PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 2, 1976

LEVEL SETS OF POLYNOMIALS IN n REAL VARIABLES

MORRIS MARDEN AND PETER A. MCCOY

The methods used in studying the zeros of a polynomial in a single complex variable are here adapted to investigating the level surfaces of a real polynomial in E^n, with respect to their intersection and finite or asymptotic tangency with certain cones. Special attention is given to the equipotential surfaces generated by an axisymmetric harmonic polynomial in E^3.

A principal interest is the application of reasoning used by Cauchy [**2**, p. 123] in obtaining bounds on the zeros of polynomials in one complex variable. We thereby seek the level sets

$$L_\alpha(H) = \{X \in E^n \mid H(X) = \alpha\}$$

generated from the real polynomials

$$H(X) - \alpha = \sum_{0 \leqq j_1 + \cdots + j_n \leqq n} \alpha_{j_1 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, \tag{1}$$

$$X = (x_1, x_2, \cdots, x_n),\ r = |X| = [x_1^2 + x_2^2 + \cdots + x_n^2]^{1/2}.$$

It is convenient to introduce direction numbers $\lambda_j = x_j r^{-1}$, $1 \leqq j \leqq n$, connected by $\lambda_1^2 + \cdots + \lambda_n^2 = 1$ and cones $\Lambda_j : \lambda_j =$ constant, about the jth axis. On the intersection of the cones Λ_j, these polynomials become

$$H(r\Lambda_j) - \alpha = \sum_{k=0}^{n} r^k A_k(\Lambda_j)$$

where

$$A_k = A_k(\Lambda_j) = \sum_{j_1 + \cdots j_n = k} \alpha_{j_1 \cdots j_n} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \qquad 0 \leqq k \leqq n.$$

At the origin the level set $L_\alpha(H)$ has νth order contact with Λ_j, if $A_k(\Lambda_j) = 0$ for $0 \leqq k \leqq \nu - 1$ but $A_\nu(\Lambda_j) \neq 0$ and $A_n(\Lambda_j) \neq 0$. For such sets we introduce the ratios

$$M_\nu = M_\nu(\Lambda_j) = \max_{\nu \leqq k \leqq n-1} |A_k / A_n|$$

$$m_\nu = m_\nu(\Lambda_j) = \min_{\nu+1 \leqq k \leqq n} (|A_\nu|/(|A_\nu| + |A_k|))$$

$$\mu_\nu = \mu_\nu(\Lambda_j) = \max_{\nu+1 \leqq k \leqq n} |A_k/A_\nu|.$$

Then, by considering points common to the level set $L_\alpha(H)$ and the cone Λ_j exterior to the unit ball $r > 1$ (about the origin), we deduce an inequality

$$(2) \qquad |H(r\Lambda_j) - \alpha| > |A_n| r^n - \sum_{k=\nu}^{n-1} |A_k| r^k \geqq |A_n| r^n \left[1 - M_\nu \sum_{k=1}^{n-\nu} r^{-k}\right] =$$

$$(2a) \qquad |A_n| r^n [1 - (M_\nu(1 - r^{\nu-n})(r-1)^{-1}] > |A_n| r^n (r - 1 - M_\nu)/(r-1)$$

from which it is clear that, if $r \geqq 1 + M_\nu$, $L_\alpha(H)$ does not intersect Λ_j. Likewise, if we consider the reciprocal polynomial associated with (1), derived by setting $1/r = \zeta > 1$, the inequalities

$$(3) \qquad |\zeta^n[H(\zeta^{-1}\Lambda_j) - \alpha]| = \left|\sum_{k=\nu}^{n} \zeta^{n-k} A_k\right|$$

$$\geqq \zeta^{n-\nu}|A_\nu| - \sum_{k=\nu+1}^{n} \zeta^{n-k}|A_k|$$

$$\geqq \zeta^{n-\nu}|A_\nu| \left[1 - \mu_\nu \sum_{k=\nu+1}^{n} \zeta^{\nu-k}\right]$$

