

GLOBAL ANALYSIS AND PERIODIC SOLUTIONS OF SECOND ORDER SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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We establish the existence of global closed connected sets of solutions of nonlinear operator equations with linearized part a polynomial in λ , bifurcating from characteristic values of odd multiplicity. These results are then applied to finding large periodic solutions of systems of second order nonlinear differential equations.

Introduction. Using Leray-Schauder degree theory P. H. Rabinowitz [10] has shown the existence of global continua of solutions bifurcating from characteristic values of odd multiplicity of the linearized part of completely continuous nonlinear operator equations. By purely local bifurcation results, unrelated to those of Rabinowitz, the author [13] has extended M. S. Berger's [1] variational techniques to find small periodic solutions of systems of second order nonlinear differential equations. In this paper we combine the two methods and develop Rabinowitz's global analysis for nonlinear operators whose linearized part is a polynomial in λ and apply these results to the existence of large periodic solutions of second order differential equations of the form

$$\begin{aligned}u'' + A_1 v' + B_1 u + F_1(w, w', w'') &= 0 \\v'' + A_2 u' + B_2 v + F_2(w, w', w'') &= 0\end{aligned}$$

where $w = (u, v)$.

1. Global analysis. We consider the existence of continua of solutions of equations of the form

$$(1.1) \quad x = \lambda Lx + G(\lambda, x)$$

where L is a completely continuous linear map of a real Banach space X into itself and $G(\lambda, x)$ is a completely continuous map [12, p. 9] of $\mathbf{R} \times X \rightarrow X$ satisfying $G(\lambda, x) = o(\|x\|)$ for x near zero, uniformly on bounded λ intervals. A solution of Eq. (1.1) is a point $(\lambda, x) \in \mathbf{R} \times X$ satisfying Eq. (1.1) and will be called trivial if $x = 0$. A $\lambda_0 \in \mathbf{R}$ is said to be a bifurcation point if every neighborhood of $(\lambda_0, 0)$ contains nontrivial solutions of Eq. (1.1). The closure of the set of nontrivial solutions of (1.1) will be denoted S .

A $\lambda_0 \in \mathbf{R}$ will be called a characteristic value of a linear operator L if there exists a nonzero $x_0 \in X$ such that $x_0 = \lambda_0 Lx_0$. The set of

characteristic values of L will be denoted $r(L)$. The domain, range and null space of an operator B will be denoted by $D(B)$, $R(B)$ and $N(B)$ respectively. For convenience we will let $N_k(B) = N(B^k)$, $N_\infty(B) = \bigcup_{i=1}^\infty N(B^i)$, and $R_k(B) = R(B^k)$. The smallest integer $k > 0$ such that $N_k(B) = N_\infty(B)$ will be denoted $\alpha(B)$. The algebraic multiplicity, $M(\lambda_0, L)$, of $\lambda_0 \in r(L)$ is $\dim N_\infty(I - \lambda_0 L)$.

Suppose that G is not globally defined. Then by employing an obvious modification of E. N. Dancer's work [4, Corollary 2] we can obtain the following extension of Rabinowitz's Theorem 1.12 in [10].

THEOREM 1.1. *Let $P = \{\lambda_1, \dots, \lambda_p\}$ be a set of consecutive characteristic values of L , $\lambda_1 < \dots < \lambda_p$, and suppose $\sum_{i=1}^p M(\lambda_i, L)$ is odd. If Ω is a bounded open set in $\mathbf{R} \times X$ containing $[\lambda_1, \lambda_p] \times \{0\}$ and $G(\lambda, x)$ is completely continuous on $\bar{\Omega}$ then S possesses a maximal subcontinuum C_P such that $(\lambda_j, 0) \in C_P$ for some $\lambda_j \in P$ and C_P either*

- (i) *meets $\partial\Omega$ or*
- (ii) *meets $(\hat{\lambda}, 0) \in \Omega$, where $\hat{\lambda} \in (r(L) - P)$.*

By a subcontinuum of S we mean a subset of S which is closed and connected in $\mathbf{R} \times X$. By $\partial\Omega$ we mean of course the boundary of Ω .

With Theorem 1.1 we can obtain global results for $G(\lambda, x)$ globally defined but now L is just a bounded linear map of X into X such that for all $\lambda \in \mathbf{R}$, $R(I - \lambda L)$ is closed and $N_\infty(I - \lambda L)$, $N_\infty(I - \lambda L^*) < \infty$, L^* the conjugate of L . With these assumptions on L we have the lemma

LEMMA 1.2. *If $k = \alpha(I - \lambda L)$ then $R_k(I - \lambda L)$ is closed, $X = N_k(I - \lambda L) \oplus R_k(I - \lambda L)$ and $I - \lambda L$ is a one-one map of $R_k(I - \lambda L)$ onto $R_k(I - \lambda L)$ with bounded inverse.*

Proof. See [14].

Lemma 1.2 implies that if $1/\lambda$ is in the spectrum of L then $\lambda \in r(L)$ and $r(L)$ is discrete.

