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v-PREHOMOMORPHISMS ON INVERSE SEMIGROUPS

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A mapping θ of an inverse semigroup S into an inverse semigroup T is called a v-prehomomorphism if, for each $a, b \in S$, $(ab)\theta \leqq a\theta b\theta$ and $(a^{-1})\theta = (a\theta)^{-1}$. The congruences on an E-unitary inverse semigroup $P(G, \mathscr{X}, \mathscr{Y})$ are determined by the normal partition of the idempotents, which they induce, and by v-prehomorphisms of S into the inverse semigroup of cosets of G.

Inverse semigroups, with v-prehomomorphisms as morphisms, constitute a category containing the category of inverse semigroups, and homomorphisms, as a coreflective subcategory. The coreflective map $\eta: S \rightarrow V(S)$ is an isomorphism if the idempotents of S form a chain and the converse holds if S is E-unitary or a semilattice of groups. Explicit constructions are given for all v-prehomomorphisms on S in case S is either a semilattice of groups or is bisimple.

0. Introduction. A mapping θ of an inverse semigroup S into an inverse semigroup T is called a v-prehomomorphism if, for each $a, b \in S$, $(ab)\theta \leqq a\theta b\theta$ and $(a^{-1})\theta = (a\theta)^{-1}$. Thus, if S and T are semilattices, a v-prehomomorphism is just an isotone mapping of S into T. N. R. Reilly and the present author have shown that the E-unitary covers of an inverse semigroup S are determined by v-prehomomorphisms with domain S. In the first section of this paper, we show that the congruences on an E-unitary inverse semigroup $S = P(G, \mathscr{X}, \mathscr{Y})$ are determined by the normal partition of the idempotents, which they induce, and by v-prehomomorphisms of S into the inverse semigroup of cosets of G. The remainder of the paper is concerned with the problem of constructing v-prehomomorphisms on an inverse semigroup S.

In §2, it is shown that inverse semigroups and v-prehomomorphisms constitute a category which contains the category of inverse semigroups and homomorphisms as a coreflective subcategory. Thus, for each inverse semigroup S, there is an inverse semigroup $V(S)$ and a v-prehomomorphism $\eta: S \rightarrow V(S)$ with the property that every v-prehomomorphism with domain S is the composite of η with a homomorphism with domain $V(S)$. It is shown that η is an isomorphism if the idempotents of S form a chain and that the converse holds if S is E-unitary or a semilattice of groups.

Section 3 is concerned with the situation when S is a simple inverse semigroup. It is shown that, in this case, $V(S)$ is also simple, but it need not be bisimple even if S is bisimple. Indeed, if S is

E-unitary, it is shown that $V(S)$ is bisimple if and only if the idempotents of S form a chain. Despite the fact that the structure of $V(S)$, for S bisimple, is not completely determined, an explicit method of construction can be given for all v-prehomomorphisms with domain S; this is done.

Section 4 is concerned with the situation when S is a semilattice of groups and the pattern here is similar to that in §3. It is shown that $V(S)$ need not be a semilattice of groups; on the other hand, an explicit method is given for constructing all v-homomorphisms with domain S.

1. Congruences on E-unitary inverse semigroups.

Let G be a group. Then it was shown in [11] that the set $\mathscr{K}(G)$ of all cosets X of G modulo subgroups of G is an inverse semigroup under the multiplication $*$ where

$$X * Y = \text{smallest coset containing } XY .$$

(Note that, if $X = Ha$, $Y = Kb$, then

$$X * Y = [H \vee aKa^{-1}]ab$$

where, for subgroups U, V of G, $U \vee V$ denotes the subgroup generated by U and V.) It was further shown in [6] that every subdirect product of an inverse semigroup S by G is determined by a mapping θ of S into $\mathscr{K}(G)$, where θ is a v-prehomomorphism in the sense of the following definition.

DEFINITION 1.1. Let S and T be inverse semigroups then a mapping $\theta: S \rightarrow T$ is a *v-prehomomorphism* if the following hold

(i) $a^{-1}\theta = (a\theta)^{-1}$ for each $a \in S$;

(ii) $(ab)\theta \leqq a\theta b\theta$ for each $a, b \in S$.

We shall consider in detail the problem of constructing the v-prehomomorphisms of one inverse semigroup into another later in this paper. Here we shall show that the congruences on an E-unitary inverse semigroup $S = P(G, \mathscr{X}, \mathscr{Y})$ are also determined by v-prehomomorphisms of S into $\mathscr{K}(G)$.

LEMMA 1.2. *Let $S = P(G, \mathscr{X}, \mathscr{Y})$ be an E-unitary inverse semigroup and let ρ be a congruence on S. For each $\boldsymbol{a} = (a, g) \in S$ set*

$$\boldsymbol{a}\theta_\rho = \{h \in G: (a, g)\rho(b, h) \quad \text{for some} \quad (b, h) \in S\} .$$

Then $\theta = \theta_\rho$ is a v-prehomomorphism of S into $\mathscr{K}(G)$. Further $\theta \leqq \boldsymbol{\sigma}$ where $\boldsymbol{a\sigma} = g$ for each $\boldsymbol{a} = (a, g)$ and where $\theta \leqq \boldsymbol{\sigma}$ means $\boldsymbol{a}\theta \leqq \boldsymbol{a\sigma}$ for each $\boldsymbol{a} \in S$.

Proof. We use the fact [2] that $X \subseteqq G$ is a coset if and only if $X = XX^{-1}X$; note that $X \subseteqq XX^{-1}X$ holds for any $X \subseteqq G$. Thus, suppose that $h_1, h_2, h_3 \in \boldsymbol{a}\theta$ with, say, $(a, g)\rho(b_i, h_i)$, $i = 1, 2, 3$. Then

$$(a, g) = (a, g)(a, g)^{-1}(a, g)\rho(b_1, h_1)(b_2, h_2)^{-1}(b_3, h_3) = (u, h_1h^{-1}h_3)$$

for some $u \in \mathscr{Y}$. Hence $h_1h_2^{-1}h_3 \in \boldsymbol{a}\theta$. It follows that $\boldsymbol{a}\theta \in \mathscr{K}(G)$. Thus $h \in \boldsymbol{a}^{-1}\theta$ implies $h^{-1} \in \boldsymbol{a}^{-1}\theta$. It follows, using the fact that $\boldsymbol{a} = (\boldsymbol{a}^{-1})^{-1}$, that $(\boldsymbol{a}\theta)^{-1} = \boldsymbol{a}^{-1}\theta$. Next, suppose $k_1 \in \boldsymbol{a}\theta$, $k_2 \in \boldsymbol{b}\theta$ with $\boldsymbol{a}\rho(c_1, k_1)$, $\boldsymbol{b}\rho(c_2, k_2)$, say. Then $\boldsymbol{ab}\rho(c_1 \wedge k_1c_2, k_1k_2)$ consequently $k_1k_2 \in \boldsymbol{ab}\theta$. Hence $\boldsymbol{a}\theta\boldsymbol{b}\theta \subseteqq (\boldsymbol{ab})\theta$ and so, since $(\boldsymbol{ab})\theta$ is a coset, $\boldsymbol{a}\theta * \boldsymbol{b}\theta \subseteqq (\boldsymbol{ab})\theta$; that is, $(\boldsymbol{ab})\theta \leqq \boldsymbol{a}\theta\boldsymbol{b}\theta$. It follows that θ is a v-prehomomorphism of S into $\mathscr{K}(G)$.

