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LOCALIZATION IN FULLY BOUNDED NOETHERIAN RINGS

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The paper defines and studies links between the prime ideals of a noncommutative fully bounded noetherian ring, and their role as obstructions to localizability: a localization with properties similar to those of the localization of a commutative ring at a prime ideal, can be constructed if and only if the equivalence class determined by the links is finite. For rings with polynomial identity, the links are described in more detail via an inductive procedure over the PI-degree, and several examples are constructed.

I. Preliminaries. The attempt to localize a noncommutative noetherian ring at its prime ideals leads to the study of *classical* sets of prime ideals (i.e. finite incomparable sets $\{P_1, \cdots, P_n\}$ such that the associated torsion theory has the Ore condition and the Artin Rees property) and in particular of *clans* (i.e., minimal classical sets). It was shown in [16] that a prime ideal belongs to at most one clan, and that the existence of *enough clans* (i.e. each prime ideals belongs to a clan) amounts to localizability at all prime ideals; cf. this paper for more detail.

The very existence of *nontrivial clans* (i.e., clans with more than one member) is evidence of the presence of links between prime ideals which constitute obstructions to localizability. The purpose of this paper is to define and study these links explicitely.

To do so, we restrict attention to FBN-rings (right- and left-fully bounded noetherian rings), where we have these fundamental results of [9] on Krull dimension κ available: Every uniform module is α-smooth for some ordinal α, i.e., all nonzero submodules have the same Krull dimension α. Every finitely generated α-smooth module has an (essentially unique) α-composition series (called basic series in [9]). The α-composition factors, also called α-critical modules, are characterized as the uniform nonsingular R/P-modules, for the various prime ideals P of the ring R with $\kappa(R/P) = \alpha$. Any $R-R$-bimodule which is finitely generated on both sides, has the same Krull dimension on both sides.

The FBN-assumption is natural for other reasons: It makes the Gabriel correspondence between spec R and the collection of indecomposable injective modules one-to-one, via $E(R/P) = E_P^{n_P}$; i.e., prime ideals "separate modules" [11]. It ensures that any localizable semiprime ideal is automatically classical (combine [9], 3.6 with [8], 4.4(5)). Recall also that primitive ideals of FBN-rings are maximal with simple artinian factorrings, and that noetherian PI-rings (i.e. noetherian rings with polynomial identity) are FBN.

A useful criterion from [8]: A semiprime ideal S of a right-noetherian ring R is right-localizable (i.e., the multiplicative set $\mathscr{C}(S)$ has the right-Ore condition) if and only if $\mathscr{C}(S)$ operates regularly on the injective hull $E(R/S)$.

Some terminology: module means right-module; but ideal, noetherian, Ore etc. means left- *and* right-ideal, -noetherian, -Ore etc., unless specified otherwise. $\kappa(M)$ is the Krull dimension [5], and $E(M)$ is the injective hull, of the module M. $J(R)$ is the Jacobson radical, and $\mathscr{C}(S)$ is the multiplicative set of modulo S regular elements, of a ring R. The collection of prime ideals P of an FBN-ring R with $\kappa(R/P) = \alpha$ will be called the α-stratum of spec R: note that distinct prime ideals in the same stratum are incomparable ([5], 7.2).

II. FBN-rings.

LEMMA 1. *Let R be an* FBN*-ring, P and Q prime ideals in the α-stratum, and $\bar{R} = R/QP$. Then $\mathscr{C}(\bar{Q} \cap \bar{P})$ is Ore in $\bar{R}$, and the corresponding quotient ring A is artinian, with two maximal ideals PA and QA and $J(A)^2 = 0$.*

Proof (cf. [9], proof of 5.2). Let $E = E(R/(Q \cap P)_R)$ which is α-smooth, and $F = \mathrm{ann}_E(QP) = E(\bar{R}/\bar{Q} \cap \bar{P})_{\bar{R}})$. $\mathscr{C}(\bar{Q} \cap \bar{P})$ operates regularly on F: if $0 \neq x \in F$ and $c \in \mathscr{C}(Q \cap P)$, let $\overline{xR}$ be the top factor of an α-composition series of xR; then $\mathrm{ann}_R(\overline{xR}) = P'$ is a prime ideal with $\kappa(R/P') = \alpha$. $FQP = 0$ implies $QP \subset P'$ hence Q or $P \subset P'$ hence Q or $P = P'$ since all three lie in the α-stratum; hence $c \in \mathscr{C}(Q \cap P) = \mathscr{C}(Q) \cap \mathscr{C}(P) \subset \mathscr{C}(P')$. Thus c operates regularly on the non-singular R/P'-module $\overline{xR}$, hence $xc \neq 0$.

By the criterion cited in the preliminaries, $\mathscr{C}(\bar{Q} \cap \bar{P})$ is an Ore-set. The quotient ring A is noetherian, semilocal with maximal ideals PA and QA and $J(A) = (Q \cap P)A$, hence $J(A)^2 = 0$ since $(Q \cap P)^2 \subset QP$; hence A is artinian.