THEOREM 1.3. *Let L and G be as described above and let $P = \{\lambda_1, \dots, \lambda_p\}$ be a set of consecutive characteristic values of L , $\lambda_1 < \dots < \lambda_p$. Suppose $\sum_{i=1}^p M(\lambda_i, L)$ is odd. Then S contains a maximal subcontinuum C_P such that $(\lambda_j, 0) \in C_P$ for some $\lambda_j \in P$ and C_P is either*

- (i) *unbounded in $\mathbf{R} \times X$ or*
- (ii) *meets $(\hat{\lambda}, 0)$, where $\hat{\lambda} \in (r(L) - P)$.*

Proof. The theorem will be proved by repeated applications of Theorem 1.1. For each positive integer n define $I_n = (\lambda_1 - n, \lambda_p + n)$ and $S_n = \{x \in X \mid \|x\| < n\}$. Let $\lambda_1, \dots, \lambda_p, \dots, \lambda_s$ be the characteristic values of L contained in \bar{I}_n . Then by Lemma 1.3, $X = N \oplus R$, where

$$N = N_{k_1}(I - \lambda_1 L) \oplus \dots \oplus N_{k_s}(I - \lambda_s L) \\ k_i = \alpha(I - \lambda_i L), \quad i = 1, \dots, s$$

and

$$R = R((I - \lambda_1 L)^{k_1} \dots (I - \lambda_s L)^{k_s}).$$

Thus $x \in X$ can be uniquely expressed as $x = u + v$ where $u \in N$ and $v \in R$. Moreover, for all $\lambda \in R$, $I - \lambda L: N \rightarrow N$, $R \rightarrow R$ and $G = G_N + G_R$ where $G_N \in N$ and $G_R \in R$. Thus our problem is equivalent to that of finding solutions $(\lambda, u + v) \in R \times (N \oplus R)$ of the system of equations

$$(1.2) \quad u - \lambda Lu = G_N(\lambda, u + v)$$

$$(1.3) \quad v - \lambda Lv = G_R(\lambda, u + v).$$

Since $I - \lambda L$ has a bounded inverse on R for all $\lambda \in \bar{I}_n$ and $(I - \lambda L)^{-1}$ is continuous in λ for all $1/\lambda$ in the resolvent of L (as a mapping of $R \rightarrow R$) [5, p. 257], $(\lambda, u + v) \in \bar{\Omega}_n = \overline{\bar{I}_n \times S_n}$ is a solution of (1.2), (1.3) if and only if $(\lambda, u + v)$ is a solution of the system

$$(1.4) \quad u = \lambda Lu + G_N(\lambda, u + v)$$

$$(1.5) \quad v = (I - \lambda L)^{-1} G_R(\lambda, u + v).$$

As $(I - \lambda L)^{-1} G_R(\lambda, u + v)$ is a completely continuous map of $\bar{\Omega}_n$ into R and $\dim N < \infty$, Eqs. (1.4), (1.5) satisfy the hypothesis of Theorem 1.1. Thus for some $1 \leq j \leq p$ and each integer n , $\bar{\Omega}_n$ contains a maximal closed connected subset C_n of S such that $(\lambda_j, 0) \in C_n$ and C_n either meets the boundary of Ω_n or meets $(\hat{\lambda}, 0)$, $\hat{\lambda} \in (r(L) - P)$. As the union of connected sets containing a common point is connected, an application of Zorn's lemma [8, p. 62] will show that S contains a unique maximal subcontinuum C_P meeting $(\lambda_j, 0)$. Thus $\bigcup_{n=1}^{\infty} C_n \subset C_P$ and our theorem is proved.

REMARK. If G is not globally defined but is instead completely continuous on the closure of an open bounded set $\Omega \subset R \times X$, containing $[\lambda_1, \lambda_p] \times \{0\}$, and L and P are as in Theorem 1.3, then by replacing the Ω_n by Ω we can readily prove that if $\sum_{i=1}^p M(\lambda_i, L)$ is odd, S possesses a maximal subcontinuum $C_P \subset \bar{\Omega}$ such that $C_P \cap (P \times \{0\}) \neq \emptyset$ and either (i) meets $\partial\Omega$ or (ii) meets $(\hat{\lambda}, 0) \in \bar{\Omega}$, where

$\hat{\lambda} \in (r(L) - P)$.

If we consider (1.4) and (1.5) and modify Rabinowitz's proof of Theorem 1.16 in [1], (as was done in [4]) for C_P in a sufficiently large Ω_n we can readily verify

THEOREM 1.4. *Suppose the hypotheses of Theorem 1.3 are satisfied and (i) does not occur. Then C_P meets a point $(\hat{\lambda}, 0)$ such that $\hat{\lambda} \in (r(L) - P)$ and $M(\hat{\lambda}, L)$ is odd.*

REMARK. Similarly we could show that when a G is not globally defined and (i) of the above remark does not obtain then C_P meets a $(\hat{\lambda}, 0)$ and $M(\hat{\lambda}, L)$ is odd.

The existence of continua of nontrivial solutions can also be shown for completely continuous operator equations of the form

$$(1.6) \quad x = (\lambda^n A_n + \cdots + \lambda A_1)x + G(\lambda, x)$$

where $G(\lambda, x)$ is $o(\|x\|)$ for x near zero, uniformly on bounded λ intervals and A_1, \dots, A_n are completely continuous linear operators of a real Banach space X into itself. We will call λ_0 a characteristic value of an operator polynomial $A(\lambda) = \lambda^n A_n + \cdots + \lambda A_1$ if there exists a nonzero vector $x_0 \in X$ such that $A(\lambda_0)x_0 = x_0$. The set of such characteristic values will be denoted $r(A)$.