Finally, if $\boldsymbol{a} = (a, g)$ then $g \in \boldsymbol{a}\theta$ so that $\boldsymbol{a}\theta \leqq \{g\} = \boldsymbol{a}\boldsymbol{\sigma}$; thus $\theta \leqq \boldsymbol{\sigma}$.

Suppose now that π is a normal partition on the idempotents of S. Then Reilly and Scheiblich [10] have shown that π^* defined by $(a, b) \in \pi^*$ if and only if $a^{-1}ea\pi b^{-1}eb$ for all $e^2 = e \in S$ is the largest congruence on S which induces the normal partition π. The prehomomorphism κ_π corresponding to π^* is given by $(a, g)\kappa_\pi = \{h \in G\colon$ for some $b \in \mathscr{Y}$ such that $h^{-1}b \in \mathscr{Y}$, $b\pi a$ and $gh^{-1}f\pi\ f$ for all $f \leqq b\}$.

Note that, if $\mathscr{X} = \mathscr{Y}$, then

$$(a, g)\kappa_\pi = \{h \in G\colon gh^{-1}f\pi f \text{ for all } f \leqq a\}$$

while, if $\pi = \Delta$ is the identity partition,

$$(a, g)\kappa_\pi = \{h \in G\colon gh^{-1}f = f \text{ for all } f \leqq a\} .$$

If ρ is a congruence on S, we shall denote by π_ρ the normal partition, on the idempotents, induced by ρ.

LEMMA 1.3. *Let ρ be a congruence on $S = P(G, \mathscr{X}, \mathscr{Y})$ and let $\boldsymbol{a} = (a, g)$, $\boldsymbol{b} = (b, h) \in S$. Then, if $\pi = \pi_\rho$, $\theta = \theta_\rho$*

(i) $\kappa_\pi \leqq \theta$;

(ii) *$a\pi b$ implies $(a, 1)\theta = (b, 1)\theta$;*

(iii) *$(\boldsymbol{a}, \boldsymbol{b}) \in \rho$ if and only if $a\pi b$ and $\boldsymbol{a}\theta = \boldsymbol{b}\theta$.*

Proof. (i) Suppose $x \in \boldsymbol{a}\theta$; thus $(a, g)\rho(y, x)$ for some $y \in \mathscr{Y}$. Then, since $\rho \subseteqq \pi^*$, $(a, g)\pi^*(y, x)$; thus $x \in \boldsymbol{a}\kappa_\pi$. It follows that $\boldsymbol{a}\theta \subseteqq \boldsymbol{a}\kappa_\pi$; that is $\boldsymbol{a}\kappa_\pi \leqq \boldsymbol{a}\theta$. Hence $\kappa_\pi \leqq \theta$.

(ii) If $a\pi b$ then $(a, 1)\rho(b, 1)$ since π is the normal partition induced by ρ. Thus, by definition $(a, 1)\theta = (b, 1)\theta$.

(iii) Suppose $(\boldsymbol{a}, \boldsymbol{b}) \in \rho$ then, since ρ induces π, $a\pi b$ and, from the definition of θ, $\boldsymbol{a}\theta = \boldsymbol{b}\theta$. Conversely, suppose $a\pi b$ and $\boldsymbol{a}\theta = \boldsymbol{b}\theta$. Then $h \in \boldsymbol{a}\theta$ so that $(a, \boldsymbol{g})\rho(c, h)$ for some $c \in \mathscr{Y} \cap h\mathscr{Y}$. We now have the following string of equivalences

$$\begin{aligned}(a, g) = (a, 1)(a, g)\rho(b, 1)(a, g) &\quad \text{since } a\pi b \text{ and } \rho \text{ induces } \pi \\ &\rho(b, 1)(c, h) \\ &= (c, 1)(b, h) \\ &\rho(b, 1)(b, h) = (b, h)\end{aligned}$$

since $(a, g)\rho(c, h)$ implies $(a, 1)\rho(c, 1)$ and $a\pi b$ implies $(a, 1)\rho(b, 1)$. Hence $(a, g)\rho(b, h)$.

Lemma 1.3 shows that ρ is determined by the normal partition π_ρ and the v-prehomomorphism θ_ρ. We now turn to the converse situation where we start with a normal partition and a v-prehomomorphism. We require the following lemma which will be of crucial importance later in the paper.

LEMMA 1.4. *Let θ be a v-prehomomorphism of an inverse semigroup S into an inverse semigroup T, and let $a, b \in S$. If $a^{-1}a \geqq bb^{-1}$ or $a^{-1}a \leqq bb^{-1}$ then $a\theta b\theta = (ab)\theta$.*

Proof. Suppose $a^{-1}a \geqq bb^{-1}$. Then

$$\begin{aligned}a\theta b\theta = a\theta(bb^{-1}b)\theta &= a\theta(a^{-1}abb^{-1}b)\theta \quad \text{since } a^{-1}a \geqq bb^{-1} \\ &= a\theta(a^{-1}ab)\theta \\ &\leqq a\theta(a^{-1})\theta(ab)\theta \quad \text{since } \theta \text{ is a } v\text{-prehomomorphism} \\ &= a\theta(a\theta)^{-1}(ab)\theta \quad \text{since } (a\theta)^{-1} = (a^{-1})\theta \\ &\leqq (ab)\theta\ .\end{aligned}$$

But by hypothesis, $(ab)\theta \leqq a\theta b\theta$.

The other case is similar.

COROLLARY 1.5. *Let G be a group and S an inverse semigroup and suppose that θ is a v-prehomomorphism of S into $\mathscr{K}(G)$. Then, for each $a \in S$, $a\theta$ is a coset modulo $(aa^{-1})\theta$.*

Proof. By Lemma 1.4, $(aa^{-1})\theta = a\theta(a^{-1})\theta = a\theta(a\theta)^{-1}$. But $a\theta$ is a coset modulo $a\theta(a\theta)^{-1}$. Hence the result.