REMARK. The kernel of $\bar{R} \to \bar{A}$, i.e. the torsion radical of $\bar{R}$ for the $(\bar{Q} \cap \bar{P})$-torsion theory, is $\mathrm{ann}_{\bar{R}}(F) = \mathrm{ann}_{\bar{R}}\, \mathrm{ann}_E(QP) =$

$\text{ann}_R \text{ann}_E (QP)/QP$, since F is an injective cogenerator for A.

PROPOSITION 2. *In the situation of Lemma* 1, *the following are equivalent*:

(1) *there is a short exact sequence* $0 \to X \to Y \to Z \to 0$ *of* R-*modules such that* Y *is uniform*, X *and* Z *are critical, and* $\text{ann}_R(X) = P$, $\text{ann}_R(Z) = Q$;

(2) $\text{Ext}^1_A(A/QA, A/PA) \neq 0$;

(3) $\text{Tor}^A_1(A/QA, A/PA) \neq 0$;

(4) $(Q \cap P)I \subset QP$ *implies* $I \subset P$, *for every ideal* I *of* R.

DEFINITION. If the conditions of Proposition 2 hold, we say that a (*right*-) *link* exists from P to Q, and write $P \rightsquigarrow Q$.

REMARK. Since (3) is left-right-symmetric (as $IA = AI$ for every ideal I of R), these conditions are also equivalent to their left-right-analogues with the roles of P and Q interchanged, i.e., to the existence of a left-link from Q to P.

Proof. If $0 \to X \to Y \to Z \to 0$ is given as in (1), then up to isomorphism $X \subset Y \subset E_P$ since X is a uniform nonsingular R/P-module and since Y is uniform; similarly $Z \subset E_Q$. Under localization, $0 \to X \otimes A \to Y \otimes A \to Z \otimes A \to 0$ stays exact, $X \otimes A$ and $Z \otimes A$ are simple A-modules of type A/PA and A/QA respectively, and the sequence is nonsplit since $Y \otimes A$ stays uniform. Hence $\text{Ext}^1_A(A/QA, A/PA) \neq 0$, i.e., (2).

Conversely if (2) holds, then there is a nonsplit exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, where X and Z are simple with annihilators PA and QA respectively. Nonsplitting forces Y to be uniform as A-module, hence X, Y, and Z are uniform as R-modules; and the annihilators of X and Z in R are just P and Q, since these prime ideals are closed in the $(Q \cap P)$-torsion theory. Moreover X and Z are $(Q \cap P)$-torsionfree hence nonsingular R/P- respectively R/Q-modules, hence critical; and we have recovered the situation (1).

The equivalence of (2) and (3) follows from the homological duality isomorphism ([1], 120), using that $\text{hom}_{A/PA}(-, A/PA)$ is a duality between finitely generated right- and left-modules over the simple artinian ring A/PA, carrying the right-module A/PA into the left-module A/PA.

One easily computes $\text{Tor}^A_1(A/QA, A/PA) = (QA \cap PA)/QPA = (Q \cap P)A$, hence $\text{Tor}^A_1(A/QA, A/PA) = 0$ if and only if there is $c \in \mathscr{C}(Q \cap P)$ with $(\bar{Q} \cap \bar{P})c = 0$ in $\bar{R} = R/QP$. But this is equivalent to the existence of an ideal $I \not\subset P$ with $(Q \cap P)I \subset QP$ (take $I = RcR + P$ and use the Lemma in [7]), establishing the equivalence of (3) and (4).

REMARKS. (1) One can deduce that the existence of a self-link from P to P is equivalent to $P \supsetneqq P^{(2)}$, the symbolic square of P [13]. Analogously the existence of a link from P to Q may be interpreted to mean that $Q \cap P$ is strictly larger than an appropriately defined symbolic product.

(2) If P and Q are maximal ideals (i.e., in the O-stratum), then $A = R/QP$ (since the latter is already artinian), and a link $P \rightsquigarrow Q$ exists if and only if $Q \cap P \supsetneqq QP$.

(3) The links between finitely many prime ideals $P_1, \cdots, P_n$ (in the same stratum of spec R) can be visualized in an artinian ring: the proofs of Lemma 1 and Proposition 2 go through for $\bar{R} = R/S^2$ where $S = P_1 \cap \cdots \cap P_n$, and its artinian quotient ring A with maximal ideals P_iA and $J(A)^2 = 0$. Hence a link $P_i \rightsquigarrow P_j$ exists in R if and only if $P_jA \cap P_iA \supsetneqq P_jP_iA$.