$$(3a) \qquad = \zeta^{n-\nu}|A_\nu|[1 - (\mu_\nu(1 - \zeta^{\nu-n}))(\zeta - 1)^{-1}]$$

$$> \zeta^{n-\nu}|A_\nu|(\zeta - 1 - \mu_\nu)/(\zeta - 1)$$

imply that $H(\zeta^{-1}\Lambda_j) \neq \alpha$ for $\zeta \geqq 1 + \mu_\nu$. Thus we infer that $H(r\Lambda_j) \neq \alpha$ for

$$r \leqq (1 + \mu_\nu)^{-1} = m_\nu,$$

which brings us to

THEOREM 1. *If the level set $L_\alpha(H)$ has νth order contact with the cone Λ_j at the origin and if it intersects the cone at any additional finite points, then it does so at a distance r from the origin where*

$$(4) \qquad m_\nu(\Lambda_j) < r < 1 + M_\nu(\Lambda_j).$$

By use of inequalities (3a) and (2a), we replace inequality (4) in Theorem 1 by

$$r_1 \leqq r \leqq r_2 \tag{4$'$}$$

where r_1 is the larger positive root of the equation

$$1-(1+\mu_\nu)r+\mu_\nu r^{n-\nu+1}=0$$

and r_2 the larger positive root of the equation

$$r^{n+1-\nu}-(1+M_\nu)r^{n-\nu}+M_\nu=0,$$

$r=1$ being a root of both equations.

A natural question arising from this theorem is that of determining the point of tangency of the level sets with the cones Λ_j. Let us consider the kth term in the polynomial (1),

$$r^k A_k(r^{-1}X)=\sum_{j_1+\cdots+j_n=k}\alpha_{j_1\cdots j_n}x_1^{j_1}\cdots x_n^{j_n}.$$

As this sum is composed of homogeneous polynomials of degree k, we may apply Euler's Identity [**1**, p. 141] to find that

$$X\cdot\nabla[r^kA_k(r^{-1}X)]=kr^kA_k(r^{-1}X), \tag{5}$$

where the left side is the scalar product of vector X and the gradient of the bracket. On account of this relation, the orthogonality condition

$$X\cdot\nabla H(X)=0$$

becomes

$$\sum_{k=\nu}^{n} kr^kA_k(\Lambda_j)=0. \tag{6}$$

Let us define

$$m^*(\Lambda_j)=\min_{2\leqq k\leqq n}\,[(kA_k+A_\nu)/(A_\nu)]$$

$$M^*(\Lambda_j)=\max_{\nu\leqq k\leqq n-1}\,(kA_k/nA_n).$$

Theorem 1 and equation 6 lead to

COROLLARY 1.1. *If the level set $L_\alpha(H)$ has νth order contact with the* *cone Λ_j at its vertex and is tangent to the cone at a positive distance r from the origin, then*

$$m_\nu^*(\Lambda_j) < r < 1 + M_\nu^*(\Lambda_j). \tag{7}$$

As equation (6) may be viewed as

$$\partial[H(X) - \alpha]/\partial r = 0, \tag{8}$$

we may use Rolle's Theorem to conclude

COROLLARY 1.2. *If the ray $(\lambda_1, \cdots, \lambda_n) \in \bigcap_{j=1}^n \Lambda_j$, the level surface $L_\alpha(H)$ has a finite tangental contact point between successive pairs of intersections of $L_\alpha(H)$ with the ray.*

The influence of the coefficient $A_n = A_n(X)$ on the structure of $L_\alpha(H)$ near infinity is found by selecting a sequence of points $\{X_k\}$, $r_k = |X_k| \to \infty$, such that $H(X_k) = \alpha$. Each of these points is located on a cone $\Lambda_j^{(k)}$. This leads to the bound $r_k < 1 + M_\nu(\Lambda_j^{(k)})$ and the limit $A_n(\Lambda_j^{(k)}) \to 0$ due to $r_k \to \infty$. From the continuity of A_n, the sequence $\Lambda_j^{(k)}$ converges to the cone Λ_j, where $A_n(\Lambda_j) = 0$. We conclude that $L_\alpha(H)$ is asymptotic to a set imbedded in the null cones of A_n. Level sets which are asymptotic to these cones are unbounded. Hence