LEMMA 1.6. *Let π be a normal partition on the set $\mathscr{Y}$ of idempotents of $P(G, \mathscr{X}, \mathscr{Y}) = S$ and let $\theta\colon S \to \mathscr{K}(G)$ be a v-prehomomorphism such that*

(i) $\kappa_\pi \leqq \theta \leqq \sigma$

(ii) *$a\pi b$ implies $(a, 1)\theta = (b, 1)\theta$ for $a, b \in \mathscr{Y}$.*

Then ρ defined by

$$(a, g)\rho(b, h) \text{ if and only if } a\pi b \text{ and } (a, g)\theta = (b, h)\theta$$

is a congruence on S which induces π. Further $\theta = \theta_\rho$.

Proof. The relation ρ is clearly an equivalence on S. Suppose that $(a, g)\rho(b, h)$ and let $(c, k) \in S$. Then $(a, g)\theta = (b, h)\theta$ implies $(a, g)\kappa_\pi = (b, h)\kappa_\pi$ since $\kappa_\pi \leqq \theta$ and then, since $a\pi b$, Lemma 1.3 implies $(a, g)\pi^*(b, h)$. Hence $(a, g)(c, k)\pi^*(b, h)(c, k)$. It follows from this that $(a \wedge gc, 1)\pi^*(b \wedge hc, 1)$ so that $(a \wedge gc)\pi(b \wedge hc)$.

Next $(a, g)\theta = (b, h)\theta$ implies $\square \neq (a, g)\theta(c, k)\theta \subseteqq (a \wedge gc, gk)\theta \cap (b \wedge hc, hk)\theta$ since θ is a v-prehomomorphism. By Corollary 1.5, $(a \wedge gc, gk)\theta$ is a coset modulo $(a \wedge gc, 1)\theta$ and $(b \wedge hc, hk)\theta$ is a coset modulo $(b \wedge hu, 1)\theta$. Hence, to prove $(a \wedge gc, gk)\theta = (b \wedge hc, hk)\theta$ it suffices to prove that $(a \wedge gc, 1)\theta = (b \wedge hc, 1)\theta$. But, since

$$(a \wedge gc)\pi(b \wedge hc) ,$$

this is immediate from condition (ii) in the statement of the lemma. It follows that ρ is right compatible. A similar argument shows that it is left compatible; thus ρ is a congruence on S.

Now $(a, 1)\rho(b, 1)$ if and only if $a\pi b$ and $(a, 1)\theta = (b, 1)\theta$. By condition (ii), $a\pi b$ implies $(a, 1)\theta = (b, 1)\theta$. Hence $(a, 1)\rho(b, 1)$ if and only if $a\pi b$; that is, ρ induces π.

Finally, suppose that $h \in (a, g)\theta_\rho$. Then $(b, h)\rho(a, g)$ for some $b \in \mathscr{Y}$ so that $(b, h)\theta = (a, g)\theta$. But $\theta \leqq \boldsymbol{\sigma}$ implies $h \in (b, h)\theta$. Hence $(a, g)\theta_\rho \subseteqq (a, g)\theta$. On the other hand, if $h \in (a, g)\theta$, then, since $\kappa_\pi \leqq \theta$, $h \in (a, g)\kappa_\pi$ so that $(b, h)\pi^*(a, g)$ for some $b \in \mathscr{Y}$. This implies $(b, 1)\pi^*(a, 1)$ so that $b\pi a$ and, consequently, $(b, 1)\theta = (a, 1)\theta$. But, since $\theta \leqq \boldsymbol{\sigma}$, $h \in (b, h)\theta$; thus $h \in (b, h)\theta \cap (a, g)\theta$. Since, by Corollary 1.5, each of these is a coset modulo $(b, 1)\theta = (a, 1)\theta$, it follows that $(b, h)\theta = (a, g)\theta$. Hence, since $a\pi b$, $(b, h)\rho(a, g)$ so that $h \in (a, g)\theta_\rho$. We have thus shown that $(a, g)\theta \subseteqq (a, g)\theta_\rho$; therefore $(a, g)\theta_\rho = (a, g)\theta$.

In order to simplify the statement of the next result, we introduce some notation. Suppose that S is an inverse semigroup and G is a group. Then $\pi(S)$ denotes the lattice of normal partitions on the idempotents of S while $\operatorname{Pre}(S, G)$ denotes the partially ordered set of v-prehomomorphisms of S into G. If $S = P(G, \mathscr{X}, \mathscr{Y})$ is E-unitary then we shall denote by $\mathscr{B}(S)$ the subset, under the cartesian ordering, of $\pi(S) \times \operatorname{Pre}(S, G)$ consisting of all pairs (π, θ) such that

(i) $\kappa_\pi \leqq \theta \leqq \boldsymbol{\sigma}$

(ii) $a\pi b$ implies $(a, 1)\theta = (b, 1)\theta$.

under the ordering $(\pi, \theta) \leqq (\rho, \psi)$ if and only if $\pi \subseteqq \rho$, $\theta \geqq \psi$.

THEOREM 1.7. *Let $S = P(G, \mathscr{X}, \mathscr{Y})$ be an E-unitary semigroup. Then the mapping ϕ defined by*

$$\rho\phi = (\pi_\rho, \theta_\rho)$$

is an isomorphism of the lattice of congruence on S onto $\mathscr{B}(S)$.

Proof. This follows easily from Lemmas 1.2, 1.3, 1.6.

COROLLARY 1.8. *Let π be a normal partition on $P(G, \mathscr{X}, \mathscr{Y})$. Then the lattice of congruences on S with normal partition π is antiisomorphic to the set of v-prehomomorphisms θ of S into $\mathscr{K}(G)$ which satisfy*

(i) $\kappa_\pi \leqq \theta \leqq \sigma$;

(ii) *if $a\pi b$ then $(a, 1)\theta = (b, 1)\theta$, for $a, b \in \mathscr{Y}$.*

2. The category of v-prehomomorphisms. In this section, we show that inverse semigroups, with v-prehomomorphisms as morphisms, form a category having the category of inverse semigroups and homomorphisms as a coreflective subcategory.

LEMMA 2.1. *Let S and T be inverse semigroups and let $\theta: S \rightarrow T$ be a v-prehomomorphism of S into T. Then*

(i) *θ maps idempotents of S to idempotents of T;*

(ii) *θ is isotone; that is, $a \leqq b$ implies $a\theta \leqq b\theta$, for $a, b \in S$.*

Proof. (i) Let $e^2 = e \in S$; then

$$e\theta = e^2\theta \leqq e\theta e\theta \leqq e\theta e\theta e\theta = e\theta(e^{-1})\theta e\theta = e\theta(e\theta)^{-1}e\theta = e\theta .$$

Hence $e\theta = e\theta e\theta$.