The geometric multiplicity of $\lambda_0 \in r(A)$ is $\dim N(I - A(\lambda_0))$. To define its algebraic multiplicity we consider the associated matrix of operators $W + V: X^n \rightarrow X^n$, where

$$(1.7) \quad W = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & A_n \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & A_1 \end{bmatrix}$$

and

$$(1.8) \quad V = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ I & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & I & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & I & 0 \end{bmatrix}.$$

If $x = (x_1, \dots, x_n)$, $x_1, \dots, x_n \in X$, then the operator equation $x = \lambda(W + V)x$ is the system of equations

$$\begin{aligned}
x_1 &= \lambda A_n x_n \\
x_2 &= \lambda (A_{n-1} x_n + x_1) \\
&\vdots \\
x_n &= \lambda (A_1 x_n + x_{n-1}) .
\end{aligned}$$

Thus it is clear $\lambda_0 \in r(A)$ if and only if $\lambda_0 \in r(W + V)$. We therefore define the algebraic multiplicity $M(\lambda_0, A)$ of λ_0 to be the

$$\dim N_\infty(I - \lambda_0(W + V)) .$$

REMARK. Our definition is an analog of the one given for finite dimensional square matrices. The algebraic multiplicity of a characteristic value λ_0 of a matrix A is the multiplicity of the eigenvalue $\beta_0 = 1/\lambda_0$ as a root of the characteristic equation $\det |\beta I - A| = 0$ [7, p. 104]. Correspondingly, the algebraic multiplicity of a characteristic value λ_0 of a matrix polynomial $A(\lambda)$ is the multiplicity of $\beta_0 = 1/\lambda_0$ as a root of the equation

$$\det |\beta^n I - A_n - \dots - \beta^{n-1} A_1| = 0 .$$

Moreover, by the Jordan canonical form the algebraic multiplicity of a characteristic value λ_0 of a matrix A , is equal to $\dim N_\infty(I - \lambda_0 A)$. Similarly, the algebraic multiplicity of a characteristic value λ_0 of $A(\lambda)$ is $\dim N_\infty(I - \lambda_0(W + V))$. Indeed $\dim N_\infty(I - \lambda_0(W + V))$ is equal to the multiplicity of $\beta_0 = 1/\lambda_0$ as a root of the characteristic equation $\det |\beta I - W - V| = 0$. But

$$\det |\beta I - W - V| = \det |\beta^n I - A_n - \dots - \beta^{n-1} A_1|$$

since by elementary row operations we have

$$\begin{aligned}
&\det |\beta I - W - V| \\
&= \begin{vmatrix} \beta I & 0 & \cdot & \cdot & \cdot & -A_n \\ -I & \beta I & \cdot & \cdot & \cdot & -A_{n-1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & & & -I & \beta I - A_1 & \end{vmatrix} \\
&= \begin{vmatrix} \beta I & 0 & \cdot & \cdot & \cdot & -A_n \\ 0 & \beta I & & & & -\beta^{-1} A_n - A_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \beta I - \beta^{-n+1} A^n - \dots - A_1 \end{vmatrix} \\
&= \det |\beta^n I - A_n - \dots - \beta^{n-1} A_1| .
\end{aligned}$$

The closure of the set of nontrivial solutions of Eq. (1.6) will be denoted by S . A $\lambda_0 \in \mathbf{R}$ will be a bifurcation point if every neighborhood of $(\lambda_0, 0)$ contains nontrivial solutions of Eq. (1.6). It is clear that if λ_0 is a bifurcation point of Eq. (1.6) then $\lambda_0 \in r(A)$. To see this we note that by the Fredholm theory [11, Chaps. IV, V] if $\lambda_0 \notin r(A)$ then $I - A(\lambda)$ is invertible for all λ near λ_0 with uniformly bounded inverse. Writing (1.6) for $x \neq 0$ in the form

$$x/\|x\| = (I - A(\lambda))^{-1}G(\lambda, x)/\|x\|$$

we see λ_0 is not a bifurcation point of Eq. (1.6) as $G = o(\|x\|)$.

We can show

THEOREM 1.5. *Let $P = \{\lambda_1, \dots, \lambda_p\}$ be a set of consecutive characteristic values of $A(\lambda)$, $\lambda_1 < \dots < \lambda_p$, and suppose $\sum_{i=1}^p M(\lambda_i, A)$ is odd. Then S contains a maximal subcontinuum C_P such that $(\lambda_j, 0) \in C_P$ for some $\lambda_j \in P$ and C_P is either*

- (i) *unbounded in $\mathbf{R} \times X$ or*
- (ii) *meets $(\hat{\lambda}, 0)$ where $\hat{\lambda} \in (r(A) - P)$.*

Proof. Showing the existence of C_P is equivalent to showing the existence of a maximal continuum of solutions C'_P of

$$(1.9) \quad x = \lambda(W + V)x + \hat{G}(\lambda, x)$$

in X^n , where

$$(1.10) \quad \hat{G}(\lambda, x) = \hat{G}(\lambda, x_1, \dots, x_n) = (0, \dots, 0, G(\lambda, x_n)).$$

