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ON THE DIFFERENTIABILITY OF MULTIFUNCTIONS

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A new concept of differential for a multifunction is introduced. Here by a multifunction we mean a map from a Banach space X to some specified family of non void subsets of a Banach space Y. The comparison with another definition due to Lasota and Strauss shows that if a multifunction admits both differentials, these must be equal. The results are applicable to the perturbation theory for multivalued differential equations in a Banach space

$$\dot{x} \in F(x)$$

in a neighborhood of a rest position.

1. Introduction. The concept of differentiability for multifunctions has been considered by many authors from different points of view (**[1]**, **[3]**, **[8]**, **[11]**, **[13]**, **[17]**, **[18]**). Of all these approaches, that developed by Lasota and Strauss **[17]** seems to be more useful in perturbation theory for ordinary differential equations in the real Euclidean space R^n. Further applications along this same direction were obtained in **[10]** (see also **[9]**). In the present paper, moving from an idea of Bridgland **[3]**, a new concept of differentiability for a multifunction is studied. This notion seems to be useful in perturbation theory. In **[7]** an application to problems of stability for multivalued differential equations in Banach spaces is given.

The definitions and the main properties of a multivalued differential (i.e. the differential of a multifunction or, in particular, of a function) are contained in §§2 and 3. Now, it is perhaps better to start by giving an answer to the preliminary question: where one may encounter a multivalued differential. To this end we recall the well known Theorems 1.1 and 1.2, due to Lyapunov.

THEOREM 1.1 (**[20]** p. 222). *Let f: $R^n \to R^n$, $f(0) = 0$, be continuously differentiable in a neighborhood of the origin with Fréchet differential f' at the origin. Let all the eigenvalues of f' have negative real parts, i.e. the origin is asymptotically stable for*

$$(1.1) \qquad \dot{x} = f'(x).$$

Then the origin is asymptotically stable for

$$(1.2) \qquad \dot{x} = f(x).$$

Of the possible extensions of the above result we mention the following two:

(I) f is single valued but not Fréchet differentiable at the origin. However f has a *multivalued differential* D at the origin, and so the variational equation which corresponds to (1.2) becomes

$$\dot{x} \in D(x). \tag{1.3}$$

(II) f is multivalued and admits at the origin a *multivalued differential* D. Thus, instead of (1.2) we have

$$\dot{x} \in f(x), \tag{1.4}$$

with corresponding variational equation (1.3).

In either case the problem arises whether the knowledge of a certain property of (1.3), for instance that the origin is a global attractor for this equation, implies that (1.2) (or (1.4)) possesses a similar, possibly weaker, property (see **[17]**).

THEOREM 1.2 (**[20]** p. 285). *Let f: $R^+ \times R^n \to R^n$, $R^+ = [0, \infty)$, be continuously differentiable, periodic in t with period $p > 0$. Let the equation*

$$\dot{x} = f(t, x) \tag{1.5}$$

have a periodic solution y of period p. Let all the characteristic numbers of the variational equation (along the periodic solution y)

$$\dot{x} = f'(t, y(t))x \tag{1.6}$$

have moduli strictly less than 1. *Then the periodic solution y is asymptotically stable for* (1.5).

A possible extension of Theorem 1.2 is the following:

(III) f is single valued but not Fréchet differentiable along y. However f has a *multivalued differential* D at any point $(t, y(t))$. Thus (1.6) is replaced by

$$\dot{x} \in D(t, y(t); x). \tag{1.7}$$

Then the problem arises whether, the fact that all solutions of (1.7) approach the origin for $t \to \infty$, implies that the periodic solution y is asymptotically stable for (1.5) (see **[10]**).

In §2 the definition of the multivalued differential D_x for a multifunction is introduced. Several elementary consequences of this definition are reviewed in §3. In the following one we consider, in infinite dimension, another definition of differential Δ_x for a multifunction. (This was introduced by Lasota and Strauss **[17]** for mappings from R^n to R^n.) In Section 5 we consider γ-Lipschitz maps (γ is the Hausdorff measure of noncompactness **[22]**). Then we show that the multivalued differential of a γ-Lipschitz map, with constant k, is γ-Lipschitz with the same constant. In the subsequent paper **[7]** an application of the above theory to a problem of stability, by the first approximation method, for a multivalued differential equation in Banach space is presented.

2. Notation and preliminaries. Let Y be a Banach space. For any $a \in Y$, define $S(a,r) = \{y : \|y - a\| < r\}$ $r > 0$, $\bar{S}(a,r) = \{y : \|y - a\| \leqq r\}$, $r \geqq 0$. We write $S, \bar{S}$ in place of $S(0,1)$, $\bar{S}(0,1)$. Denote by: $\mathscr{B}(Y)$ (resp. $\mathscr{C}(Y)$, $\mathscr{C}_0(Y)$, $\mathscr{K}(Y)$, $\mathscr{K}_0(Y)$) the family of all non void bounded (resp. bounded closed, bounded closed convex, compact, compact convex) subsets of Y; $N = \{1, 2, \cdots\}$; $\bar{A}$ the closure, $\overline{\text{co}}\,A$ the closed convex hull of $A \subset Y$. Let $A, B \in B(Y)$. Define

$$d(A,B) = \inf\{t > 0 : A \subset B + tS,\ B \subset A + tS\}.$$

We review a number of well known properties of d, some of which will be used in the sequel. We have:

$$d(A,B) \geqq 0, \qquad d(A,A) = 0$$
$$d(A,B) = d(B,A)$$
$$d(A,B) \leqq d(A,C) + d(C,B).$$

To conclude that d is a metric one has to show that $d(A,B) = 0$ implies $A = B$. This is not true in $\mathscr{B}(Y)$, but it is in $\mathscr{C}(Y)$. The restriction of d to couples of elements of $\mathscr{C}(Y)$ is called the *Hausdorff metric* in $\mathscr{C}(Y)$. We write $\|A\|$ in place of $d(A, 0)$.