(ii) Suppose $a \leqq b$; thus $a = eb$ for some $e^2 = e \in S$. Then $a\theta = (eb)\theta \leqq e\theta b\theta \leqq b\theta$ since, by (i), $e\theta$ is an idempotent of T.

COROLLARY 2.2. *Inverse semigroups, with v-prehomomorphisms as morphisms, constitute a category.*

Proof. We need only show that the composite of v-prehomomorphisms is again a v-prehomomorphism. Thus, let $\theta: S \rightarrow T$ and $\phi: T \rightarrow U$ be v-prehomomorphisms and let $a, b \in S$. Then $(ab)\theta \leqq a\theta b\theta$ whence, since ϕ is isotone, $(ab)\theta\phi \leqq (a\theta b\theta)\phi \leqq a\theta\phi b\theta\phi$. Further $(a^{-1})\theta\phi = (a\theta^{-1})\phi = (a\theta\phi)^{-1}$. Hence $\theta\phi$ is a v-prehomomorphism.

It is a straightforward matter to show that, as a subcategory of the category of inverse semigroups and v-prehomomorphisms, the category of inverse semigroups and homomorphisms is closed under limits and has solution sets. Hence, by the adjoint functor theorem, it is a coreflective subcategory. This may be shown directly since the inequality in the definition of a v-prehomomorphism can be written as an equality. Thus $\theta: S \rightarrow T$ is a v-prehomomorphism if and only if, for each $a, b \in S$

(i)′ $(ab)\theta = (ab)\theta(ab)\theta^{-1}a\theta b\theta$
(ii) $(a^{-1})\theta = (a\theta)^{-1}$.

THEOREM 2.3. *Let S an inverse semigroup. Then there is an inverse semigroup $V(S)$ and a v-prehomomorphism $\eta: S \to V(S)$ with the following property: given any v-prehomomorphism $\theta: S \to T$ there is a unique homomorphism $\psi: V(S) \to T$ such that $\theta = \eta\psi$.*

Proof. Let ρ be the congruence on the free inverse semigroup $FI(S)$ on S, generated by the relations

$$ab = ab.(ab)^{-1}.a.b$$
$$a = a.a^{-1}.a$$

for all $a, b \in S$, where juxtaposition denotes the product in S and denotes that in $FI(S)$; let $V(S) = FI(S)/\rho$. Then the mapping $\eta: S \to V(S)$ defined by $a\eta = a\rho^{\natural}$ is, by the definition of ρ, a v-prehomomorphism. Further, because of the universal property of $FI(S)$, any v-prehomomorphism $\theta: S \to T$ factors uniquely through a homomorphism $\psi: V(S) \to T$ as $\theta = \eta\psi$.

The following proposition gives some properties of $V(S)$ for an arbitrary inverse semigroup.

PROPOSITION 2.4. *Let S be an inverse semigroup. Then*

(i) *$\eta: S \to V(S)$ is one-to-one and S is a homomorphic retract of $V(S)$; if $\theta: V(S) \to S$ is the retraction then, for each $w \in V(S)$*

$$w\theta\eta = \min\{u \in V(S): w\theta = u\theta\}$$

i.e. for each $s \in S$, $w\theta = s$ implies $w \geqq s\eta$;

(ii) *$V(S)/\sigma \approx S/\sigma$ where σ denotes the minimum group congruence;*

(iii) *is S has an identity 1, then 1η is the identity of $V(S)$; if S has a zero 0, then 0η is the zero of $V(S)$.*

Proof. (i) The identity mapping $1_S: S \to S$ is a homomorphism. Hence it factors through η: $1_S = \eta\theta$ for some homomorphism θ. This means that η is one-to-one and θ is onto.

Now let $w = s_1\eta s_2\eta \cdots s_n\eta \in V(S)$. Then $w\theta = s_1s_2 \cdots s_n$ but $s_1\eta \cdots s_n\eta \geqq (s_1 \cdots s_n)\eta$. Hence

$$w\theta\eta = \min\{u \in V(S): w\theta = u\theta\}\ .$$

(ii) Let G and H be respectively the maximal group homomorphic images of S and $V(S)$, with α, β the corresponding canonical homomorphisms, and consider the diagram

$$\begin{array}{ccc} V(S) & \xrightarrow{\beta} & H \\ \uparrow{\scriptstyle\eta} & & \\ S & \xrightarrow[\alpha]{} & G \end{array}$$

Since α is a v-prehomomorphism of S into a group, there is a unique homomorphism $\psi: H \rightarrow G$ such that $\alpha = \eta\beta\psi$. On the other hand, any v-prehomomorphism of S into a group is actually a homomorphism. Hence there is a unique homomorphism $\chi: G \rightarrow H$ such that $\eta\beta = \alpha\chi$. Thus

$$\alpha 1_G = \alpha = \alpha\chi\psi \qquad \text{whence, since } \alpha \text{ is onto, } \chi\psi = 1_G$$

and

$$\eta\beta\psi\chi = \eta\beta = \eta\beta 1_H \quad \text{whence } \psi\chi = 1_H .$$

It follows that χ and ψ are inverse isomorphisms so that $G \approx H$.

(iii) Each element of $V(S)$ has the form $s_1\eta \cdots s_n\eta$ with $s_1, \cdots, s_n \in S$. Hence, to prove that 1η is the identity of $V(S)$, it suffices to show that $1\eta s\eta = s\eta = s\eta 1\eta$ for each $s \in S$. Now, $1^{-1}1 = 1 \geqq ss^{-1}$ and $11^{-1} = 1 \geqq s^{-1}s$ so, by Lemma 1.4, $s\eta 1\eta = (s1)\eta = s\eta = (1s)\eta = 1\eta s\eta$.

The case when S has a zero is treated similarly.

It follows from Theorem 2.3 that the problem of describing the v-prehomomorphisms with domain S is the same as that of describing homomorphisms with domain $V(S)$. In particular each v-prehomomorphism is a homomorphism if and only if η is a homomorphism, thus an isomorphism, of S into $V(S)$. Since $V(S)$ is generated, as an inverse semigroup, by S this occurs if and only if η is an isomorphism of S onto $V(S)$.

PROPOSITION 2.5. *Let S be an inverse semigroup whose idempotents form a chain. Then $\eta: S \rightarrow V(S)$ is an isomorphism.*

Proof. Let $a, b \in S$; then either $a^{-1}a \geqq bb^{-1}$ or $bb^{-1} \geqq a^{-1}a$. Hence by Lemma 1.4, $(ab)\eta = a\eta b\eta$. Thus η is a homomorphism and therefore an isomorphism.