We will show Eq. (1.9) satisfies the hypotheses of Theorem 1.3. By (1.10) $\hat{G}(\lambda, x)$ is completely continuous and $\hat{G}(\lambda, x) = o(\|x\|)$ for x near zero, uniformly on bounded λ intervals. In addition, $\dim N_\infty(I - \lambda(W + V))$, $\dim N_\infty(I - \lambda(W + V)^*) < \infty$ and $R(I - \lambda(W + V))$ is closed for all real λ . Indeed, one can readily verify $V^{n-1} = 0$. Thus as $(W + V)^m = K_m + V^m$ for all positive integers m , where K_m is completely continuous, it follows that $(V + W)^{n-1}$ is completely continuous. Moreover

$$N_\infty(I - \lambda(W + V)) \subset N_\infty(I - \lambda^{n-1}(W + V)^{m-1}).$$

Thus by the Riesz-Schauder theorem [11, p. 183 and p. 219], $\dim N_\infty(I - \lambda(W + V))$ is finite. Similarly as $((W + V)^*)^{n-1}$ is completely continuous, $\dim N_\infty(I - \lambda(W + V)^*) < \infty$. Lastly, as $(V + W)^{n-1}$ is completely continuous it readily follows that $R(I - \lambda(W + V))$ is closed [6, p. 705].

Hence Eq. (1.9) satisfies the hypothesis of Theorem 1.3 and the result follows.

An argument analogous to that of the proof of Theorem 1.5 will show that if G is not globally defined and $A(\lambda)$ and P are as in Theorem 1.5 then

THEOREM 1.6. *If Ω is a bounded open set in $\mathbf{R} \times X$ containing $[\lambda_1, \lambda_p] \times \{0\}$ and $G(\lambda, x)$ is completely continuous on $\bar{\Omega}$ and $\sum_{i=1}^p M(\lambda_i, A)$ is odd then S possesses a maximal subcontinuum $C_P \subset \bar{\Omega}$ such that $(\lambda_j, 0) \in C_P$ for some $\lambda_j \in P$ and C_P either*

(i) *meets $\partial\Omega$ or*

(ii) *meets $(\hat{\lambda}, 0) \in \Omega$, where $\hat{\lambda} \in (r(A) - P)$.*

By arguing as in the proof of Theorem 1.5 and applying Theorem 1.4 or the remark following Theorem 1.4 we can verify

THEOREM 1.7. *Suppose the hypotheses of Theorem 1.5 (1.6) are satisfied and (i) does not occur. Then C_P meets a point $(\hat{\lambda}, 0)$ such that $\hat{\lambda} \in (r(A) - P)$ and $M(\hat{\lambda}, A)$ is odd.*

Theorem 1.6 can be used to show the existence of bifurcation points of operator equations which are not completely continuous. Suppose A_1, \dots, A_n are closed linear operators of X into X such that $\bigcap_{i=1}^n D(A_i) = X$ and λ_0 is a characteristic value of $\lambda^n A_n + \dots + \lambda A_1$. Let us further assume $G(\lambda, x)$ is a continuous map of a neighborhood of $(\lambda_0, 0) \in \mathbf{R} \times X$ into X satisfying

$$(1.11) \quad \|G(\lambda, x_1) - G(\lambda, x_2)\| \leq h(x_1, x_2) \|x_1 - x_2\|$$

for $(\lambda, x_1), (\lambda, x_2)$ near $(\lambda_0, 0)$ and where $h(a, b)$ is a function independent of λ tending to zero as both a and b tend to zero. If we define W and V by (1.7) and (1.8) and let S be the closure of the set of nontrivial solutions of

$$(1.12) \quad x = (\lambda^n A_n + \dots + \lambda A_1)x + G(\lambda, x)$$

we can prove

THEOREM 1.8. *If $I - \lambda_0(W + V)$ is a closed Fredholm operator of index zero [4] and $M(\lambda_0, A)$ is odd then λ_0 is a bifurcation point of Eq. (1.12) and S contains a maximal subcontinuum meeting $(\lambda_0, 0)$.*

Proof. As in the proof of Theorem 1.5 it suffices to show λ_0 is a bifurcation point of

$$(1.13) \quad x = \lambda(W + V)x + \hat{G}(\lambda, x)$$

in X^n , where \hat{G} is defined by (1.10). By hypothesis one can readily

verify that $X^n = N_p(I - \lambda_0(W + V)) \oplus R_p(I - \lambda_0(W + V))$ where $p = \alpha(I - \lambda_0(W + V))$ [14]. Thus for $x \in X^n$, $x = u + v$ where $u \in N_p(I - \lambda_0(W + V))$ and $v \in R_p(I - \lambda_0(W + V))$ and $\hat{G} = G_N + G_R$ where $G_N \in N_p(I - \lambda_0(W + V))$ and $G_R \in R_p(I - \lambda_0(W + V))$. Moreover $I - \lambda(W + V)$ is invertible on $R_p(I - \lambda_0(W + V))$ for all λ near λ_0 . Thus finding solutions of Eq. (1.12) is equivalent to finding solutions of the system

$$(1.14) \quad \begin{aligned} u &= \lambda(W + V)u + G_N(\lambda, u + v) \\ v &= (I - \lambda(W + V))^{-1}G_R(\lambda, u + v). \end{aligned}$$

An application of the contraction mapping principle [5, p. 260] to Eq. (1.14) shows the existence of a uniquely determined continuous function $v(\lambda, u)$ such that $v(\lambda, u) \equiv (I - \lambda(W + V))^{-1}G_R(\lambda, u + v(\lambda, u))$. Hence our problem is reduced to that of solving the equation

$$u = \lambda(W + V)u + G_N(\lambda, u + v(\lambda, u)).$$

As the equation satisfies the hypotheses of Theorem 1.6 there is a unique maximal subcontinuum of meeting $(\lambda_0, 0)$ and the result follows.

