwise then there is a subsequence n_k and functions $v_k \in A$ such that $Re\, v_k \geqq u_{n_k}$ and $\phi(v_k) \to 0$.*
3. *If u_n is a sequence of continuous functions on X such that $1 \geqq u_n \geqq 0$ and $u_n \to 0$ pointwise then there is a subsequence n_k and functions $g_k \in A$ such that $|g_k| \leqq e^{-u_{n_k}}$ and $\phi(g_k) \to 1$.*

Proof. (1) ⇒ (2) If M_ϕ is weakly compact and u_n is a sequence as in (2) then $\sup_{\mu \in M_\phi} \int u_n d\mu \to 0$ (see [4]). Hence if ε_n is a sequence of strictly positive numbers tending to zero then $\exists v_n \in A$ such that $\operatorname{Re} v_n \geqq u_n$ and $\phi(v_n) \leqq \sup_{\mu \in M_\phi} \int u_n d\mu + \varepsilon_n$ (see [6 p. 82]). Clearly $\phi(v_n) \to 0$.

(2) ⇒ (3) Write $g_k = e^{-v_k}$ and observe that $|g_k| = e^{-\operatorname{Re} v_k} \leqq e^{-u_{n_k}}$ and $\phi(g_k) = e^{-\phi(v_k)} \to 1$.

(3) ⇒ (1) If M_ϕ is not weakly compact then following [4] there is a sequence of functions $u_n \in \mathscr{C}(X)$ and a sequence of measures $\mu_n \in M_\phi$ as well as $\varepsilon > 0$ such that

(i) $0 \leqq u_n \leqq 1$ and $u_n \to 0$ pointwise

(ii) $\int u_n d\mu_n > \varepsilon$.

Let now g_k be as in (3) then

$$|\phi(g_k)| \leqq \int |g_k|\, d\mu_{n_k} \leqq \int e^{-u_{n_k}} d\mu_{n_k} \leqq 1 - \frac{e-1}{e} \int u_{n_k} d\mu_{n_k} \leqq 1 - \frac{e-1}{e}\varepsilon$$

and this contradicts $\phi(g_k) \to 1$.

REMARK. The conditions (2) and (3) are of course related to the conditions of being enveloped and being a core measure. The difference is that the sequence u_n is supposed to be uniformly bounded.

COROLLARY. *If A satisfies one of the following conditions then for all $\phi \in M_A$, M_ϕ is weakly compact.*

(1) *If $1 \geqq u_n \geqq 0$; $u_n \in \mathscr{C}(X)$ and $u_n \to 0$ pointwise then there is a subsequence n_k and $v_k \in A$ such that v_k are uniformly bounded,* $\operatorname{Re} v_k \geqq u_{n_k}$ *and $v_k \to 0$ on X.*

(2) *If $1 \geqq u_n \geqq 0$; $u_n \in \mathscr{C}(X)$ and $u_n \to 0$ pointwise then there is a subsequence n_k and $g_k \in A$ such that $|g_k| \leqq e^{-u_{n_k}}$ and $g_k \to 1$ on X.*

3. The D.P. property for some uniform algebras. In the following theorem we say that a measure ν is singular to A if for all ϕ and all $\mu \in M_\phi$, the measure ν is singular with respect to μ.

THEOREM. *A has the Dunford-Pettis property if*

(1) *for all $\phi \in M_A$, the set M_ϕ is weakly compact,*

(2) *if λ is orthogonal to A and λ is singular to A then $\lambda = 0$.*

Proof. Of course we only have to prove that A^* has the D.P. property, since it follows from the definition that a Banach space is a Dunford-Pettis space as soon as its dual is a Dunford-Pettis space. We first prove the following lemma.

LEMMA. *If $(E_\beta)_{\beta \in B}$ is a family of Banach spaces all having the* D.P. *property and if*

$$\left(\sum_\beta \oplus E_\beta\right)_{l_1} = E = \left\{e = (e_\beta)_{\beta \in B} \mid e_\beta \in E_\beta ;\ \sum_\beta \| e_\beta \| = \| e \| < \infty\right\}$$

then E has the D.P. *property.*

Proof. $\forall \beta$ let $P_\beta : E \to E_\beta$ be the canonical projection.

Let $e_n \in E$ such that $e_n \to 0$ weakly and $\| e_n \| \leqq 1$; $e_n^* \in E^*$ such that $e_n^* \to 0$ weakly and $\| e_n^* \| \leqq 1$; $P_\beta e_n = e_{n,\beta}$; $P_\beta^* e_n^* = e_{n,\beta}^*$; $t_{n,\beta} = e_{n,\beta}^*(e_{n,\beta})$.

Only a denumerable part of the numbers $t_{n,\beta}$ can be different from zero so we can take $B = N$. We first prove that the sum $e_n^*(e_n) = \sum_\beta t_{n,\beta}$ converges uniformly in n, i.e.

($*$) for all $\varepsilon > 0$ there is N such that $\forall n$ we have $\sum_{\beta > N} | t_{n,\beta} | < \varepsilon$. If this is not the case then we start a well-known procedure. Let $\varepsilon > 0$ be such that ($*$) does not hold for this ε, take $\delta_n > 0$ such that $\sum_{n=1}^\infty \delta_n \leqq \varepsilon/4$. Let $n_1 = 1$, $N_0 = 0$, N_1 such that $\sum_{\beta > N_1} \| e_{n_1,\beta} \| \leqq \delta_1$.