The structure of any artinian ring A with $J(A)^2 = 0$, is given by a finite family of primary (hence local up to Morita equivalence) artinian rings (A_i, M_i) with $M_i^2 = 0$ and of bimodules X_{ij} $(i \neq j)$ over the simple artinian rings A_i/M_i and A_j/M_j, with zero-multiplication $X_{ij}X_{jk} = 0$ [14]. In such a representation a link $P_i \rightsquigarrow P_j$ exists if and only if $X_{ji} \neq 0$. Since for suitable A_i the X_{ij} can be chosen zero or nonzero at will ([15], Lemma 1), any finite directed graph can be realized as the graph of links between the prime ideals of an artinian ring.

Note on the other hand that for a bounded HNP-ring, the occuring graphs are very restricted: they are directed circuits, corresponding to the cycles of [3].

LEMMA 3. *The following are equivalent for prime ideals P, Q in the α-stratum of an* FBN-*ring R:*

(1) *there is a nonzero homomorphism ϕ: $E_P \rightarrow E_Q$;*

(2) *there exists $e \in E_P$ with $\mathrm{ann}_R(eR) \subset Q$;*

(3) *Q is the annihilator of an α-composition factor of E_P.*

Proof. If $0 \neq \phi$: $E_P \rightarrow E_Q$ is given, then there is $e \in E_P$ with $\phi(e) \neq 0$ and $\mathrm{ann}_R(\phi(e)R) = Q$. Then $\phi(e)\,\mathrm{ann}\,(eR) = 0$ hence $\mathrm{ann}\,(eR) \subset Q$.

If $e \in E_P$ with $\operatorname{ann}(eR) \subset Q$ is given, consider an α-composition series $0 = B_0 \subsetneqq \cdots \subsetneqq B_n = eR$; then the $\operatorname{ann}_R(B_i/B_{i-1}) = P_i$ are prime ideals of R with $\kappa(R/P_i) = \alpha$. Then $eR \cdot P_n \cdots P_1 = 0$ hence $P_n \cdots P_1 \subset \operatorname{ann}(eR) \subset Q$ hence $P_i \subset Q$ for some i.

If finally Q is the annihilator of the α-composition factor A/B of E_P, then A/B is a uniform nonsingular R/Q-module hence embedds into E_Q, and the natural map $A \to A/B$ extends to a nonzero homomorphism $E_P \to E_Q$.

PROPOSITION 4. *Let P, Q be distinct prime ideals in the α-stratum of an* FBN-*ring R. Then a finite chain of links $P \rightsquigarrow P_1 \rightsquigarrow \cdots \rightsquigarrow Q$ exists if and only if there is a finite chain of nonzero homomorphisms $E_P \to E_{P'} \to \cdots \to E_Q$.*

Proof. If $P \rightsquigarrow Q$ is given, one has $0 \to X \to Y \to Z \to 0$ with $X \subset Y \subset E_P$ and $Z \subset E_Q$ as observed in the proof of Proposition 2, and $Y \to Z$ extends to a nonzero homomorphism $E_P \to E_Q$.

Conversely, associate with every Q' in the α-stratum, for which a nonzero homomorphism $E_P \to E_{Q'}$ exists, an α-composition series of shortest length for a finitely generated submodule of E_P, for which Q' is the annihilator of a factor (necessarily the top factor); this is possible by Lemma 3. We proceed by induction over this shortest length.

Thus let $0 = B_0 \subsetneqq \cdots \subsetneqq B_{n-2} \subsetneqq B_{n-1} \subsetneqq B_n$ be a α-composition series of shortest length in E_P with $\operatorname{ann}(B_n/B_{n-1}) = Q$. The inclusion $B_{n-1}/B_{n-2} \subset B_n/B_{n-2}$ is essential, since otherwise

$$B_{n-1}/B_{n-2} \oplus B/B_{n-2} \subset B_n/B_{n-2}$$

hence $B/B_{n-2} \neq 0$ embedds into B_n/B_{n-1} hence is critical with annihilator Q, producing an α-composition series $0 = B_0 \subsetneqq \cdots \subsetneqq B_{n-2} \subsetneqq B$ of shorter length for Q, a contradiction. Thus

$$E(B_n/B_{n-2}) = E(B_{n-1}/B_{n-2}) = E_{Q'}$$

where $Q' = \operatorname{ann}(B_{n-1}/B_{n-2})$ has an α-composition series

$$0 = B_0 \subsetneqq \cdots \subsetneqq B_{n-2} \subsetneqq B_{n-1}$$

of shorter length hence a finite chain $P \rightsquigarrow P_1 \cdots \rightsquigarrow Q'$ by induction. Moreover a link $Q' \rightsquigarrow Q$ exists from the exact sequence

$$0 \longrightarrow B_{n-1}/B_{n-2} \longrightarrow B_n/B_{n-2} \longrightarrow B_n/B_{n-1} \longrightarrow 0$$

and Proposition 2 (4), and we are done.

REMARK. The proceeding proof motivates the notation $P \rightsquigarrow\rightsquigarrow Q$, if $\kappa(R/P) = \kappa(R/Q)$ and if a nonzero homomorphism $E_P \to E_Q$ exists. Note that all the intermediate primes in both chains of Proposition 4 belong automatically to the α-stratum.