2. Second order systems. Let us consider autonomous systems of ordinary differential equations of the form

$$(2.1) \quad w'' + A_1 w' + A_2 w + F(w, w', w'') = 0$$

where w denotes the n -vector $(w_1(t), \dots, w_n(t))$ of real-valued functions, $w' = (dw_1/dt, \dots, dw_n/dt)$, A_1 and A_2 are real $n \times n$ matrices, F is a continuous function of $3n$ variables satisfying $F(x) = o(|x|)$ ($|x| = (x_1^2 + \dots + x_{3n}^2)^{1/2}$) and for any matrix A and vector v by Av we mean the vector $(A \cdot v^t)^t$ with t denoting the transpose. We wish to determine when the existence of periodic solutions of the linearized part,

$$(2.2) \quad w'' + A_1 w' + A_2 w = 0,$$

implies the existence of periodic solutions of (2.1) and the relationships between the periodic solutions of Eq. (2.2) and those of Eq. (2.1).

Using classical results from the theory of ordinary differential equations [3], one can readily verify that Eq. (2.2) has a solution of period $2\pi\lambda_0$ for some $\lambda_0 > 0$ if and only if λ_0 is a characteristic value of the matrix polynomial $A(\lambda) = i\lambda A_1 + \lambda^2 A_2$ [13]. From the results of §1 and [13], [1], one might expect Eq. (2.1) to have periodic solutions when $A(\lambda)$ has real characteristic values of odd algebraic multiplicity. However it will be shown in §3 that unless we make further restrictions on A_1 , A_2 and F , (2.1) may have no periodic solutions even though $A(\lambda)$ has only simple characteristic values.

Let us therefore assume A_1 and A_2 are of the special form

$$(2.3) \quad A_1 = \begin{bmatrix} 0 & \hat{A}_1 \\ \bar{A}_1 & 0 \end{bmatrix}$$

$$(2.4) \quad A_2 = \begin{bmatrix} \hat{A}_2 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}$$

where \hat{A}_1 is an $r \times q$ matrix, \bar{A}_1 a $q \times r$ matrix, \hat{A}_2 a nonsingular $r \times r$ matrix, and \bar{A}_2 a $q \times q$ matrix with $0 \leq q < n$ and $0 < r \leq n$. Thus if

$$w = (u, v) = (u_1, \dots, u_r, v_1, \dots, v_q).$$

system (2.1), can be expressed

$$(2.5) \quad u'' + \hat{A}_1 v' + \hat{A}_2 u + F_1(u, v, u', v', u'', v'') = 0$$

$$(2.6) \quad v'' + \bar{A}_1 u' + \bar{A}_2 v + F_2(u, v, u', v', u'', v'') = 0.$$

We further assume

$$F_1(u, -v, -u', v', u'', -v'') = F_1(u, v, u', v', u'', v'')$$

$$F_2(u, -v, -u', v', u'', -v'') = -F_2(u, v, u', v', u'', v'').$$

If λ_1 is a positive characteristic value of $A(\lambda)$ let $S(\lambda_1)$ be the set of all different positive characteristic values, $\lambda_1, \dots, \lambda_N$ of $A(\lambda)$ satisfying $\lambda_1/\lambda_j = \text{integer}$ ($j = 1, \dots, N$). We define $\mathcal{M}(\lambda_1) = \sum_{\lambda_j \in S(\lambda_1)} M(\lambda_j, A)$.

THEOREM 2.1. Suppose $S(\lambda_1)$ has N elements, $\mathcal{M}(\lambda_1)$ is odd, and that $|F(x) - F(y)| \leq h(x, y)|x - y|$ for $x, y \in \mathbf{R}^{3n}$ near zero, where $h(x, y) \rightarrow 0$ as $(x, y) \rightarrow 0$. Then for each sufficiently small δ there exists a nontrivial periodic solution $w_\delta(t) = (u_\delta(t), v_\delta(t))$ of Eqs. (2.5), (2.6) with period $2\pi\lambda_\delta$ such that $w_\delta(t) \rightarrow 0$ and $\lambda_\delta \rightarrow \lambda_1$, as $\delta \rightarrow 0$.

Proof. We first introduce a change of variables and set $t = \lambda s$, where λ is a real constant to be determined. The resulting system is then

$$(2.7) \quad \omega'' + \lambda A_1 \omega' + \lambda^2 A_2 \omega + \lambda^2 \hat{F}(\lambda, \omega) = 0$$

where

$$\hat{F}(\lambda, \omega) = (\hat{F}_1(\lambda, \omega), \hat{F}_2(\lambda, \omega)) = (F_1(\omega, \lambda^{-1}\omega', \lambda^{-2}\omega''), F_2(\omega, \lambda^{-1}\omega', \lambda^{-2}\omega''))$$

and $\omega = (\mu, \nu) = (\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_q)$. It suffices to determine the 2π periodic solutions of (2.7) as they correspond to $2\pi\lambda$ periodic solution of (2.1).