Since $e_{n,1}, \cdots, e_{n,N_1} \to 0$ weakly we can find $\bar{n}_2$ such that for all $n \geqq \bar{n}_2 \geqq n_1$ we have $\sum_{\beta=1} | e_{n,j}^*(e_{n_1,j}) | \leqq \delta_2$. Let now $n_2 \geqq \bar{n}_2$ be such that $\sum_{\beta>N_1}|t_{n_2,\beta}|>\varepsilon$ and $N_2>N_1$ such that $\sum_{\beta>N_2}\|e_{n_2,\beta}\|\leqq\delta_2$. Continuing this procedure we find two strictly increasing sequences (n_k, N_k) such that

(1) $\sum_{\beta>N_k}\|e_{n_k,\beta}\|\leqq\delta_k$

(2) $\forall n\geqq n_k$ the sum $\sum_{\beta=1}^{N_{k-1}}|e^*_{n,j}(e_{n_{k-1},\beta})|\leqq\delta_k$

(3) $\sum_{\beta>N_{k-1}}|t_{n_k,\beta}|>\varepsilon$.

Let now

$$e^*=(\gamma_1 e^*_{1,1};\cdots;\gamma_{N_1}e^*_{1,N_1};\gamma_{N_1+1}e^*_{n_2,N_1+1};\cdots;\gamma_{N_2}e^*_{N_2};\gamma_{N_2+1}e^*_{n_3,N_2+1};\cdots)$$

where γ_β is such that if $N_{k-1}+1\leqq\beta\leqq N_k$ then $\gamma_\beta e^*_{n_k,\beta}(e_{n_k,\beta})=|t_{n_k,\beta}|$. Clearly $e^*\in E^*$ and $\|e^*\|\leqq 1$. For all $k\geqq 2$

$$\begin{aligned}e^*(e_{n_k}) &= \sum_{j=1}^{k-1}\sum_{\beta=N_{j-1}+1}^{N_j}\gamma_\beta e^*_{n_j,\beta}(e_{n_j,\beta})\\ &\quad+\sum_{\beta=N_{k-1}+1}^{N_k}|t_{n_k,\beta}|\\ &\quad+\sum_{\beta>N_k}\gamma_\beta e^*_\beta(e_{n_k,\beta})\,.\end{aligned}$$

So

$$\begin{aligned}|e^*(e_{n_k})| &\geqq -\sum_{j=1}^{k-1}\delta_j+\sum_{\beta=N_{k-1}+1}^{N_k}|t_{n_k,\beta}|-\delta_k\\ &\geqq -\sum_{j=1}^{k}\delta_j+\sum_{\beta>N_{k-1}}|t_{n_k,\beta}|-2\delta_k\\ &\geqq \varepsilon-2\sum_{j=1}^{\infty}\delta_j\geqq\varepsilon/2\,.\end{aligned}$$

But this contradicts $e_{n_k}\to 0$ weakly. This proves that $(*)$ is verified and hence $\lim_{n\to\infty}\sum_\beta t_{n,\beta}=\sum_\beta\lim t_{n,\beta}=0$, since each of the E_β has the D.P. property.

REMARK. If $E_n=l_2^n$ (i.e. the n-dimensional Hilbert space) then $E=(\Sigma\oplus E_n)_{l_1}$ has the D.P. property but E^* has not, because as easily seen, the space E^* has a complemented subspace isometric to l_2, this contradicts D.P. (see [4]).

Proof of the theorem. For each $\pi\in\Pi$ we select $\phi_\pi\in\pi$ and $m_\pi\in M_\phi$ dominant. By [6 p. 144] all m_π are mutually singular. Select now probability measures $(m_\beta)_{\beta\in B}$ such that $\{m_\pi\mid\pi\in\Pi\}\cup\{m_\beta\mid\beta\in B\}$ is a maximal farmily of mutually singular measures. (This can be done using Zorn's lemma.) An application of the Radon-Nikodym theorem yields:

$$M(X)=\mathscr{C}(X)^*=\Big(\sum_{\alpha\in\Pi\cup B}\oplus L^1(m_\alpha)\Big)_{l_1}\,.$$

For each π define N_π as the set $\{\pi \in L^1(m_\pi) \mid \mu \perp A\}$. The abstract F. and M. Riesz theorem [6] and hypothesis 2 give that

$$A^\perp = \Big(\sum_{\pi \in \Pi} \oplus N_\pi\Big)_{l_1}$$

and hence

$$A^* = \Big(\sum_{\pi \in \Pi} \oplus L^1(m_\pi)/N_\pi\Big)_{l_1} \oplus \Big(\sum_{\beta \in B} \oplus L^1(m_\beta)\Big)_{l_1}.$$

In [2] and [1] it is proved that the spaces $L^1(m_\pi)/N_\pi$ have the Dunford-Pettis property. By the preceding lemma and Grothendieck's result that an L^1 space is a Dunford-Pettis space we have that A^* has the D.P. property.

REMARK. (1) If $D = \{z \mid |z| < 1\}$ and A is the so-called disc-algebra i.e. $A = \{f \mid f$ analytic on D, continuous on $\bar{D}\}$ then A satisfies all requirements hence A and A^* have the D.P. property.

(2) If K is a compact set which is finitely connected then by Wilken's theorem $R(K)$ satisfies hypothesis 2 and by [6, p. 145, paragraph 3], $R(K)$ also satisfies hypothesis 1. Consequently $R(K)$ as well as $R(K)^*$ have the Dunford-Pettis property.

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