THEOREM 2. *The level set $L_\alpha(H)$ is unbounded if and only if it is asymptotic to a set imbedded in a cone Λ_j such that $A_n(\Lambda_j) = 0$.*

Let us turn our attention to the influence of the algebraic sign of the coefficients of these polynomials on their level sets. It is of course clear that, if a level set $L_\alpha(H)$ has contact with a cone Λ_j on p spheres, then $L_\alpha(H)$ has contact with these same spheres on each cone Λ_l for which the coefficients A_k agree term wise. A more explicit conclusion is obtained thru the use of Descartes' rule of signs in

THEOREM 3. *If the number of variations in sign of the terms in the sequence of coefficients*

$$A_0(\Lambda_j), \cdots, A_n(\Lambda_j) \tag{9}$$

generated from the polynomial $H(X) - \alpha$ on the cone Λ_j is p, then the number of intersections of surface $L_\alpha(H)$ and cone Λ_j is p or is less than p by an even positive integer. If the number of permanences in sign for (9) *is q, then surface $L_\alpha(H)$ and cone $\Lambda_j^* = (-\Lambda_j)$ have at most of q intersections.*

A sufficient condition for such an intersection is found in

COROLLARY 3.1. *If Λ_j is a cone for which the signs of the coefficients $A_0(\Lambda_j)$ and $A_n(\Lambda_j)$ are opposite, then the level set $L_\alpha(H)$ has positive contact with Λ_j.*

Additional connections between these level sets and the coefficients A_k are found in the equation

$$H(r\lambda\Lambda_l)-\alpha=\sum_{k=0}^{n}A_k(\Lambda_l)\lambda^k r^k=\sum_{k=0}^{n}A_k(\Lambda_j)r^k=H(r\Lambda_j)-\alpha$$

which hold on cones Λ_l and Λ_j for which $\lambda^k A_k(\Lambda_l)=A_k(\Lambda_j)$, $A_k(\Lambda_j)$, $0\leqq k\leqq n$, for some real constant λ. This equation establishes a relation between the intersections of level sets with cones about jth and lth axes, as stated in

THEOREM 4. *Let the level set $L_\alpha(H)$ meet the cone Λ_j at the positive distances $r_1,\cdots,r_p$. Then $L_\alpha(H)$ meets each cone Λ_l for which there exists a positive constant λ such that*

$$\lambda^k A_k(\Lambda_l)=A_k(\Lambda_j),\qquad 0\leqq k\leqq n$$

at the distances $\lambda r_1,\lambda r_2,\cdots,\lambda r_p$.

Let us now focus our attention upon equipotential surfaces generated by axisymmetric harmonic polynomials in E^3. These surfaces arise when the coefficients $A_k(\Lambda_j)$ reduce to $P_k(\cos\theta)$, the Legendre polynomial of degree k in $\cos\theta=xr^{-1}$ and the polynomial $H(X)-\alpha$ becomes the real harmonic polynomial of degree n

$$H(r,\theta)-\alpha=\sum_{k=0}^{n}a_k r^k P_k(\cos\theta),\qquad a_n\neq 0. \tag{10}$$

Elementary reasoning based on the fact that on the cone $\theta=\theta_0$, $H(r,\theta)-\alpha$ is a polynomial of degree n in the variable r leads us to geometrical properties of these surfaces which are summarized in

THEOREM 5. *For each axisymmetric harmonic polynomial H, every finite point of E^3 belongs to some equipotential of H. In particular, if the equipotential surfaces $L_\alpha(H)$ and $L_\beta(H)$ have contact with a cone on the spheres $r=r_0$ and $r=R_0$, respectively, then for each choice of λ between α and β the equipotential surface $L_\lambda(H)$ has contact with this cone between these spheres.*

Although equipotential surfaces generated from distinct harmonic polynomials of degree n with common zeroth order contact at the origin have no more than $n-1$ common circles of intersection on any fixed cone, near infinity these surfaces have nearly identical structure. To bring forth this asymptotic property, we apply Theorem 2 to equation (10) to conclude

THEOREM 6. *An equipotential surface generated from an axisymmetric harmonic polynomial of degree n is unbounded if and only if it is asymptotic to at least one of the cones* $\theta = \theta_j$ *for which* $P_n(\cos\theta_j) = 0$, $1 \leqq j \leqq n$.