COROLLARY 2.6. *Let S be an ω-bisimple inverse semigroup. Then $\eta: S \rightarrow V(S)$ is an isomorphism. Thus every v-prehomomorphism with domain S is a homomorphism.*

The next result and its corollaries give partial converses to Proposition 2.5.

THEOREM 2.7. *Let S be an E-unitry inverse semigroup. Then $\eta: S \to V(S)$ is an isomorphism if and only if the idempotents of S form a chain.*

Proof. Suppose $S = P(G, \mathscr{X}, \mathscr{Y})$ where $\mathscr{X}$ is a down directed partially ordered set having $\mathscr{Y}$ as an ideal and subsemilattice and where G acts on $\mathscr{X}$ in such a way that $\mathscr{X} = G \cdot \mathscr{Y}$; this is possible by [4], Theorem 2.6. Let $\bar{\mathscr{X}}$ denote the set of finitely generated up ideals of $\mathscr{X}$. Then G acts on $\bar{\mathscr{X}}$ by $g \cdot A = \{ga: a \in A\}$ and $\bar{\mathscr{X}}$ is a semilattice under $\cup$. Hence we may form the semidirect product $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$ of $\bar{\mathscr{X}}$ by G.

For each $(a, g) \in S$ define

$$(a, g)\phi = (A, g) \quad \text{where} \quad A = \{x \in \mathscr{X}: x \geqq a\}\ .$$

Then, for $(a, g), (b, h) \in S$ with $(a, g)\phi = (A, g)$, $(b, h)\phi = (B, h)$,

$$(a, g)\phi(b, h)\phi = (A \cup gB, gh)$$

while $[(a, g)(b, h)]\phi = (C, gh)$ where $C = \{x \in \mathscr{X}: x \geqq a > gb\} \subseteqq A \cup gB$. The partial order on $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$ is defined by $(U, u) \leqq (V, v)$ if and only if $u = v$ and $V \subseteqq U$. Hence $[(a, g)(b, h)]\phi \leqq (a, g)\phi(b, h)\phi$. Further, it is easy to see that $(a, g)^{-1}\phi = [(a, g)\phi]^{-1}$. Thus ϕ is a v-prehomomorphism of S into $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$.

Suppose now that $\eta: S \to V(S)$ is an isomorphism, then ϕ also is a homomorphism. Let $e, f \in \mathscr{Y}$ and set $(e, 1)\phi = (U, 1)$, $(f, 1)\phi = (V, 1)$. Then, from the definition of ϕ, $(e, 1)\phi(f, 1)\phi = (U \cup V, 1)$. On the other hand, since ϕ is a homomorphism, $(e, 1)\phi(f, 1)\phi = (e \wedge f, 1)\phi$. Hence $U \cup V = \{x \in \mathscr{X}: x \geqq e \wedge f\}$. This implies $e \wedge f \in U$ or $e \wedge f \in V$; that is $e \wedge f \geqq e$ or $e \wedge f \geqq f$. Thus either $f \geqq e$ or $e \geqq f$. It follows that the idempotents of S form a chain.

The converse is immediate from Proposition 2.5.

COROLLARY 2.8. *Let S be a semilattice. Then $V(S)$ is a semilattice; further $\eta: S \to V(S)$ is an isomorphism if and only if S is a chain.*

Proof. The fact that $V(S)$ is a semilattice is immediate from Lemma 2.1, since $V(S)$ is generated by $S\eta$. The other assertion is immediate from Theorem 2.7.

PROPOSITION 2.9. *Let S be an inverse semigroup and suppose that S admits an idempotent separating homomorphism onto an E-unitary inverse semigroup. Then $\eta: S \to V(S)$ is an isomorphism* *if and only if the semilattice of idempotents of S is a chain.*

Proof. Let $\theta: S \to P$ be an idempotent separating homomorphism of S onto an E-unitary inverse semigroup P and suppose that $\eta_S: S \to V(S)$ is an isomorphism. Then $\theta\eta_P = \eta_S\psi$ for some homomorphism $V(S) \to V(P)$. Thus, for idempotents $\bar{e} = e\theta, \bar{f} = f\theta$ in P, $(\bar{e}\bar{f})\eta_P = \bar{e}\eta_P\bar{f}\eta_P$. As in the proof of Theorem 2.6, this implies $\bar{e} \geqq \bar{f}$ or $\bar{f} \geqq \bar{e}$. Hence the idempotents of P, thus of S, form a chain.

COROLLARY 2.10. *Let S be a semilattice of groups then $\eta: S \to V(S)$ is an isomorphism if and only if the idempotents of S form a chain.*

Let E be a semilattice and let $\alpha \in T_E([8])$ with domain $\alpha = \{x \in E: x \leqq e\}$; if f is in the domain of α and $g\alpha = g$ for all $g \leqq f$, we shall say that f is a nontrivial fixpoint of α. If α has no nontrivial fixpoints we shall say that α is fixpoint free. We shall say that E is *locally rigid* if each non idempotent of T_E is fixpoint free. It is easy to see that T_E is E-unitary if and only if E is *locally rigid.*

COROLLARY 2.11. *Let S be an inverse semigroup whose semilattice of idempotents is locally rigid. Then $\eta: S \to V(S)$ is an isomorphism if and only if the idempotents form a chain.*

It remains an open question whether $\eta: S \to V(S)$ an isomorphism implies that the idempotents of S form a chain. In the next two sections, we consider situations when S has special structure. Here more definitive results may be given.

3. Simple and bisimple inverse semigroups.

PROPOSITION 3.1. *Let S be a simple inverse semigroup. Then $V(S)$ is a simple inverse semigroup.*

Proof. Let $w = s_1\eta \cdots s_r\eta \in V(S)$; then $w \in V(S)^1 s_i\eta V(S)^1$ for $1 \leqq i \leqq r$. On the other hand, $w \geqq (s_1 \cdots s_r)\eta$ so that $(s_1 \cdots s_r)\eta \in V(S)^1 w V(S)^1$. But, since S is simple, $s_i = u_i(s_1 \cdots s_r)v_i$ for some $u_i, v_i \in S^1$, so that $s_i\eta \leqq u_i\eta(s_1 \cdots s_r)\eta v_i\eta$ so that $s_i\eta \in V(S)^1 w V(S^1)$, $1 \leqq i \leqq r$. It follows that $w \mathscr{J} s_i\eta$, $1 \leqq i \leqq r$. This shows

(i) every element of $V(S)$ is $\mathscr{J}$-equivalent to some $s\eta$, $s \in S$

(ii) is $s, t \in S$ then $s\eta \mathscr{J} (st)\eta \mathscr{J} t\eta$.