The following lemma is fundamental.

LEMMA 2.1 (Rådström **[21]**). *Let A, B, C be non void subsets of Y. Suppose B closed and convex C bounded and $A + C \subset B + C$. Then $A \subset B$.*

LEMMA 2.2. *Let $A, A_1, B, B_1 \in \mathscr{B}(Y)$. Then*

(i) $d(tA, tB) = t d(A,B)$ $t \geqq 0$

(ii) $d(A + B, A_1 + B_1) \leqq d(A, A_1) + d(B, B_1)$.

If $A, B \in \mathscr{C}_0(Y)$ and $C \in \mathscr{B}(Y)$ we have

(iii) $d(A + C, B + C) = d(A,B)$.

Proof. Property (i) is obvious. To prove (ii) let $t > d(A, A_1)$, $t_1 > d(B, B_1)$. Then

$$A \subset A_1 + tS \qquad B \subset B_1 + t_1 S$$
$$A_1 \subset A + tS \qquad B_1 \subset B + t_1 S$$

and $A + B \subset A_1 + B_1 + (t + t_1)S$, $A_1 + B_1 \subset A + B + (t + t_1)S$ which imply $d(A + B, A_1 + B_1) \leqq t + t_1$. Letting $t \to d(A, A_1)$, $t_1 \to d(B, B_1)$ we get (ii).

Let us prove (iii). By (ii) $d(A + C, B + C) \leqq d(A, B)$. Suppose the strict inequality holds and let t be such that $d(A + C, B + C) < t < d(A, B)$. Then

$$A + C \subset B + C + tS \subset \overline{B + tS} + C$$

$$B + C \subset A + C + tS \subset \overline{A + tS} + C$$

and, since $\overline{B + tS}$, $\overline{A + tS}$ are closed convex while C is bounded, Lemma 2.1 yields $A \subset \overline{B + tS}$, $B \subset \overline{A + tS}$. On the other hand

$$\overline{B + tS} = \bigcap_{n=1}^{\infty} [(B + tS) + 2^{-n}S], \qquad \overline{A + tS} = \bigcap_{n=1}^{\infty} [(A + tS) + 2^{-n}S],$$

thus if we choose n such that $t + 2^{-n} < d(A, B)$ we obtain $A \subset B + (t + 2^{-n})S$, $B \subset A + (t + 2^{-n})S$. These imply $d(A, B) \leqq t + 2^{-n} < d(A, B)$, a contradiction.

Property (iii) is proved in **[21]** under different hypotheses (see also **[8]**).

Let X, Y be Banach spaces. Let U be a non void open subset of X.

DEFINITION 2.3. $F: U \to \mathcal{B}(Y)$ is said to be *upper semicontinuous* (= u.s.c.) at $x \in U$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(x + h) \subset F(x) + \epsilon S$, when $\|h\| < \delta$. F is said to be *continuous* at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(x + h) \subset F(x) + \epsilon S$ and $F(x) \subset F(x + h) + \epsilon S$, when $\|h\| < \delta$.

DEFINITION 2.4. $F: X \to \mathcal{B}(Y)$ is said to be *homogeneous* if $F(tx) = tF(x)$, $t \geqq 0$, $x \in X$.

The following definition of differentiability is suggested by an idea due to Bridgland **[3]**.

DEFINITION 2.5. $F: U \to \mathcal{B}(Y)$ is said to be *differentiable* at $x \in U$ if there exist a map $D_x: X \to \mathcal{C}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > 0$ such that

$$d(F(x + h), F(x) + D_x(h)) = o(h) \quad \text{when} \quad \|h\| < \delta.$$

(Here $o(h)$ denotes a nonnegative function such that $\lim_{h\to 0} o(h)/\|h\| = 0$.) D_x is called the *(multivalued) differential* of F at x.

REMARK 2.6. Let F be a map from U to $\mathcal{K}_0(Y)$. In **[18]** Martelli and Vignoli define F to be differentiable at $x \in U$ if there exists a map $S_x: X \to \mathcal{K}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > 0$ such that

$$F(x+h) = F(x) + S_x(h) + R(h), \quad \text{when} \quad \|h\| < \delta,$$

and $\lim_{h\to 0}\|R(h)\|/\|h\| = 0$. We shall see later that the existence of S_x implies that of D_x and $S_x = D_x$. The converse is false. To see this define $F: (-\pi/4, \pi/4) \to \mathcal{K}_0(Y)$, $Y = R^n$, by

$$F(t) \begin{cases} = \bar{S} & \text{if} \quad t = 0 \\ = (1 + t^2 \sin 1/t)\bar{S} & \text{if} \quad 0 < |t| < \pi/4. \end{cases}$$

Then F is differentiable at 0 and $D_0 = 0$. But S_0 does not exist for the existence of S_0 implies that, in a neighborhood of 0 the diameter of $F(t)$ is not less than the diameter of $F(0)$, which is clearly impossible.