Having established these properties of equipotentials, let us now estimate the growth of these surfaces in a neighborhood of infinity. To accomplish this, consider an unbounded equipotential surface generated from an nth degree harmonic polynomial with νth order contact at the origin.

At large distances from the origin this surface either coincides with or approaches some cone $\theta = \theta_j$, $P_n(\cos\theta_j) = 0$. In the latter case assuming the equipotential meets the cone $\theta = \theta_0$ for $\theta_0 > \theta_j$ we select $\epsilon(\epsilon > 0)$ sufficiently small so that $P_n'(\cos\theta)P_n(\cos\theta) \neq 0$ for $0 \leqq \theta_j < \theta + \epsilon < \pi$. Let us now apply the Mean Value Theorem on the interval $J_\theta = [\theta_j, \theta]$, $\theta_j < \theta < \theta_j + \epsilon$ to find $\eta \in J_\theta$ so that

$$P_n'(\cos\eta) = [P_n(\cos\theta) - P_n(\cos\theta_j)]/(\cos\theta - \cos\theta_j).$$

We then use the relations

$$\begin{aligned} -\cos\theta + \cos\theta_j &= 2\sin((\theta+\theta_j)/2)\sin((\theta-\theta_j)/2) \\ &> (\sin\theta_j)(\theta-\theta_j)\sin(\epsilon/2)/\epsilon \end{aligned}$$

to deduce that

$$\begin{aligned} \left|\frac{P_k(\cos\theta)}{P_n(\cos\theta)}\right| &= \left|\frac{P_k(\cos\theta)}{(\cos\theta-\cos\theta_j)}\,\frac{(\cos\theta-\cos\theta_j)}{(P_n(\cos\theta)-P_n(\cos\theta_j)}\right| \\ &\leqq \frac{K(\epsilon)}{(\theta-\theta_j)|P_n'(\cos\eta)|} \end{aligned}$$

From this estimate we find that on the equipotential surface $L_\alpha(H)$,

$$r(\theta) \leqq 1 + \max_{\nu \leqq k \leqq n-1} \left|\frac{a_k P_k(\cos\theta)}{a_n P_n(\cos\theta)}\right| < 1 + M_\epsilon/(\theta - \theta_j)$$

for $\theta_j < \theta < \theta_j + \epsilon$ and $\nu + 1 < n$ establishing

THEOREM 7. *If an equipotential surface generated by an axisymmetric harmonic polynomial in E^3 is unbounded and in neighborhood of infinity meets the cone $\theta = \theta_0$ at a distance $r = r(\theta_0)$, then*

$$r(\theta_0) = \mathbf{0}\{\max_{1 \leqq j \leqq n} (1/|\theta_0 - \theta_j|)\}$$

where $P_n(\cos\theta_j) = 0$, $1 \leqq j \leqq n$ for $\theta_0 \neq \theta_j$.

We now turn to some analytic results on the zeros $r_1, \cdots, r_{n-\nu}$ of the function

$$H(r, \theta_0) - \alpha = \sum_{k=\nu}^{n} a_k r^k P_k(\cos\theta_0), \qquad r > 0,$$

from which we infer that

$$H(r, \theta_0) - \alpha = a_n P_n(\cos\theta_0) r^\nu (r^{n-\nu} + s_{n-1} r^{n-\nu-1} + \cdots + s_{\nu+1} r + s_\nu)$$

where $s_k = a_k P_k(\cos\theta_0)/a_n P_n(\cos\theta_0)$, $\nu \leqq k \leqq n-1$. The coefficients s_k of the equation