Even 2π periodic functions are of the form $\mu = \mu_0 + \mu^*$, where $\mu^* = 1/2\pi \int_0^{2\pi} \mu(s) ds$, the mean value of μ , and μ_0 is of mean value

zero. Thus Eq. (2.7) can be expressed equivalently as

$$(2.8) \quad \hat{A}_2 \mu^* + \frac{1}{2\pi} \int_0^{2\pi} \hat{F}_1(\lambda, \mu_0(s) + \mu^*, \nu(s)) ds = 0$$

$$(2.9) \quad \begin{aligned} \mu_0'' + \lambda \hat{A}_1 \nu' + \lambda^2 \left[\hat{A}_2 \mu_0 + \hat{F}_1(\lambda, \mu_0 + \mu^*, \nu) \right. \\ \left. - \frac{1}{2\pi} \int_0^{2\pi} \hat{F}_1(\lambda, \mu_0(s) + \mu^*, \nu(s)) ds \right] = 0 \end{aligned}$$

$$(2.10) \quad \nu'' + \lambda \bar{A}_1 \mu_0' + \lambda^2 [\bar{A}_2 \nu + \hat{F}_2(\lambda, \mu_0 + \mu^*, \nu)] = 0.$$

To apply Theorem 1.8 we introduce the appropriate Banach spaces. Let \mathcal{C}_1 be the space of twice continuously differentiable, 2π periodic, even r -vector functions of mean value zero and \mathcal{C}_2 the space of twice continuously differentiable, 2π periodic, odd q -vector functions.

Then let \mathcal{C} be the Banach space defined by $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ with the norm

$$|||\omega||| = |||(\omega_1, \dots, \omega_n)||| = \max_{\substack{j=1, \dots, n \\ k=0, 1, 2}} \{||\omega_j^{(k)}||\}$$

where $||\omega_j^{(k)}|| = \sup_{0 \leq t \leq 2\pi} |\omega_j^{(k)}(t)|$. Note that there is a $c \geq 0$ such that for all $\omega \in \mathcal{C}$, $|||\omega||| \leq c \max_{j=1, \dots, n} \{||\omega_j''||\}$. Indeed if $f(t)$ is a twice continuously differentiable 2π periodic function of mean value zero then there exist t_1 and t_2 such that $f(t_1) = f'(t_2) = 0$. Therefore for each t

$$|f(t)| = |f(t) - f(t_1)| \leq |t - t_1| |f'(\bar{t})| \leq |t - t_1| |\bar{t} - t_2| |f''(t_0)|,$$

where \bar{t} lies between t_1 and t , and t_0 lies between \bar{t} and t_2 . Hence $||f(t)|| \leq 2\pi ||f'(t)|| \leq 4\pi^2 ||f''(t)||$. Thus

$$(2.10a) \quad |||\omega||| \leq c \max_{j=1, \dots, n} \{||\omega_j''||\}.$$

Let H_1 be the Hilbert space of absolutely continuous, 2π periodic, even r -vector functions of mean value zero such that $\int_0^{2\pi} (\mu_0'(s))^2 ds < \infty$ and H_2 the Hilbert space of absolutely continuous, 2π periodic, odd q -vector functions such that $\int_0^{2\pi} (\nu'(s))^2 ds < \infty$. Let $H = H_1 \times H_2$ with inner product defined by $\langle x, y \rangle = \int_0^{2\pi} x'(s) \cdot y'(s) ds$, $x = (\mu_0, \nu)$. And lastly let X^* be the space of the μ^* with Euclidean norm. For convenience we express $(\mu_0 + \mu^*, \nu)$ as $x + \mu^*$.

\hat{A}_2 is nonsingular and we consider now finding solutions of the operator equations

$$(2.11) \quad \mu^* + \hat{A}_2^{-1} \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{F}_1(\lambda, \mu_0(s) + \mu^*, \nu(s)) ds \right] = 0$$

$$(2.12) \quad x - \mathcal{A}(\lambda)x - \lambda^2 \mathcal{F}(\lambda, x + \mu^*) = 0$$

in $R \times X^* \times \mathcal{C}$ where $\mathcal{A}(\lambda) = \lambda \mathcal{A}_1 + \lambda^2 \mathcal{A}_2$ and for $\lambda \in R$ and $x \in \mathcal{C}$, $\mathcal{A}_1 x$, $\mathcal{A}_2 x$ and $\mathcal{F}(\lambda, x)$ are defined to be the unique element in \mathcal{C} such that

$$(2.13) \quad \langle \mathcal{A}_1 x, y \rangle = \int_0^{2\pi} A_1 x'(s) \cdot y(s) ds, \quad \langle \mathcal{A}_2 x, y \rangle = \int_0^{2\pi} A_2 x(s) \cdot y(s) ds$$

and

$$(2.14) \quad \langle \mathcal{F}(\lambda, x + \mu^*), y \rangle = \int_0^{2\pi} F(\lambda, x(s) + \mu^*) \cdot y(s) ds,$$

for all $y \in H$.