By Proposition 4, $P \rightsquigarrow Q$ and $P \rightsquigarrow\rightsquigarrow Q$ generate the same equivalence relation on spec R. By the remark following Proposition 2, the very same equivalence relation arises from left-links. The equivalence class of P in this equivalence relation will be written class (P).

THEOREM 5. *A prime ideal P of an* FBN-*ring R belongs to a clan if and only if* class (P) *is finite; then* clan (P) = class (P).

Proof. Assume class (P) to be finite, and put $S = \bigcap$ class (P). If S is not right-localizable, there exists $c \in \mathscr{C}(S)$ and $0 \neq e \in E_Q$ for some $Q \in$ class (P) with $ec = 0$, by Jategaonkar's criterion. If the top factor $\overline{eR}$ of an α-composition series of eR is annihilated by Q', then $Q' \in$ class (Q) = class (P) by Lemma 3, hence $c \in \mathscr{C}(S) \subset \mathscr{C}(Q')$, hence c operates regularly on the nonsingular R/Q'-module $\overline{eR}$; a contradiction to $ec = 0$. Thus and by left-right-symmetry, S is localizable hence classical, hence a union of clans ([16], Theorem 4); in particular P belongs to a clan.

Assume conversely that P belongs to a clan; put $S = \bigcap$ clan (P) and select $P' \in$ clan (P) with maximal $\kappa(R/P') = \alpha$. If $P' \rightsquigarrow\rightsquigarrow Q$ then by Lemma 3 there is $e \in E_{P'}$ with ann $(eR) \subset Q$; by [8], 4.4(5) there is n with $eS^n = 0$. Thus $S^n \subset$ ann$(eR) \subset Q$ hence there is $P'' \in$ clan(P) with $P'' \subset Q$. Consequently

$$\kappa(R/P'') \geqq \kappa(R/Q) = \kappa(R/P') = \alpha\ ;$$

and by maximality of α, $\kappa(R/P'') = \alpha$ hence $Q = P'' \in$ clan (P). This shows class $(P') \subset$ clan (P); hence class (P') is finite and therefore by the preceeding consideration classical; therefore class (P') = clan (P).

COROLLARY 6. *Every* clan *of an* FBN-*ring is contained in a stratum.*

If $\bar{R}$ is a factorring of R, then a link $\bar{P} \rightsquigarrow \bar{Q}$ in $\bar{R}$ gives rise to a link $P \rightsquigarrow Q$ in R, but not necessarily vice versa: we write $\bar{R}$-class$(P) \rightsquigarrow \bar{R}$-class$(Q)$ if there exist $P' \in \bar{R}$-class(P) and $Q' \in \bar{R}$-class(Q) with $P' \rightsquigarrow Q'$.

THEOREM 7. *Let R be an* FBN-*ring with prime radical N, let $\bar{R} = R/N$, and let P, Q, be distinct prime ideals in the α-stratum of* spec R. *Then* $\bar{R}$-class $(P) \rightsquigarrow \bar{R}$-class (Q), *if and only if* $\bar{R}$-class $(P) = \bar{R}$-class (Q) *or there exist* $P'' \in \bar{R}$-class (P) *and* $Q'' \in \bar{R}$-class (Q) *such that* $\text{rt-ann}_R(N/Q''N) \subset P''$.

Proof. By Proposition 2 (4), the links in R and R/N^2 are the same; hence assume without loss of generality $N^2 = 0$.

1. $\bar{R} = R/N$ is again an FBN-ring, and the indecomposable injective $\bar{R}$-module corresponding to the prime ideal $\bar{P}$ is $U_P = \text{ann}_{E_P}(N)$. The exact sequence $0 \to N \to R \to \bar{R} \to 0$ yields under the functor $\hom_R(-, E_P)$ the exact sequence $0 \to U_P \to E_P \to H_P \to 0$ where $H_P = \hom_R(N, E_P) = \hom_{\bar{R}}(N, U_P)$, using $N^2 = 0$. The injective hull of H_P as $\bar{R}$-module is the direct sum of α_i-smooth indecomposable injective U_{Q_i}, since $\bar{R}$ is FBN; as $H_P \cap U_{Q_i} \neq 0$ by essentiality and as H_P is a factor of the α-smooth module E_P, $\alpha_i \leqq \alpha$.