$$r^{n-\nu} + s_{n-1} r^{n-\nu-1} + \cdots + s_{\nu+1} r + s_\nu = 0$$

are symmetric functions of its roots r_k. Thus,

$$\begin{aligned} s_{n-1} &= -(r_1 + \cdots + r_{n-\nu}) \\ s_{\nu+1} &= (-1)^{n-\nu-1}(r_2 r_3 \cdots r_{n-\nu} + r_1 r_3 \cdots r_{n-\nu} + \cdots + r_1 r_2 \cdots r_{n-\nu-1}) \\ s_\nu &= (-1)^{n-\nu}(r_1 \cdots r_{n-\nu}). \end{aligned}$$

From these symmetric functions we find

$$(11) \qquad \begin{aligned} r_1 + \cdots + r_{n-\nu} &= -(a_{n-1} P_{n-1}(\cos\theta_0)/a_n P_n(\cos\theta_0)) \\ 1/r_1 + \cdots + 1/r_{n-\nu} &= -(a_{\nu+1} P_{\nu+1}(\cos\theta_0)/a_\nu P_\nu(\cos\theta_0)). \end{aligned}$$

which bring us to

THEOREM 8. *Let the equipotential surface generated by the harmonic polynomial*

$$H(r, \theta) - \alpha = \sum_{k=\theta}^{n} a_k r^k P_k(\cos\theta)$$

having νth order contact with the cone $\theta = \theta_0$ at the origin meet this cone in $n - \nu$ additional finite circles at the distances $r_1, \cdots, r_{n-\nu}$. Let M_a, M_g and M_h be respectively the arithmetic, geometric and harmonic means of $r_1 \cdots r_{n-\nu}$ and let $b_k = |a_k / a_n|$ and $\tau_k(\theta_0) = |P_k(\cos\theta_0)/P_n(\cos\theta_0)|$. If $P_n(\cos\theta_0)P_{\nu+1}(\cos\theta_0) \neq 0$ and $a_n a_{\nu+1} \neq 0$, then

$$M_a = (n - \nu)^{-1} b_{n-1} \tau_{n-1}(\theta_0),$$
$$M_g = [b_\nu \tau_\nu(\theta_0)]^{1/(n-\nu)},$$
$$M_h = (n - \nu)(b_\nu / b_{\nu+1})[\tau_\nu(\theta_0)/\tau_{\nu+1}(\theta_0)].$$

Bounds on the circles of intersection having maximum and minimum radii are found in

COROLLARY 8.1. *The maximum circle of intersection of the equipotential surface $L_\alpha(H)$ and the cone $\theta = \theta_0$ lies exterior to the sphere about the origin with a radius max $\{M_a, M_h\}$ and the minimum circle of intersection not on the origin lies interior to the sphere about the origin with a radius* $\min\{M_a, M_h\}$.

When contact at the origin is zeroth order, from the facts that the distances $r_1, \cdots, r_{n-\nu}$ are positive, $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$ and equations (11) we deduce

COROLLARY 8.2. *For an equipotential surface $L_\alpha(H)$ having zeroth order contact with the cone $\theta = \theta_0$ at the origin to intersect this cone in n finite circles, it is necessary that $0 \leqq \theta_0 < \pi/2$ if $a_1/a_0 < 0$ and $\pi/2 < \theta_0 < \pi$ if $a_1/a_0 > 0$.*

REFERENCES

1. O. D. Kellogg, *Foundations of Potential Theory*, Frederick Ungar Pub. Co., New York, 1929.
2. M. Marden, *Geometry of Polynomials*, Math. Surveys, No. **3**, Amer. Math. Soc., Providence, R.I., 1966.
3. G. Szegö, *Orthogonal Polynomials*, Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, R.I., 1939.

Received October 3, 1974. Research of the first author partly supported by a grant of the University of Wisconsin–Milwaukee Graduate School from W.A.R.F. funds.

CALIFORNIA POLYTECHNIC STATE UNIVERSITY — SAN LIUS OBISPO
AND
U.S. NAVAL ACADEMY, ANAPOLIS