It is clear that if $(\lambda, x + \mu^*)$ is a solution of Eqs. (2.11) and (2.12) then it is a solution of Eq. (2.7) as we have

$$\int_0^{2\pi} (x'' + \lambda A_1 x' + \lambda^2 A_2 x + \lambda^2 \hat{F}(\lambda, x + \mu^*)) \cdot y = 0$$

for all $y \in H$.

Now \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{F} satisfy the equations

$$d\mathcal{A}_1 x/ds = -A_1 x, \quad d^2 \mathcal{A}_2 x/ds^2 = -A_2 x$$

and

$$\partial^2 \mathcal{F}(\lambda, x + \mu^*)/\partial s^2 = -\hat{F}(\lambda, x + \mu^*) + \frac{1}{2\pi} \int_0^{2\pi} \hat{F}_1.$$

Thus an application of the Ascoli-Arzelà theorem will verify that \mathcal{A}_1 and \mathcal{A}_2 are compact maps of \mathcal{C} into \mathcal{C} . By Eq. (2.10a) it follows $(\hat{A}_2^{-1} [1/2\pi \int_0^{2\pi} \hat{F}_1], \mathcal{F})$ is a continuous nonlinear map of $R \times X^* \times \mathcal{C}$ into $X^* \times \mathcal{C}$ satisfying Eq. (1.11).

In the course of proving theorem 1.5 it was shown that $I - \lambda_0(W + V)$ is a closed Fredholm operator of index zero whenever A_1, \dots, A_n are compact. Thus by Theorem 1.8 the proof of Theorem 2.1 will be complete once we have shown that λ_1 is a characteristic value of odd algebraic multiplicity of the linearized part of (2.11), (2.12),

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{A}(\lambda) \end{bmatrix}$$

(which is a mapping of $X^* \times \mathcal{C}$ into $X^* \times \mathcal{C}$). It suffices to show $M(\lambda_1, \mathcal{A})$ is odd. To this end we prove the lemma

LEMMA 2.2. *Under the hypotheses of Theorem 2.1, $M(\lambda_1, \mathcal{A})$ is odd.*

Proof. If $\lambda_k \in S(\lambda_1)$ and $(a + ib, c + id)$ with $a, b \in \mathbf{R}^r$ and $c, d \in \mathbf{R}^r$ is a characteristic vector corresponding to λ_k , then by equating components we have the equalities

$$\begin{aligned}\lambda_k^2 \hat{A}_2 a - \lambda_k \hat{A}_1 d &= a \\ \lambda_k^2 \bar{A}_2 d + \lambda_k \bar{A}_1 a &= d \\ \lambda_k^2 \hat{A}_2 b + \lambda_k \hat{A}_1 c &= b \\ \lambda_k^2 \bar{A}_2 c - \lambda_k \bar{A}_1 b &= c.\end{aligned}$$

On the other hand it is clear that for all integers $p > 0$

$$(2.15) \quad \mathcal{A}_1(\alpha \cos ps, \beta \sin ps) = 1/p(\hat{A}_1 \beta \cos ps, -\bar{A}_1 \alpha \sin ps)$$

$$(2.16) \quad \mathcal{A}_2(\alpha \cos ps, \beta \sin ps) = 1/p^2(\hat{A}_2 \alpha \cos ps, \bar{A}_2 \beta \sin ps).$$

Thus if $p_k = \lambda_1/\lambda_k$, $(b \cos p_k s, c \sin p_k s)$ is a characteristic vector corresponding to λ_1 if and only if (b, c) is a characteristic vector corresponding to the characteristic value λ_k of $\lambda \tilde{A}_1 + \lambda^2 A_2$ where

$$\tilde{A}_1 = \begin{bmatrix} 0 & \hat{A}_1 \\ -\bar{A}_1 & 0 \end{bmatrix}.$$

Moreover from Eqs. (2.15) and (2.16) it follows that for

$$L = \begin{bmatrix} 0 & \mathcal{A}_2 \\ I & \mathcal{A}_1 \end{bmatrix}$$

if $(y, x) \in \mathcal{C} \times \mathcal{C}$ is an element of $N_\infty(I - \lambda_1 L)$ then y and x must be of the form

$$y = \sum_{j=1}^N (\alpha_j \cos p_j s, \beta_j \sin p_j s) \quad p_j = \lambda_1/\lambda_j$$

and

$$x = \sum_{j=1}^N (\gamma_j \cos p_j s, \delta_j \sin p_j s) \quad p_j = \lambda_1/\lambda_j$$

and the coefficient vector $(\alpha_j, \beta_j, \gamma_j, \delta_j)$ for $j = 1, \dots, N$ is an element of $N_\infty(I - \lambda_j Q)$, where

$$Q = \begin{bmatrix} 0 & A_2 \\ I & \tilde{A}_1 \end{bmatrix}.$$

Conversely, if $(\alpha, \beta, \gamma, \delta) \in N_\infty(I - \lambda_j Q)$ then $(\alpha \cos p_j s, \beta \sin p_j s, \gamma \cos p_j s, \delta \sin p_j s) \in N_\infty(I - \lambda_1 L)$. Hence

$$(2.17) \quad \dim N_{\infty}(I - \lambda_1 L) = \sum_{j=1}^N \dim N_{\infty}(I - \lambda_j Q).$$