A given link $P \rightsquigarrow Q$ yields a nonzero homomorphism $\phi: E_P \to E_Q$ by Proposition 4. ϕ has either nonzero restriction $U_P \to U_Q$, in which case $\bar{P} \rightsquigarrow \rightsquigarrow \bar{Q}$ in $\bar{R}$, i.e., $\bar{R}$-class $(P) = \bar{R}$ class (Q); or zero restriction to U_P, in which case it induces a nonzero homomorphism $\bar{\phi}: H_P \to U_Q$. Then $\bar{\phi}$ extends to the $\bar{R}$-injective hull $\bigoplus U_{Q_i}$ of H_P, and some component $\phi_i: U_{Q_i} \to U_Q$ of this extension is nonzero. For such i, $\alpha_i \geqq \kappa(\bar{R}/\bar{Q}) = \alpha$ hence $\alpha_i = \alpha$, hence $\bar{Q}_i \rightsquigarrow \rightsquigarrow \bar{Q}$ or $Q_i \in \bar{R}$-class (Q).

By essentiality there exists $0 \neq \beta \in H_P \cap U_{Q_i}$ with $\beta Q_i = 0$; this means $\beta(Q_i N) = 0$ hence induces $0 \neq \bar{\beta}$: $N/Q_i N \to U_P$. Consequently the right-$\bar{R}$-module $N/Q_i N$ is not $\bar{P}$-torsion, hence the ideal rt-ann $(N/Q_i N)$ is contained in P. This proves one direction of the theorem.

2. Assume conversley rt-ann $(N/QN) \subset P$. Then the right-module N/QN cannot be $\bar{P}$-torsion: otherwise and since it is finitely generated as left-module, rt-ann (N/QN) would be $\bar{P}$-dense hence $\not\subset P$. Consequently there exists $0 \neq \beta: N/QN \to U_P$; and we may choose βR uniform; notice $\beta Q = 0$. Then for any $0 \neq \gamma \in \beta R$ we have $B = \text{ann}(\gamma R) \supset Q$; and since $\alpha = \kappa(R/Q) \geqq \kappa(R/B) \geqq \text{lt-}\kappa(N/BN)$, which equals by [9], 2.2 $\text{rt-}\kappa(N/BN) \geqq \kappa(N/\ker \gamma)$, which equals α since $N/\ker \gamma$ embedds into the α-smooth module U_P, one gets $\alpha = \kappa(R/B)$ hence $B = Q$ by [5], 7.5 and 7.2. This shows that βR is a uniform prime hence α-criterical module ([9], 2.5).

Pick $e \in E_P$ which maps to $\beta \in H_P$ under the surjection $E_P \to H_P$, and take an α-composition series of $eR \cap U_P$, all whose factors have annihilators in $\bar{R}$-class (P) by Lemma 3, applied to the $\bar{R}$-module U_P. As $\beta R \cong eR/(eR \cap U_P)$ is α-critical with annihilator Q, one has an α-composition series of eR whose top factor has annihilator Q while all other factors have annihilators in $\bar{R}$-class (P).

Now take among all α-composition series of finitely generated submodules of E_P with these properties, one of shortest length, say $0 = B_0 \subsetneqq \cdots \subsetneqq B_n$. As in the proof of Proposition 4,

$$0 \longrightarrow B_{n-1}/B_{n-2} \longrightarrow B_n/B_{n-2} \longrightarrow B_n/B_{n-1} \longrightarrow 0$$

yields a link $\bar{R}\text{-class}(P) \ni \text{ann}(B_{n-1}/B_{n-2}) \rightsquigarrow \text{ann}(B_n/B_{n-1}) = Q$ hence $\bar{R}$-class $(P) \rightsquigarrow \bar{R}$-class (Q), as desired.

3. It remains to deduce the same conclusion from the assumption $\bar{R}$-class $(P) = \bar{R}$-class (Q). But then $\bar{R}$-class (P) has at least the two members P and Q, and then a link $\overline{P'} \rightsquigarrow \overline{P''}$ between suitable P', $P'' \in \bar{R}$-class (P) obviously exists.

REMARKS. (1) Note that the criterion of Theorem 7 really involves only R/N-modules: $\text{rt-ann}_{\bar{R}}(N/Q''N) \subset \bar{P}$.

(2) If $\text{rt-ann}_R(N/QN) \subset P$ and $\kappa(R/P) = \kappa(R/Q)$, then

$$\begin{aligned}\kappa(R/Q) &\geqq \kappa(R/\text{lt-ann}(N/QN)) = \text{lt-}\kappa(N/QN) = \text{rt-}\kappa(N/QN)\\ &= \kappa(R/\text{rt-ann}(N/QN)) \geqq \kappa(R/P)\end{aligned}$$

by [9], 2.1 and 2.2; hence equality holds throughout. The first equality implies $Q = \text{lt-ann}(N/QN)$, and the second one that P is a minimal prime ideal over $\text{rt-ann}(N/QN)$.

Hence for given Q, such prime ideals P exist if and only if $\text{lt-ann}(N/QN) = Q$, ann in this case there are only finitely many of them (since a noetherian ring has only finitely many minimal prime ideals). This means that any Q is directly linked in R to only finitely many $\bar{R}$-classes; consequently if $|\bar{R}\text{-class}(P)| \leqq \aleph$ for some infinite cardinal $\aleph$ and all P, then also $|R\text{-class}(P)| \leqq \aleph$.