Hence $V(S)$ is simple.

The result of Proposition 3.1 does not hold if simple is replaced by 0-simple. For example, we have

EXAMPLE 3.2. Let $S = M_2$ be the Brandt semigroup of 2×2 matrix units with non zero elements:a, a^{-1}, $e = aa^{-1}$, $f = a^{-1}a$. Then, by Lemma 1.4, $a\eta^{-1} = (a^{-1})\eta$, $e\eta = (aa^{-1})\eta = a\eta(a\eta)^{-1}$, $f\eta = (a^{-1}a)\eta = (a\eta)^{-1}a\eta$. Hence $V(S)$ has exactly one nonzero generator $a\eta$ and so is a homomorphic image of F_1^0 where F_1 denotes the free inverse semigroup on one generator, a.

On the other hand, the mapping $\theta: S \rightarrow F_1^0$ defined by $a\theta = a$, $a^{-1}\theta = a^{-1}$, $e\theta = aa^{-1}$, $f\theta = a^{-1}a$, $0\theta = 0$, is easily seen to be a v-prehomomorphism of S into F_1^0. Hence $\theta = \eta\psi$ for a unique homomorphism $\psi: V(S) \rightarrow F_1^0$. It follows that η is an isomorphism so that $V(S) \approx F_1^0$, which is not 0-simple.

In a similar way, the result of Proposition 3.1 does not hold if simple is replaced by bisimple. Indeed we have the following proposition.

PROPOSITION 3.3. *Let S be an E-unitary bisimple inverse semigroup. Then the following statements are equivalent*:

(1) *$\eta: S \rightarrow V(S)$ is an isomorphism*;

(2) *$V(S)$ is bisimple*;

(3) *the idempotents of S are totally ordered.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Suppose that $S = P(G, \mathscr{X}, \mathscr{Y})$ and, as in Theorem 2.7, consider the v-prehomomorphism ϕ of S into $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$. Then, by hypothesis, the inverse subsemigroup T of $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$ generated by $S\phi$ is bisimple.

Let $e, f \in \mathscr{Y}$ with $U = \{x \in \mathscr{X}: x \geqq e\}$, $V = \{x \in \mathscr{X}: x \geqq f\}$. Then $(U \cup V, 1) = e\phi f\phi$ so that $(U \cup V, 1)$ is $\mathscr{D}$-equivalent to $e\phi$ in T, thus in $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$. The form of Green's relations on $P(G, \bar{\mathscr{X}}, \bar{\mathscr{X}})$, [2], then implies that $U \cup V$ has a least element z. This must be either e or f so that $e \geqq f$ or $f \geqq e$. Hence the idempotents of S form a chain and (3) holds.

(3) $\Rightarrow$ (1) is immediate from Proposition 2.5.

Despite the fact that, when S is bisimple, $V(S)$ need not be bisimple and its structure is not completely determined, one can give a direct method for constructing all v-prehomomorphisms with domain S. Before doing this we need to introduce some terminology.

A *partial semigroup* is a pair (R, P), where R is a set and P is a nonempty subset of R, together with a map $P \times R \rightarrow R$, written as multiplication, such that, for $a, b \in P$, $c \in R$, $ab \in P$ and $a(bc) = (ab)c$. If (R, P) and (U, Q) are partial semigroups a morphism $\phi: (R, P) \rightarrow$

(U, Q) is a mapping $\phi: R \rightarrow U$ such that $P\phi \subseteq Q$ and $(ab)\phi = a\phi b\phi$ for $a \in P, b \in R$.

PROPOSITION 3.4. *Let S be a bisimple inverse semigroup and let e be an idempotent of S; set $R = \{x \in S: xx^{-1} = e\}$, $P = R \cap eSe$. Suppose that T is an inverse semigroup and let f be an idempotent of T; set $U = \{x \in T: xx^{-1} = f\}$ and $Q = U \cap fTf$. If ϕ is a morphism $(R, P) \rightarrow (U, Q)$ then $\theta: S \rightarrow T$ defined by*

$$s\theta = (a\phi)^{-1}b\phi \quad if \quad s = a^{-1}b$$

is a v-prehomomorphism of S into T such that $e\theta = f$.

Conversely, each such is constructed in this way.

Proof. We show first that θ is well defined. Suppose that $a^{-1}b = c^{-1}d$. Then, [9], $c = ga$, $d = gb$ for some $g \in P$ such that $gg^{-1} = g^{-1}g = e$. Thus

$$\begin{aligned} c\phi^{-1}d\phi &= (g\phi a\phi)^{-1}g\phi b\phi \\ &= (a\phi)^{-1}g\phi^{-1}g\phi b\phi \\ &= (a\phi)^{-1}(g^{-1}gb)\phi = (a\phi)^{-1}b\phi \end{aligned}$$

since $g^{-1}g = e$ is a left identity for R.

Next, let $a^{-1}b, c^{-1}d \in S$ and choose $u, v \in P$ such that $ub = vc$ and $Pb \cap Pc = Pub$; this is possible since S is bisimple, see [9]. Then $a^{-1}bc^{-1}d = (ua)^{-1}vd$. Thus

$$\begin{aligned} (a^{-1}bc^{-1}d)\phi &= (ua)\phi^{-1}(vd)\phi \\ &= (a\phi)^{-1}(u\phi)^{-1}(v\phi)d\phi \\ &= (a\phi)^{-1}(u\phi)^{-1}(v\phi)c\phi(c\phi)^{-1}d\phi \text{ since } c\phi\mathscr{R}d\phi \\ &= (a\phi)^{-1}(u\phi)^{-1}(ub)\phi(c\phi)^{-1}d\phi \text{ since } ub = vc \\ &= (a\phi)^{-1}(u\phi)^{-1}(u\phi)b\phi(c\phi)^{-1}d\phi \\ &\leqq [(a\phi)^{-1}b\phi][(c\phi)^{-1}d\phi] \text{ since } (u\phi)^{-1}u\phi \text{ is idempotent ,} \end{aligned}$$

while, by definition $s^{-1}\theta = (s\theta)^{-1}$ for each $s \in S$. Hence θ is a v-prehomomorphism of S into T, and, since e, f are the unique idempotents in R, U, $e\theta = f$.