3. Properties of differentiable multifunctions. In this section several elementary properties of differentiable multifunctions are reviewed. Let U be a non void open subset of X. The following theorem shows that the differential D_x is well defined.

THEOREM 3.1. *The multivalued differential D_x of $F: U \to \mathcal{B}(Y)$ at $x \in U$ if it exists is unique.*

Proof. Let δ correspond to D_x. Let there exist D_x^1 and $\delta_1 > 0$ such that $d(F(x+h), F(x) + D_x^1(h)) = o^1(h)$, when $\|h\| < \delta_1$. Trivially $D_x(0) = D_x^1(0) = 0$, being both D_x and D_x^1 homogeneous. Let $u \neq 0$. Let $t > 0$ be such that $t\|u\| < \delta$, δ_1. Then, by Lemma 2.2 (iii),

$$\begin{aligned} d(D_x(tu), D_x^1(tu)) &= d(D_x(tu) + F(x), D_x^1(tu) + F(x)) \\ &\leqq d(D_x(tu) + F(x), F(x+tu)) \\ &\quad + d(F(x+tu), D_x^1(tu) + F(x)) \\ &\leqq o(tu) + o^1(tu). \end{aligned}$$

Thus $d(D_x(u), D_x^1(u)) \leqq o(tu)/t + o_1(tu)/t$ and, letting $t \to 0$, $d(D_x(u), D_x^1(u)) = 0$. Since $D_x(u), D_x^1(u)$ are bounded closed we have $D_x(u) = D_x^1(u)$.

REMARK 3.2. Suppose $F: U \to \mathcal{K}_0(Y)$ has the differential S_x. Then D_x exists and $S_x = D_x$. In fact, if $\|h\| < \delta$,

$$\begin{aligned} d(F(x+h), F(x)+S_x(h)) &\leqq d(F(x+h), F(x)+S_x(h)+R(h)) \\ &\quad + d(F(x)+S_x(h)+R(h), F(x)+S_x(h)) \\ &\leqq \|R(h)\| \end{aligned}$$

and, since $\lim_{h\to 0}\|R(h)\|/\|h\| = 0$, we have $d(F(x+h), F(x)+S_x(h)) = o(h)$. By the uniqueness of D_x it follows $D_x = S_x$.

THEOREM 3.3. *If $F: U \to \mathcal{B}(Y)$ is differentiable at x it is there continuous.*

Proof. Let $\epsilon > 0$. Since F is differentiable at x there exists $\delta > 0$ such that $d(F(x+h), F(x)+D_x(h)) = o(h)$, when $\|h\| < \delta$. Furthermore, since D_x is u.s.c. at the origin and $D_x(0) = 0$, there exists $0 < \delta_1 < \delta$ such that $D_x(h) \subset \epsilon S$ if $\|h\| < \delta_1$. For $\|h\| < \delta_1$ we have

$$\begin{aligned} d(F(x+h), F(x)) &\leqq d(F(x+h), F(x)+D_x(h)) + d(F(x)+D_x(h), F(x)) \\ &\leqq o(h) + \|D_x(h)\| \\ &\leqq o(h) + \epsilon \end{aligned}$$

and F is continuous at x.

THEOREM 3.4. *Let U be a non void open and convex subset of X. The multifunction $F: U \to \mathscr{C}(Y)$ is constant if and only if, for every $x \in U$, $D_x = 0$.*

Proof. Let us prove the sufficiency of the condition (the necessity is trivial). For every $x \in U$ there exists $\delta > 0$ such that $d(F(x+h), F(x)) = o(h)$ if $\|h\| < \delta$. Let $x, x_1 \in U$. We have

$$|d(F(x_1), F(x+h)) - d(F(x_1), F(x))| \leqq d(F(x+h), F(x))$$

$$|d(F(x_1), F(x+h)) - d(F(x_1), F(x))|/\|h\| \leqq o(h)/\|h\|, \quad 0 < \|h\| < \delta.$$

Let $h \to 0$. Then the real valued functional $x \mapsto d(F(x_1), F(x))$, having zero Fréchet differential for every $x \in U$, must be constant. Since it vanishes for $x = x_1$ it is identically zero.

For $A, B \in \mathscr{C}(Y)$ define $d^*(A, B) = \inf\{t > 0 \colon A \subset B + tS\}$. We have:

$$d^*(A,B) \geqq 0, \qquad d^*(A,B) = 0 \quad \text{if and only if} \quad A \subset B$$
$$d^*(A,B) \leqq d^*(A,C) + d^*(C,B)$$
$$d^*(A,B) \leqq d(A,B).$$

If $A = \{a\}$, $B = \{b\}$ then $d^*(A,B) = d(A,B) = \|a - b\|$.

Given a map $F\colon U \to \mathscr{C}(Y)$, a single valued function $f\colon U \to Y$ satisfying $f(x) \in F(x)$, $x \in U$, is called a *selection* of F.