III. Noetherian PI-rings. Let R be a noetherian PI-ring, i.e., a noetherian ring with a polynomial identity with integer coefficients which is proper on every nonzero factorring [18]. For a prime ideal Q of R, PI-degree $(Q) = n$ if R/Q satisfies S_{2n} but not S_{2n-2}. PI-degree $(R) = \max\{\text{PI-degree}(Q) : Q \in \text{spec}\, R\}$ is a well defined natural number, for any PI-ring R.

If R is a semiprime PI-ring and if Q is a prime ideal of R with

PI-degree (Q) = PI-degree $(R) = n$, then Q doesn't contain all evaluations of the Formanek polynomial [4] for $n \times n$-matrices, hence the commutative localization R_Π at the prime ideal $\Pi = Q \cap \text{centre}(R)$ of the centre inverts all elements of $\mathscr{C}(P)$ ([2], [21], [18]); thus such Q is classical, i.e. $\{Q\}$ is a clan.

Define $N_n = \bigcap \{Q \in \text{spec}\, R: \text{PI-degree}\,(Q) \leqq n\}$, a semiprime ideal of the noetherian PI-ring R. For any factorring R/I with $I \subset N_n$ which has only prime ideals of PI-degree $\leqq n$, in particular for $R/(N_n^2 + N_{n+1})$, the R/I-classes are described by Theorem 7 via the R/N_n-bimodules $N_n/(QN_n + N_{n+1})$ and $N_n/(N_nQ + N_{n+1})$ as certain mergers of R/N_n-classes. On the other hand R/N_{n+1}-class $(P) = \{P\}$ if PI-degree $(P) = n + 1$ by the preceding paragraph, and R/N_{n+1}-class $(Q) = R/(N_n^2 + N_{n+1})$-class (Q) if PI-degree $(Q) \leqq n$, since any link $\bar{Q} \rightsquigarrow \bar{Q}'$ between prime ideals of R/N_{n+1} of PI-degree $\leqq n$ shows already in $R/(Q'Q + N_{n+1})$ hence a fortiori in $R/(N_n^2 + N_{n+1})$; cf. Proposition 2.

Summarizing: in passing from R/N_n to $R/(N_n^2 + N_{n+1})$ (or to R if PI-degree $(R) = n$) certain R/N_n-classes merge to form the $R/(N_n^2 + N_{n+1})$-classes (or R-classes) according to Theorem 7; in passing from $R/(N_n^2 + N_{n+1})$ to R/N_{n+1} certain new singleton R/N_{n+1}-classes appear but no mergers occur.

Climbing in this way from the commutative ring R/N_1 in finitely many steps to the noetherian PI-ring R, provides a description of its classes. Combining this with Remark (2) after Theorem 7 yields:

COROLLARY 8. *For every noetherian* PI-*ring, the classes are at most countable.*

Unfortunately the step from countable to finite classes, i.e., clans, is still difficult; we have only a rather trivial positive result and a couple of counterexamples:

PROPOSITION 9. *If an* FBN-*ring satisfies* INC *with respect to its centre, then it has enough clans. In particular this applies to any noetherian ring which is integral over its centre, and to every prime noetherian* PI-*ring of Krull dimension one.*

Proof. For an FBN-ring with centre C, the intersection $Q \cap C$ is the same for all $Q \in \text{class}\,(P)$: if $a \in Q \cap C$ and $P \rightsquigarrow Q$, then by Proposition 2 (4) for the ideal $I = aR$, $(Q \cap P)I = a(Q \cap P) \subset QP$ hence $a \in aR \subset P$. Therefore if R satisfies INC (i.e., if prime ideals of R over the same ideal of C are incomparable [10]), then all $Q \in \text{class}\,(P)$ are minimal over $(P \cap C)R$ hence finite in number; so

Theorem 5 yields the result.

If R is noetherian and integral over C, it is automatically fully bounded [20] and has INC [6]. If R is a prime noetherian PI-ring of Krull dimension one, all nonzero prime ideals are maximal and have nonzero intersection with C [17], hence INC holds again.

REMARKS. (1) Every FBN-ring of Krull dimension zero has enough clans, in fact is artinian. We do not know whether every prime FBN-ring of Krull dimension one has enough clans; apart from our result for noetherian PI-rings this is true for HNP-rings (cf. [16]), by a proof in the spirit of our approach [12].

(2) Our argument also shows the following: for any prime ideal Π of the centre C of any FBN-ring R, the prime ideals P of R with maximal $\kappa(R/P)$ among those containing Π, belong to clans. Note that such P exist for every Π.

COUNTEREXAMPLE 1. *A noetherian* PI-*ring of Krull dimension one with infinite classes.*

The example. Put $R = A \times A_\sigma$, the split extension of a commutative noetherian domain A by the bimodule A_σ, with the right-A-module structure modified by an automorphism σ of A.