Conversely, let $\theta: S \rightarrow T$ be a v-prehomomorphism such that $e\theta = f$. Then for $a \in R$, $e\theta = (aa^{-1})\theta = a\theta a\theta^{-1}$ so that $a\theta \in U$. Further, if $b \in P$ then $b = be$ implies $b^{-1}b = b^{-1}be \leqq e$ so that, by Lemma 1.4, $(ba)\theta = b\theta a\theta$; in particular $b\theta = b\theta f$ so that $b\theta \in Q$. Hence the restriction ϕ of θ to R is a morphism of (R, P) into (U, Q).

Finally, if $s = a^{-1}b \in S$ then, since $(a^{-1})^{-1}a^{-1} = aa^{-1} = bb^{-1}$, Lemma 1.4 shows that $s\theta = a\theta^{-1}b\theta = a\phi^{-1}b\phi$.

The result in Proposition 3.4 can easily be adapted to deal with the case of a 0-bisimple inverse semigroup.

Proposition 3.4 can be used to give necessary and sufficient conditions for $V(S)$ to be bisimple whenever S is a bisimple monoid. However these conditions can not be regarded as giving a completely satisfactory answer to the problem.

PROPOSITION 3.5. *Let $S = S^1$ be a bisimple inverse monoid with right unit subsemigroup R. Then $V(S)$ is bisimple if and only if S is the unique inverse monid having right unit subsemigroup R and generated as an inverse semigroup, by R. In this case $\eta: S \to V(S)$ is an isomorphism.*

Proof. Suppose that S is the unique inverse semigroup generated by R and having right unit subsemigroup R. We shall show that $V(S)$ has right unit subsemigroup $R\eta$. Then $\eta: S \to V(S)$ is an isomorphism and $V(S)$ is bisimple.

Let $x\eta y\eta$ be a right unit in $V(S)$. Then $x\eta y\eta y\eta^{-1} x\eta^{-1} = 1\eta$ so that $x\eta^{-1} x\eta = x\eta^{-1}(x\eta y\eta y\eta^{-1} x\eta^{-1}) x\eta = x\eta^{-1} x\eta y\eta y\eta^{-1} \leqq y\eta y\eta^{-1}$. Hence, by Lemma 1.4, $x\eta y\eta = (xy)\eta$ so that, since $(xy)\eta(xy)\eta^{-1} = xy(xy)^{-1}\eta$ and η is one-to-one, $x\eta y\eta \in R\eta$. Now suppose that $w = s_1\eta \cdots s_n\eta, n \geqq 2$ is a right unit of $V(S)$. Then $s_1\eta s_2\eta$ is a right unit so that $s_1\eta s_2\eta = (s_1s_2)\eta$. Repetition then gives $w = (s_1s_2 \cdots s_n)\eta$ and, as above $s_1 \cdots s_n \in R$. Hence, since each member of $R\eta$ is a right unit, we have shown that $V(S)$ has right unit subsemigroup $R\eta$.

Since S is generated by R and $V(S)$ is generated by $S\eta$, $V(S)$ is, by Proposition 3.1, a simple inverse semigroup generated by $R\eta$. Hence $V(S) \approx S$ is bisimple and then, every element of $V(S)$ is of the form $a\eta^{-1} b\eta$ with $a, b \in R\eta$. Hence η is onto so that, since $1_s = \eta\theta$ for some homomorphism $\theta: V(S) \to S$, η is an isomorphism.

Conversely, suppose $V(S)$ is bisimple and let $U(R)$ be the free inverse semigroup with right unit subsemigroup R, and generated by R. Then [4], $U(R)$ is simple and, by Proposition 3.4, the mapping $\phi: a^{-1}b \to (a\nu)^{-1} b\nu$ is a v-prehomomorphism; here ν is the embedding $R \to U(R)$. Hence $\phi = \eta\theta$ for some homomorphism θ of $V(S)$ into $U(R)$. Since $U(R)$ is generated by $R\nu$, θ is onto. Hence $U(R)$ is bisimple with right unit subsemigroup isomorphic to R and so $S \approx U(R)$ is the only inverse semigroup with right unit subsemigroup R and generated by R.

4. **Semilattices of groups.** This section follows the pattern of §3. In the first part we show that, if S is a semilattice of groups then $V(S)$ need not be a semilattice of groups. In the second part, we give a method for constructing all v-prehomomorphisms of a semilattice of groups into an inverse semigroup T

DEFINITION 4.1. Let S be a semilattice of groups. Then the *trunk* of S is the set

$$\{a \in S\colon \text{ for each } e^2 = e \in S \text{ either } aa^{-1} \leqq e \text{ or } aa^{-1} \geqq e\}\,.$$

Note that the trunk of S is an inverse subsemigroup of S. If the idempotents of S form a tree then the trunk is an ideal of S.

PROPOSITION 4.2. *Let S be a semilattice of groups whose idempotents form a tree. Then $V(S)$ is a semilattice of groups if and only if every nontrivial subgroup of S is contained in the trunk.*

Proof. Suppose that each nontrivial subgroup of S is contained in the trunk. Let $a \in S$ and suppose that a is not idempotent; thus a belongs to the trunk of S. Then, by Lemma 1.4, $a\eta b\eta = (ab)\eta$ for each $b \in S$. It follows that each element of $V(S)$ has one of the forms $a\eta$, where a is a nonidempotent in the trunk of S, or $e_1\eta e_2\eta \cdots e_r\eta$ where $e_1, e_2, \cdots, e_r$ are idempotents.

Since η is one-to-one, it follows that the non-idempotents of $V(S)$ are the elements $a\eta$ where a is a nonidempotent in the trunk of S. We show that each such $a\eta$ commutes with all the idempotents of $V(S)$. Let $e_1\eta e_2\eta \cdots e_r\eta$ be an idempotent of $V(S)$. Then

$$\begin{aligned} e_1\eta e_2\eta \cdots e_r\eta a\eta &= e_1\eta \cdots (e_r a)\eta \quad \text{by Lemma 1.4} \\ &= e_1\eta \cdots e_{r-1}\eta(ae_r)\eta \quad \text{since idempotents in} \\ &\qquad \text{are central} \\ &= (e_1\eta \cdots e_{r-1}\eta)a\eta e_r\eta \end{aligned}$$

which repeating the argument is equal to $a\eta(e_1\eta \cdots e_r\eta)$.

Hence each nonidempotent of $V(S)$ belongs to a subgroup; that is, $V(S)$ is a semilattice of groups.