THEOREM 3.5. *Let $F\colon X \to \mathscr{C}(Y)$, $F(0) = 0$, be differentiable at the origin with differential D_0. Let f be a selection of F in a neighborhood $S(0, \delta_1)$ of the origin of X. If f has Fréchet differential f_0' at the origin then f_0' is a selection of D_0.*

Proof. There exists $0 < \delta < \delta_1$ such that

$$d(F(h), D_0(h)) = o(h), \qquad \|f(h) - f_0'(h)\| = o^1(h) \quad \text{if} \quad \|h\| < \delta.$$

Trivially D_0 and f_0' are equal for $u = 0$. Let $u \neq 0$. Let $t > 0$ be such that $t\|h\| < \delta$. Then we have

$$d^*(f_0'(tu), D_0(tu)) \leqq d^*(f_0'(tu), f(tu)) + d^*(f(tu), F(tu)) + d^*(F(tu), D_0(tu))$$

and

$$d^*(f_0'(u), D_0(u)) \leqq t^{-1}\|f_0'(tu) - f(tu)\| + t^{-1}d(F(tu), D_0(tu)) \leqq o^1(tu)/t + o(tu)/t.$$

Letting $t \to 0$ we obtain $d^*(f_0'(u), D_0(u)) = 0$.

4. Comparison with another definition of differential.

In **[17]** Lasota and Strauss gave the definition of a multivalued differential Δ_x for a single-valued map $F\colon R^n \to R^n$ and used such definition to prove a perturbation theorem for ordinary differential equations in R^n. Further results along this same direction were established in **[10]** and, for difference equations, in **[9]**. In this section the definition of Δ_x is extended to maps $F\colon X \to \mathscr{K}(Y)$, where X, Y are Banach spaces. Furthermore the relationship between the multivalued differential D_x and the Lasota–Strauss differential Δ_x is considered.

Let X, Y be Banach spaces, $U \subset X$ be open and non void.

DEFINITION 4.1. $F\colon U \to \mathscr{K}(Y)$ is said to be *Lipschitzian* at $x \in U$ if $F(x)$ is singleton and there exist constants $L \geqq 0$ and $\delta > 0$ such that $d(F(x+h), F(x)) \leqq L\|h\|$ if $\|h\| < \delta$.

DEFINITION 4.2. Let $F: U \to \mathscr{K}(Y)$ be Lipschitzian at $x \in U$. A map $\varphi: X \to \mathscr{K}_0(Y)$ is said to be an *upper differential* of F at x if φ is u.s.c., homogeneous and there exists $\delta > 0$ such that

$$F(x+h) \subset F(x) + \varphi(h) \quad \text{if} \quad \|h\| < \delta.$$

Denote by $\mathscr{F}$ the set of all upper differentials of F at x. $\mathscr{F}$ may be empty. However, if $\dim(Y) < \infty$, F has at least one upper differential, namely $\varphi(h) = L\|h\|\bar{S}$.

DEFINITION 4.3. Let $F: U \to \mathscr{K}(Y)$ be Lipschitzian at $x \in U$. Suppose that $\mathscr{F} \neq \varnothing$ and, for each $h \in X$, $\bigcap_{\varphi \in \mathscr{F}} \varphi(h) \neq \varnothing$. Define the L. S. *differential* $\Delta_x: X \to \mathscr{K}_0(Y)$ by

$$\Delta_x(h) = \bigcap_{\varphi \in \mathscr{F}} \varphi(h) \qquad h \in X.$$

The above definition reduces to that given by Lasota and Strauss **[17]** for single-valued maps $F: R^n \to R^n$.

REMARK 4.4. Berge **[2]** (p. 114) defines a map $F: U \to \mathscr{K}(Y)$ to be u.s.c at $x \in U$ if for every open set $G \supset F(x)$ there exists $\delta > 0$ such that $F(x+h) \subset G$ if $\|h\| < \delta$. If F is u.s.c. in this sense it is also u.s.c. according to Definition 2.3. Conversely let F be u.s.c. at x. To prove that F is u.s.c. according to Berge's definition it is sufficient to show the existence of a positive integer n such that $F(x) + 2^{-n}S \subset G$. Indeed, in the contrary case, for every $n \in N$, we have $(F(x) + 2^{-n}S) \cap (Y \backslash G) \neq \varnothing$. This implies the existence of a sequence $\{y_n + s_n\}$, $y_n \in F(x)$, $s_n \in 2^{-n}S$ such that $y_n + s_n \in Y \backslash G$. By the compactness of $F(x)$ we can and do assume, without loss of generality, $y_n \to y \in F(x)$. Since $y_n + s_n \to y$ and $Y \backslash G$ is closed, $y \in Y \backslash G$. From the contradiction the claim follows. Since (**[2]** p. 119) the intersection of any family of u.s.c. (homogeneous) mappings is u.s.c. (homogeneous) it remains proved that Δ_x if it exists is u.s.c. (homogeneous). Clearly for any $h \in X$, $\Omega(h) = \bigcap_{\varphi \in \mathscr{F}} \varphi(h)$ belongs to $\mathscr{K}_0(Y)$, provided it is non void. Thus the existence of Δ_x is finally established if we show that, for every $h \in X$, $\Omega(h) \neq \varnothing$.