A_σ is an ideal of R of square zero, hence R satisfies S_2^2 and has prime radical $N = A_\sigma$. There is a one-to-one correspondence between the prime ideals P of A and of R; we do not distinguish between them notationally. One has $\kappa(R) = \kappa(A)$; and $N/PN = A_\sigma/PA_\sigma = (A/P)_\sigma$, hence lt-ann$(N/PN) = P$ and rt-ann$(N/PN) = \sigma^{-1}(P)$; therefore by Theorem 7, class$(P) = \{\sigma^n(P): n \in \boldsymbol{Z}\}$.

It is easy to find instances where $\kappa(A) = 1$ and class(P) is infinite; e.g. let $A = K[x]$ for a field K of characteristic zero, $\sigma(x) = x + 1$ and $P = \langle x \rangle$ (this example is mentioned in [16]).

Variations. $A = K[x, y]_{\langle x,y \rangle}$ and $\sigma(x) = x + y$, $\sigma(y) = y$ produces a local noetherian PI-ring R of Krull dimension two, with infinite classes in the middle stratum of spec R. The case of a proper endomorphism σ can also be discussed.

COUNTEREXAMPLE 2. *A prime noetherian* PI-*ring of Krull dimension two with infinite classes.*

A general construction. Let C be a commutative noetherian domain, with two noetherian subrings A and B, both containing the nonzero semiprime principal ideal $I = cC$ of C; consider the noncommutative ring $R = \begin{pmatrix} A & I \\ I & B \end{pmatrix}$. R is prime, noetherian, and satisfies all identities of 2×2-matrices over $\boldsymbol{Z}$. Its Krull dimension equals the maximum of the Krull dimensions of A and B. The centre is $A \cap B$, embedded diagonally; it is not necessarily noetherian. The Formanek centre, i.e. the collection of all evaluations of central polynomials with zero constant term for $M_2(\boldsymbol{Z})$, equals I^2 (use the central polynomial $[x, y]^2 = -\det([x, y])$).

A prime ideal of R of PI-degree two doesn't contain the Formanek centre, hence doesn't contain I. By [18], 4.16(c) these prime ideals are in one-to-one correspondence with the prime ideals of the centre $A \cap B$ not containing I; they constitute singleton clans.

The prime ideals of R of PI-degree one contain the Formanek centre hence I; explicitely there are two types $\mathscr{P} = \begin{pmatrix} P & I \\ I & B \end{pmatrix}$ and $\mathscr{Q} = \begin{pmatrix} A & I \\ I & Q \end{pmatrix}$ for arbitrary prime ideals P and Q containing I, of A and B respectively. The intersection of all these prime ideals is $N = N_1 = M_2(I)$; to apply Theorem 7 one computes

$$N/\mathscr{Q}N = \begin{pmatrix} I/I & I/I \\ I/QI & I/QI \end{pmatrix}, \quad \text{lt-ann}\,(N/\mathscr{Q}N) = \begin{pmatrix} A & I \\ I & B \cap QC \end{pmatrix}$$

and $\text{rt-ann}\,(N/\mathscr{Q}N) = \begin{pmatrix} A \cap QC & I \\ I & B \cap QC \end{pmatrix}$, using at this point $I = cC$ and cancelling c. By Theorem 7 and the remarks following it, a link into $\mathscr{Q}$ exists if and only if $B \cap QC = Q$, and in this case there is precisely a self-link $\mathscr{Q} \rightsquigarrow \mathscr{Q}$ and at least one, at most finitely many links $\mathscr{P} \rightsquigarrow \mathscr{Q}$, namely for the $\mathscr{P} = \begin{pmatrix} P & I \\ I & B \end{pmatrix}$ with $P \supset A \cap QC$ and $\kappa(A/P) = \kappa(B/Q)$.

By symmetry (transposition is an antiautomorphism of R) $\mathscr{P} \rightsquigarrow \mathscr{Q}$ implies $\mathscr{Q} \rightsquigarrow \mathscr{P}$ and vice versa. One may therefore, ignoring direction and self-links and passing to factors modulo I, represent all links by a bipartite locally finite graph between spec $\bar{A}$ and spec $\bar{B}$, where the existence of an edge is given by $\kappa(\bar{A}/\bar{P}) = \kappa(\bar{B}/\bar{Q})$ and $\bar{P} \supset \bar{A} \cap \overline{QC}$ (or equivalently $\bar{Q} \supset \bar{B} \cap \overline{PC}$).

The example (appears in [19] in a different context). Let $k = \boldsymbol{Q}(\sqrt{6})$, $K = \boldsymbol{Q}(\sqrt{2}, \sqrt{3}) = k(a)$ where $a = \sqrt{3} - \sqrt{2}$, and let $a^* = -a$ be the nontrivial automorphism of K/k. Let $C = K[x, y]$, $A = k[x + \sqrt{3}, y, \sqrt{3}\,y]$, $B = k[x + \sqrt{2}, y, \sqrt{2}\,y]$ and $I = yC$. Then R has Krull dimension two as desired; the two nonmaximal

PI-degree-one prime ideals $\begin{pmatrix} A & I \\ I & I \end{pmatrix}$ and $\begin{pmatrix} I & I \\ I & B \end{pmatrix}$ form a clan; but the maximal PI-degree-one ideals determine infinite classes.