Conversely, suppose that H is a nontrivial maximal subgroup, with identity e, not contained in the trunk of S. Then there is a maximal subgroup K, with identity f, such that $e \ngeqq f$, $f \ngeqq e$. Let $T = H \cup K \cup \{0\}$ and turn T into a semilattice of groups with linking homomorphisms $H \to \{0\}$, $K \to \{0\}$. Then the mapping $\theta\colon S \to T$ defined by

$$a\theta = \begin{cases} ae & \text{if } \ aa^{-1} \geqq e \\ af & \text{if } \ aa^{-1} \geqq f \\ 0 & \text{otherwise} \end{cases}$$

is a homomorphism of S onto T. Let H inv K denote the coproduct of H and K in the category of inverse semigroups and define $\phi\colon T \to (H \text{ inv } K)^0$ by $h\phi = h$, for $h \in H$, $k\phi = k$ for $k \in K$ and $0\phi = 0$, where we regard h and k as being contained in $H \operatorname{inv} K$. Then ϕ is a v-prehomomorphism of T into $(H \operatorname{inv} K)^0$ so that $\psi = \theta\phi$ is a v-prehomomorphism of S into $(H \operatorname{inv} K)^0$. But, [6], $H \operatorname{inv} K$ is not a semilattice of groups. Hence $V(S)$ is not a semilattice of groups.

REMARK 4.3. One can show that $V(T) \approx (H \operatorname{inv} K)^0$.

Proposition 4.2 is false without the assumption that the idempotents of S form a tree.

EXAMPLE 4.4. Let H be a nontrivial group with identity e and let $\{f\}$ be a one-element group. Construct the semilattice of groups with linking maps given by the diagram

$$\begin{tikzcd}[column sep=small] & H_1 = H \arrow[dl, "i_H"'] \arrow[dr] & \\ H = H_2 \arrow[dr] & & \{f\} \arrow[dl] \\ & \{0\} & \end{tikzcd}$$

where the unmarked maps are the obvious ones. Denote the resulting semigroup by S. Then, by Lemma 1.4, each element of $V(S)$ is in $S\eta$ or is a product of terms from $H_2\eta \cup \{f\eta\}$. Let $h_2 \in H_2$ then

$$\begin{aligned} h_2\eta f\eta &= (e_2h_1)\eta f\eta && \text{where } h_1 = h_2 \text{ in } H_1 \\ &= e_2\eta h_1\eta f\eta && \text{since } h_1h_1^{-1} \geqq e_2 \\ &= e_2\eta(h_1f)\eta && \text{since } h_1^{-1}h_1 \geqq f \\ &= e_2\eta f\eta \\ &= f\eta h_2\eta \,. \end{aligned}$$

It follows that $V(S) = S\eta \cup \{e_2\eta f\eta\} \approx S^0$ so that $V(S)$ is a semilattice of groups. However H_2 does not belong to the trunk of S.

We now turn to the problem of describing the v-prehomomorphisms on a semilattice of groups S. In order to do this we need to construct a family of semilattices of groups based on a semilattice E.

Let E be a semilattice and let $\theta: E \to T$ be an isotone mapping of E into the idempotents of an inverse semigroup T. For each $e \in E$, set $K_e = \{h \in H_{e\theta}: h(f\theta) = (f\theta)h$ for each $f \leqq e$ in $E\}$. It is clear that K_e is a subgroup of H_e. Suppose that $e \geqq f$ and define $\phi_{e,f}$ by

$$h\phi_{e,f} = h(f\theta) \qquad \text{for each } h \in K_e \,.$$

LEMMA 4.5. *Each $\phi_{e,f}$, $e \geqq f$ is a homomorphism of K_e into K_f. Further $\phi_{e,e}$ is the identity on K_e while, if $e \geqq f \geqq g$, then $\phi_{e,g} = \phi_{e,f}\phi_{f,g}$.*

Proof. This is straightforward.

It follows, from Lemma 4.5, that we can construct an inverse semigroup which is the semilattice of groups $\{K_e: e \in E\}$ with linking homomorphisms $\phi_{e,f}$, $e \geqq f$. We shall denote this semigroup by $SL(E, \theta, T)$.

PROPOSITION 4.6. *Let S be a semilattice of groups with semilattice of idempotents E. Let θ be an isotone mapping of E into the idempotents of an inverse semigroup T. Suppose that ϕ is an idempotent separating homomorphism of S into $SL(E, \theta, T)$. Then ψ defined by*

$$a\psi = a\phi$$

regarded as an element of T is a v-prehomomorphism of S into T such that $e\psi = e\theta$ for each $e^2 = e \in S$.

Conversely, each such v-prehomomorphism has this form for a unique idempotent separating homomorphism $\phi: S \rightarrow SL(E, \theta, T)$.

Proof. It is clear that ψ is a mapping of S into T such that $e\psi = e\theta$ for each $e^2 = e \in S$ and that $(a^{-1})\psi = (a\psi)^{-1}$ for each $a \in S$. Suppose that $a \in H_e$, $b \in H_f$ then $ab \in H_{ef}$ implies

$$\begin{aligned}(ab)\psi &= (ab)\phi = a\phi b\phi = a\phi\phi_{e,ef}b\phi\phi_{f,ef}\\ &= a\psi(ef)\theta b\psi(ef)\theta\\ &\leqq a\psi b\psi \quad \text{since } (ef)\theta \text{ is idempotent}.\end{aligned}$$

Hence ψ is a v-prehomomorphism.

Conversely, let ψ be a v-prehomomorphism of S into T such that $e\psi = e\theta$ for each $e^2 = e \in S$. Suppose that $h \in H_e$ and let $f \leqq e$. Then

$$h\psi f\theta = h\psi f\psi = (hf)\psi = (fh)\psi = f\psi h\psi = f\theta h\psi$$

by Lemma 1.4 since $hh^{-1} = h^{-1}h \geqq f$. Hence $h\psi \in K_e$. Further, by Lemma 1.4, $h_1\psi h_2\psi = (h_1h_2)\psi$ for $h_1, h_2 \in H_e$. Thus ϕ defined by

$$h\phi = h\psi \quad \text{regarded as a member of} \quad SL(E, \theta, T)$$

is an idempotent separating mapping of S into $SL(E, \theta, T)$ which is a homomorphism on each subgroup of S. Now let $h \in H_e$, $k \in H_f$. Then

$$
\begin{aligned}
h\phi k\phi &= h\phi\phi_{e,ef}k\phi\phi_{f,ef} \\
&= h\psi(ef)\theta k\psi(ef)\theta \\
&= h\psi(ef)\psi k\psi(ef)\psi \\
&= (hef)\psi(kef)\psi && \text{by Lemma 1.4} \\
&= (hef\,kef)\psi = (hk)\psi && \text{by Lemma 1.4}\,.
\end{aligned}
$$

Hence ϕ is a homomorphism and

$$h\psi = h\phi \quad \text{considered as a member of} \quad T\,.$$

Finally, the uniqueness of ϕ is immediate.

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