THEOREM 4.5. *Let Y be reflexive. Let $F: U \to \mathscr{K}(Y)$ be Lipschitzian at $x \in U$. If $\mathscr{F} \neq \varnothing$ the L. S. differential Δ_x of F at x exists. Moreover Δ_x is u.s.c. and homogeneous.*

Proof. After Remark 4.4 the only fact which requires a proof is that $\Omega(h) \neq \varnothing$, $h \in X$. Let $h \neq 0$ (the case $h = 0$ is trivial). There exists a positive integer k such that

$$\frac{d(F(x+h/n), F(x))}{\|h/n\|} \leqq L \quad \text{if} \quad n \geqq k.$$

Choose $y_n \in F(x+h/n)$. Since the sequence $\{(y_n - F(x))\|h/n\|^{-1}\}_{n\geqq k}$ is bounded in the reflexive Banach space Y we assume, without loss of generality that it converges weakly to some element $z \in Y$. Then

$$z \in \overline{\text{co}}\left\{\frac{y_n - F(x)}{\|h/n\|}\right\}_{n\geqq k} \subset \overline{\text{co}}\left\{\frac{F(x+h/n)-F(x)}{\|h/n\|}\right\}_{n\geqq k}.$$

Let φ be any upper differential of F at x and let $\delta > 0$ correspond. There exists $k_1 \geqq k$ such that $n \geqq k_1$ implies $\|h/n\| < \delta$. For $n \geqq k_1$ we have $F(x+h/n) \subset F(x) + \varphi(h/n)$. Therefore

$$z \in \overline{\text{co}}\left\{\frac{\varphi(h/n)}{\|h/n\|}\right\}_{n\geqq k_1} = \varphi\left(\frac{h}{\|h\|}\right)$$

and $\|h\| z \in \varphi(h)$. Since φ is arbitrary $\|h\| z \in \Omega(h)$.

If $\dim(Y) < \infty$ the hypothesis $\mathcal{F} \neq \emptyset$ in the above theorem can be omitted.

LEMMA 4.6. *Let X and Y be separable Banach spaces. Let $F\colon U \to \mathcal{K}(Y)$ be Lipschitzian at x with L. S. differential Δ_x. Then there exists a sequence $\{\varphi_n\}$ of upper differentials of F at x such that*

$$\text{(4.1)} \qquad \varphi_n(h) \supset \varphi_{n+1}(h), \qquad \Delta_x(h) = \bigcap_{n=1}^{\infty} \varphi_n(h) \qquad h \in X.$$

Proof. Let Ψ be any upper differential of F at x. The graph G_Ψ of Ψ is closed for Ψ is u.s.c. (Berge **[2]** p. 117). Since X and Y have countable bases, $X \times Y$ has the same property and, by Lindelöf theorem (Dunford and Schwartz **[12]** p. 12) there exists a sequence $\{\Psi_n\}$ of upper differentials such that $\bigcap_{\varphi\in\mathcal{F}} G_\varphi = \bigcap_{n=1}^{\infty} G_{\Psi_n}$. Then

$$G_{\Delta_x} = \bigcap_{\varphi\in\mathcal{F}} G_\varphi = \bigcap_{n=1}^{\infty} G_{\Psi_n} = G_{\bigcap_{n=1}^{\infty}\Psi_n}$$

implies $\Delta_x = \bigcap_{n=1}^{\infty} \Psi_n$. Since a finite intersection of u.s.c. (homogeneous) maps is u.s.c. (homogeneous) the sequence $\{\varphi_n\}$, $\varphi_n = \bigcap_{k=1}^{n} \Psi_k$ consists of upper differentials which satisfy the conclusions of the lemma.

The following result is useful in perturbation theory **[10]**.

THEOREM 4.7. *Let X, Y be finite dimensional Banach spaces. Let* $F: U \to \mathcal{K}(Y)$ *be Lipschitzian at* $x \in U$, *with constant L. Let* $\Delta_x: X \to \mathcal{K}_0(Y)$ *be continuous. Then the map* $V_\epsilon: h \mapsto \Delta_x(h) + \epsilon \|h\| \bar{S}$, $\epsilon > 0$, *is an upper differential of F at x.*

Proof. The map V_ϵ from X to $\mathcal{K}_0(Y)$ is continuous and homogeneous. To conclude that V_ϵ is an upper differential of F at x we need to show that there exists $\delta > 0$ such that $F(x+h) \subset F(x) + V_\epsilon(h)$ if $\|h\| < \delta$. Suppose the contrary. There exists a sequence $\{h_n\}$, $h_n \neq 0$, $h_n \to 0$ such that $F(x+h_n) \not\subset F(x) + V_\epsilon(h_n)$. Thus there exists a sequence $\{y_n\}$, $y_n \in F(x+h_n)$, satisfying $y_n - F(x) \not\subset V_\epsilon(h_n)$ or, equivalently,

$$(y_n - F(x))/\|h_n\| \not\subset V_\epsilon(h_n/\|h_n\|) \qquad n \in N.$$

Since $\{h_n/\|h_n\|\}$ and $\{(y_n - F(x))/\|h_n\|\}$ are bounded and X, Y are finite dimensional, we can and do assume (without loss of generality)