Indeed consider the class determined by the prime ideal of R which in turn is determined by the irreducible polynomial q of $\bar{B} = k[x + \sqrt{2}]$, with roots $\rho_1, \cdots, \rho_s$ in the algebraic closure of k. For $t = x + \sqrt{3}$, $q(t - a)q(t + a) \in \bar{A} \cap \overline{QC}$ since it is invariant under the automorphism *; conversely if $f(t)$ generates the $\bar{A}$-ideal $\bar{A} \cap \overline{QC}$, then $f(t) = q(t - a)g(t)$ for some $g \in \bar{C}$ hence $f(t) = f(t)^* = g(t+a)g^*(t)$ hence $f(t)^2 = g(t-a)q(t+a)q(t)g^*(t)$ with $gg^* \in \bar{A}$; therefore the roots of $f(t)$ are precisely $\rho_i \pm a$.

Consequently the prime ideals $\mathscr{P}$ and $\mathscr{Q}'$ in the class in question, correspond to the minimal polynomials of all the $\rho_i + na$ for even respectively odd integers n; since these are infinite in number, the class contains infinitely many prime ideals of both types.

Variations. With the choice of C, A, B and I made above, $\begin{pmatrix} A/I^2 & I/I^2 \\ I/I^2 & B/I^2 \end{pmatrix}$ and $\begin{pmatrix} A/I^2 & I/I^2 \\ 0 & B/I^2 \end{pmatrix}$ have also infinite classes and Krull dimension one (as does counterexample 1), and they satisfy all identities of 2×2-matrices over $\boldsymbol{Z}$.

For the choice (cf. [19]) $C = F[x]$, $A = F_1 + xF[x]$, $B = F_2 + xF[x]$, $I = xF[x]$ for finite-dimensional field extensions F/F_i, one has $\kappa(R) = \kappa(A) = \kappa(B) = 1$; hence R has enough clans by Proposition 9. $F/(F_1 \cap F_2)$ may be chosen finite-dimensional, algebraic or transcendental; then R is finitely generated as module, integral or non-integral over its centre $(F_1 \cap F_2) + xF[x]$, respectively.

References

1. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956.
2. A. W. Chatters, *Localization in P. I. rings*, J. London Math. Soc., **2** (1970), 763-768.
3. D. Eisenbud and J. C. Robson, *Hereditary noetherian prime rings*, J. Algebra **16** (1970), 86-104.
4. E. Formanek, *Central polynomials for matrix rings*, J. Algebra, **23** (1972), 129-132.
5. R. Gordon and J. C. Robson, *Krull dimension*, Memoirs Amer. Math. Soc., **133** (1973).
6. A. G. Heinicke, *A remark about noncommutative integral extensions*, Canad. Math. Bull., **13** (1970), 359-361.
7. A. V. Jategaonkar, *The torsion theory at a semiprime ideal*, A. Acad. Brasil. Cienc., **45** (1973), 197-200.
8. A. V. Jategaonkar, *Injective modules and localization in non-commutative noetherian rings*, Trans. Amer. Math. Soc., **188** (1974), 109-123.
9. A. V. Jategaonkar, *Jacobson's conjecture and modules over fully bounded noetherian rings*, J. Algebra, **30** (1974), 103-121.

10. I. Kaplansky, *Commutative Rings*, Univ. of Chicago Press, 1974.
11. G. Krause, *On fully left-bounded left-noetherian rings*, J. Algebra, **23** (1972), 88-99.
12. T. H. Lenagan, *Bounded hereditary noetherian prime rings*, J. London Math. Soc., **6** (1973), 241-246.
13. G. Michler, *Right symbolic powers and classical localization in right noetherian rings*, Math. Z., **127** (1972), 57-69.
14. B. J. Müller, *On semi-perfect rings*, Illinois J. Math., **14** (1970), 464-467.
15. ———, *The structure of quasi-Frobenius rings*, Canad. J. Math., **26** (1974), 1141-1151.
16. ———, *Localization in non-commutative noetherian rings*, Canad. J. Math., **28** (1976), 600-610.
17. L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc., **79** (1973), 219-223.
18. ———, *On rings with central polynomials*, J. Algebra, **31** (1974), 393-426.
19. W. Schelter, *Integral extensions of rings satisfying a polynomial identity*, preprint.
20. J. Shapiro, *A noncommutative analog to prime ideals*, Ph. D. thesis Rutgers Univ. 1975.
21. L. Small, *Localization in* P.I.-*rings*, J. Algebra, **18** (1971), 269-270.

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