$$h_n/\|h_n\| \to h \in X, \qquad (y_n - F(x))/\|h_n\| \to y \in Y. \tag{4.2}$$

Suppose $y \in \Delta_x(h) + (\epsilon/2)\bar{S}$. This implies $y + (\epsilon/4)\bar{S} \subset \Delta_x(h) + (3/4)\epsilon\bar{S}$ and for n sufficiently large, say $n \geqq k$, $(y_n - F(x))/\|h_n\| \in \Delta_x(h) + (3/4)\epsilon\bar{S}$. Since Δ_x is continuous at h there exists $k_1 \geqq k$ such that $\Delta_x(h) \subset \Delta_x(h_n/\|h_n\|) + (\epsilon/4)\bar{S}$ if $n \geqq k_1$. Thus

$$(y_n - F(x))/\|h_n\| \in \Delta_x(h_n/\|h_n\|) + \epsilon\bar{S} = V_\epsilon(h_n/\|h_n\|)$$

if $n \geqq k_1$, a contradiction.

Suppose $y \not\in \Delta_x(h) + (\epsilon/2)\bar{S}$. Then if ϵ_1 is such that $0 < \epsilon_1 < \epsilon/2$ we have $\bar{S}(y, \epsilon_1) \cap \Delta_x(h) = \emptyset$. By Lemma 4.6 there exists a sequence $\{\varphi_m\}$ of upper differentials of F at x satisfying (4.1).

We claim that there is $k \in N$ such that for all $m \geqq k$ we have $\bar{S}(y, \epsilon_1) \cap \varphi_m(h) = \emptyset$. Let the claim be false. Since $\varphi_1(h) \supset \varphi_2(h) \supset \cdots$, for every $m \in N$ there exists z_m in both sets $\bar{S}(y, \epsilon_1)$ and $\varphi_m(h)$. Without loss of generality we assume $z_m \to z$. Then $z \in \bar{S}(y, \epsilon_1)$ and $z \in \varphi_m(h)$ for every $m \in N$, thus $z \in \bar{S}(y, \epsilon_1) \cap \Delta_x(h)$, a contradiction. The claim is true. This implies

$$\bar{S}\left(y, \frac{\epsilon_1}{2}\right) \cap \left(\varphi_m(h) + \frac{\epsilon_1}{2}\bar{S}\right) = \emptyset, \qquad m \geqq k. \tag{4.3}$$

By (4.2), for all n sufficiently large say $n \geqq r$ we have

$$(y_n - F(x))/\|h_n\| \in \bar{S}\left(y, \frac{\epsilon_1}{2}\right), \qquad \varphi_m(h_n/\|h_n\|) \subset \varphi_m(h) + \frac{\epsilon_1}{2}\bar{S}$$

and so, by virtue of (4.3), $(y_n - F(x))/\|h_n\| \not\in \varphi_m(h_n/\|h_n\|)$ i.e. $y_n \not\in F(x) + \varphi_m(h_n)$ for all $n \geqq r$. This implies $F(x+h_n) \not\subset F(x) + \varphi_m(h_n)$, $n \geqq r$, a contradiction since φ_m is an upper differential of F.

Next theorem shows that $D_x = \Delta_x$ if both exist.

THEOREM 4.8. *Let X, Y be Banach spaces. Let $F\colon U \to \mathcal{K}_0(Y)$ be Lipschitzian at $x \in U \subset X$. Then $\Delta_x = D_x$ if both exist.*

Proof. By hypothesis there exists $\delta > 0$ such that $d(F(x+h), F(x) + D_x(h)) = o(h)$ if $\|h\| < \delta$. Let $\epsilon > 0$. Then there exists $0 < \delta_1 < \delta$ such that

$$F(x+h) \subset F(x) + D_x(h) + \epsilon \|h\| \bar{S} \tag{4.4}$$

$$F(x) + D_x(h) \subset F(x+h) + \epsilon \|h\| \bar{S} \quad \text{if} \quad \|h\| < \delta_1. \tag{4.5}$$

Let φ be any upper differential of F. This implies the existence of $0 < \delta_2 < \delta_1$ such that $F(x+h) \subset F(x) + \varphi(h)$, if $\|h\| < \delta_2$. Then $F(x+h) + \epsilon\|h\|\bar{S} \subset F(x) + \varphi(h) + \epsilon\|h\|\bar{S}$ and, by (4.5), $F(x) + D_x(h) \subset F(x) + \varphi(h) + \epsilon\|h\|\bar{S}$, if $\|h\| < \delta_2$. Thus, $D_x(h) \subset \varphi(h) + \epsilon\|h\|\bar{S}$ from which one easily obtains $D_x(h) \subset \varphi(h)$, if $\|h\| < \delta_2$. Since φ is any upper differential of F we have $D_x(h) \subset \Delta_x(h)$ and, by the homogeneity of D_x and Δ_x, the inclusion holds for all $h \in X$.

Next let us show the reverse inclusion. Let φ be any upper differential of F. Define $\varphi_1(h) = \varphi(h) \cap (D_x(h) + \epsilon\|h\|\bar{S})$, $h \in X$. We claim that φ_1 is an upper differential of F. From (4.4) and $F(x+h) \subset F(x) + \varphi(h)$, which hold for $\|h\|$ small enough, it follows that $\varphi_1(h) \neq \emptyset$ in a neighborhood of the origin and, by homogeneity, for all $h \in X$. Trivially $\varphi_1(h)$ is convex, for every $h \in X$. Furthermore, for each $h \in X$, $D_x(h)$ is compact, for it is contained in $\Delta_x(h)$, and so $\varphi_1(h)$ is compact being the intersection of $\varphi(h)$ compact, and $D_x(h) + \epsilon\|h\|\bar{S}$ closed. Thus φ_1 maps X into $\mathcal{K}_0(Y)$. Clearly φ_1 is homogeneous and satisfies $F(x+h) \subset F(x) + \varphi_1(h)$, for $\|h\|$ sufficiently small. So to conclude that φ_1 is an upper differential of F it remains to be shown that it is u.s.c. But this follows at once from a result of Berge ([**2**] p. 117) because the map $h \mapsto D_x(h) + \epsilon\|h\|\bar{S}$ from X to $\mathcal{C}_0(Y)$ is closed and $\varphi\colon X \to \mathcal{K}_0(Y)$ is u.s.c. Then there exists $\delta_3 > 0$ such that $\Delta_x(h) \subset \varphi_1(h) \subset D_x(h) + \epsilon\|h\|\bar{S}$, if $\|h\| < \delta_3$, which implies $\Delta_x(h/\|h\|) \subset D_x(h/\|h\|)$, $0 < \|h\| < \delta_3$. By homogeneity, $\Delta_x(h) \subset D_x(h)$ for every $h \in X$.

5. The differential of a γ-Lipschitz function. In this section it is shown that the differential D_x of a multifunction which is γ-Lipschitz with constant k possesses this same property. Let us introduce the following

DEFINITION 5.1. Let $A \in \mathscr{B}(Y)$. The *measure* $\gamma(A)$ *of noncompactness* of A is defined by

$$\gamma(A) = \inf\{t > 0\text{: there exists } C \in \mathscr{K}(Y) \text{ such that } A \subset C + t\bar{S}\}.$$

There are alternative (non equivalent) definitions of measures of noncompactness (**[5]**, **[14]**, **[16]**, **[22]**). That which we use seems to be flexible enough to be adapted for the measure of noncompactness in the weak topology as well **[6]**. The following theorem is well known. However we include the proofs of those statements which are proved in a different, perhaps simpler, way (see (f)–(i)).

THEOREM 5.2. *The functional* γ *has the properties*:
(a) $A \subset B$ *implies* $\gamma(A) \leqq \gamma(B)$
(b) $\gamma(A) = \gamma(\bar{A})$
(c) $\gamma(A) = 0$ *if and only if* $\bar{A}$ *is compact*
(d) $\gamma(A + B) \leqq \gamma(A) + \gamma(B)$
(e) $\gamma(sA) = s\gamma(A) \quad s \geqq 0$
(f) $\gamma(A) = \gamma(\overline{\mathrm{co}}\, A)$
(g) $\gamma(\bigcup_{u \in [0,s]} uA) = s\gamma(A)$
(h) $\gamma(S) = 1$ *if* $\dim(Y) = \infty$
(i) $\gamma(A + B) = \gamma(A)$ *if* $\gamma(B) = 0$.

Proof. (a)–(e) follow easily from the Definition 5.1. (f) Let $\epsilon > 0$. There exist $\gamma(A) < t < \gamma(A) + \epsilon$ and $C \in \mathscr{K}(Y)$ such that $A \subset C + t\bar{S}$. This implies $A \subset \overline{\mathrm{co}}\, C + t\bar{S}$ where, by Mazur theorem (Dunford and Schwartz **[12]** p. 416), $\overline{\mathrm{co}}\, C$ is compact. Thus $\overline{\mathrm{co}}\, A \subset \overline{\mathrm{co}}\, C + t\bar{S}$, being the second member convex and closed. The last inclusion shows $\gamma(\overline{\mathrm{co}}\, A) \leqq t$ and $\gamma(\overline{\mathrm{co}}\, A) \leqq \gamma(A)$. The reverse inequality is trivial.

(g) Let $\epsilon > 0$. There exist $\gamma(A) < t < \gamma(A) + \epsilon$ and $C \in \mathscr{K}(Y)$ such that $A \subset C + t\bar{S}$. This implies $A \subset (C \cup \{0\}) + t\bar{S} \subset C_1 + t\bar{S}$ where $C_1 = \overline{\mathrm{co}}(C \cup \{0\})$ is compact. Thus, for every $u \in [0, s]$, $uA \subset uC_1 + ut\bar{S} \subset sC_1 + st\bar{S}$ (since C_1 is convex and contains the origin) and we have $\bigcup_{u \in [0,s]} uA \subset sC_1 + st\bar{S}$. This implies $\gamma(\bigcup_{u \in [0,s]} uA) \leqq st$ and $\gamma(\bigcup_{u \in [0,s]} uA) \leqq s\gamma(A)$. The reverse inequality is obvious.

(h) Since $\bar{S} = \{0\} + 1\,\bar{S}$ we have $\gamma(\bar{S}) \leqq 1$. Suppose $\gamma(\bar{S}) < 1$. Then there exist $\gamma(\bar{S}) < t < 1$ and $C \in \mathscr{K}(Y)$ such that $\bar{S} \subset C + t\bar{S}$. From this

$$\bar{S} \subset \overline{\mathrm{co}}\, C + t\bar{S}$$

$$(1 - t)\bar{S} + t\bar{S} \subset \overline{\mathrm{co}}\, C + t\bar{S}$$

and, by Lemma 2.1, $(1 - t)\bar{S} \subset \overline{\mathrm{co}}\, C$. Thus $\bar{S} \subset (1 - t)^{-1}\overline{\mathrm{co}}\, C$ and since the set on the right is compact such must be $\bar{S}$. This is a contradiction since $\dim(Y) = \infty$.

(i) Let $b \in B$. Then $A \subset A + B + \{-b\}$ implies

$$\gamma(A) \leqq \gamma(A + B + \{-b\}) = \gamma(A + B) \leqq \gamma(A) + \gamma(B) = \gamma(A)$$

and (i) is true.

Denote by U a non void open subset of Y.

DEFINITION 5.3. $F\colon U \to \mathcal{K}(Y)$ is said to be *γ-Lipschitz*, with constant $k \geqq 0$, if for every $A \in \mathcal{B}(Y)$, $A \subset U$, we have $\gamma(F(A)) \leqq k\gamma(A)$.

Now we want to show that the multivalued differential of a γ-Lipschitz map is γ-Lipschitz, with the same constant.

THEOREM 5.4. *Let $F\colon U \to \mathcal{K}(Y)$ be γ-Lipschitz with constant k. Let D_x be the differential of F at $x \in U$. Then D_x is γ-Lipschitz with the same constant k.*

Proof. There exists $\delta > 0$ such that $d(F(x+h), F(x) + D_x(h)) = o(h)$ if $\|h\| < \delta$. This implies

$$F(x) + D_x(h) \subset F(x+h) + (o(h) + \|h\|^2)S \quad \text{if} \quad \|h\| < \delta.$$

Let $A \in \mathcal{B}(Y)$, $A \subset U$. Let $t > 0$ be such that $t\|A\| < \delta$. Let $\sigma(t) = \sup\{o(h)\colon h \in tA\}$. It is easy to see that $\lim_{t\to 0} \sigma(t)/t = 0$. Let $h \in tA$. We have

$$F(x) + D_x(h) \subset F(x + tA) + [\sigma(t) + t^2\|A\|^2]S$$
$$F(x) + D_x(tA) \subset F(x + tA) + [\sigma(t) + t^2\|A\|^2]S.$$

Using the properties of γ

$$\begin{aligned}\gamma(D_x(tA)) &= \gamma(F(x) + D_x(tA))\\ &\leqq \gamma(F(x+tA)) + \sigma(t) + t^2\|A\|^2\\ &\leqq k\gamma(tA) + \sigma(t) + t^2\|A\|^2.\end{aligned}$$

Thus

$$\gamma(D_x(A)) \leqq k\gamma(A) + \sigma(t)/t + t\|A\|^2$$

and, letting $t \to 0$, the desired result follows.

COROLLARY 5.5 (Daneš [4], Nussbaum [19], Sadovskiĭ [22]). *Let $F\colon U \to Y$ be a single valued γ-Lipschitz map with constant k. Let F'_x be the Fréchet differential of F at $x \in U$. Then F'_x is γ-Lipschitz with the same constant k.*

DEFINITION 5.6. Let $U = \{x \in X\colon \|x\| > r\}$, $r > 0$. $F\colon U \to \mathscr{B}(Y)$ is said to be *differentiable at infinity* if there exist a map $D_\infty\colon X \to \mathscr{C}_0(Y)$, which is u.s.c. and homogeneous, and a number $\delta > r$ such that

$$d(F(x), D_\infty(x)) = o(x) \quad \text{when} \quad \|x\| > \delta,$$

and $\lim_{x\to\infty} o(x)/\|x\| = 0$. D_∞ is called the *asymptotic differential* of F.

DEFINITION 5.7. (Krasnosel'skiĭ [15] p. 207). Let $F\colon U \to Y$ be a continuous single valued map, U being as in the above definition. Let there exist a linear map F'_∞ and a number $\delta > r$ such that $F(x) = F'_\infty(x) + z(x)$, if $\|x\| > \delta$, and $\lim_{x\to\infty} z(x)/\|x\| = 0$. Then F is said to be *asymptotically linear* and F'_∞ is called the *asymptotic derivative* of F.

THEOREM 5.8. *The asymptotic differential D_∞ of $F\colon U \to \mathscr{B}(Y)$ if exists is unique.*

Proof. Similar to that of Theorem 3.1.

THEOREM 5.9. *Let $U = \{y \in Y : \|y\| > r\}$, $r > 0$. Let $F\colon U \to \mathscr{K}(Y)$ be γ-Lipschitz with constant k. Let D_∞ be the asymptotic differential of F. Then D_∞ is γ-Lipschitz with the same constant k.*

Proof. Similar to that of Theorem 5.4.

Since a single valued continuous map is completely continuous if and only if it is γ-Lipschitz with constant $k = 0$, we have

COROLLARY 5.10 (Krasnosel'skiĭ [15] p. 207). *The asymptotic derivative F'_∞ of a completely continuous single valued map $F\colon U \to Y$ is completely continuous.*

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