For $j = 1, \dots, N$, $\dim N_{\infty}(I - \lambda_j Q)$ is equal to the multiplicity of λ_j as a root of $\det |I - A(\lambda)| = 0$. Indeed by the remark preceding Theorem 1.5 it follows that $\dim N_{\infty}(I - \lambda_j Q)$ is equal to the multiplicity of λ_j as a root of $\det |I - \lambda^2 A_2 - \lambda \tilde{A}_1| = 0$. It is easily verified that if P is a nonsingular square matrix we have the formula

$$(2.18) \quad \det \begin{vmatrix} P & Q \\ R & S \end{vmatrix} = \det \begin{vmatrix} I & 0 \\ -RP^{-1} & I \end{vmatrix} \cdot \det \begin{vmatrix} P & Q \\ R & S \end{vmatrix} \\ = \det |P| \cdot \det |S - RP^{-1}Q|.$$

Thus for all but a finite number of λ we have

$$\det |I - \lambda^2 A_2 - \lambda \tilde{A}_1| \\ = \det |I - \lambda^2 \hat{A}_2| \cdot \det |(I - \lambda^2 \bar{A}_2) - \lambda \bar{A}_1 (I - \lambda^2 \hat{A}_2)^{-1} \lambda \hat{A}_1| \\ = \det |I - A(\lambda)|.$$

However they are polynomials and so

$$\det |I - \lambda^2 A_2 - \lambda \tilde{A}_1| \equiv \det |I - A(\lambda)|.$$

Hence $\dim N_{\infty}(I - \lambda_j Q) = M(\lambda_j, A)$. Therefore by (2.17) $\dim N_{\infty}(I - \lambda_1 L)$ is odd.

With Lemma 2.2 the proof of Theorem 2.1 is complete.

If F depends on w'' then \mathcal{F} need not be completely continuous and we cannot establish the existence of large periodic solutions of (2.1). Therefore we now consider the system of ordinary differential equations.

$$(2.19) \quad w'' + A_1 w' + A_2 w + F(w, w') = 0$$

where A_1 and A_2 are as described by (2.3) and (2.4), and F is a continuous function of $2n$ variables satisfying

$$F(x) = o(|x|), \quad F = (F_1, F_2), \quad F_1(u, -v, -u', v') \\ = F_1(u, v, u', v), \quad F_2(u, -v, -u', v') = -F_2(u, v, u', v').$$

To characterize sets of nontrivial periodic solutions of (2.19) let E be the set of nontrivial 2π periodic solutions with $\lambda > 0$, of

$$(2.20) \quad \omega'' + \lambda A_1 \omega' + \lambda^2 A_2 \omega + \lambda^2 \hat{F}(\lambda, \omega) = 0$$

in $R \times X^* \times \mathcal{C}$, where $\hat{F}(\lambda, \omega) = F(\omega, \lambda^{-1} \omega')$. Clearly if $(\lambda, \mu^*, x) \in E$ then $x(t/\lambda) + \mu^*$ is a $2\pi\lambda$ periodic solution of (2.20). The closure

of E in $R \times X^* \times \mathcal{C}$ will be denoted by \bar{E} .

We note that $\hat{F}(\lambda, \omega)$ is undefined for $\lambda = 0$ so that care must be taken when dealing with solutions with λ near zero.

THEOREM 2.3. *Let λ_1 be a characteristic value of $A(\lambda)$ and suppose $S(\lambda_1)$ has N elements and $\mathcal{M}(\lambda_1)$ is odd. Then \bar{E} contains a maximal set of solutions, C_{λ_1} , such that C_{λ_1} meets $(\lambda_1, 0, 0)$, C_{λ_1} is closed and connected in $R \times X^* \times \mathcal{C}$ and either*

- (i) C_{λ_1} meets a point $(\lambda_0, 0, 0)$ such that for some positive integer m , λ_0/m is a positive characteristic value of $A(\lambda)$, $\lambda_0 \notin S(\lambda_1)$, and $\mathcal{M}(\lambda_0/m)$ is odd or
- (ii) C_{λ_1} is unbounded in $R \times X^* \times \mathcal{C}$ or
- (iii) for each $\lambda_1 > \varepsilon > 0$, C_{λ_1} meets a point of the form $(\varepsilon, \mu_\varepsilon^*, x_\varepsilon)$.

Proof. Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{F}(\lambda, x + \mu^*)$ be defined by (2.13) and (2.14). As $\partial^2 \mathcal{F}(\lambda, x + \mu^*)/\partial s^2 = -\hat{F}(\lambda, x + \mu^*) + 1/2\pi \int_0^{2\pi} \hat{F}_1$, an application of the Ascoli-Arzelà theorem will show that

$$\left(A_2^{-1} \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{F}_1 \right], \mathcal{F} \right)$$

is completely continuous. The positive characteristic values of $\mathcal{A}(\lambda)$ are all of the form βm where m is a positive integer and β is a positive characteristic value of $A(\lambda)$. Thus by repeating the proof of Theorem 2.1 and applying Theorems 1.6 and 1.7 to Eqs. (2.11) and (2.12) on $\Omega_{j,\varepsilon} = I_{j,\varepsilon} \times S_j$ for $j = 1, \dots, 0 < \varepsilon < \lambda_1$

$$I_{j,\varepsilon} = \{\lambda | \varepsilon < \lambda < j\}, S_j = \{(\mu^*, x) | \|(\mu^*, x)\| < j\},$$

there exists a unique maximal closed connected set $C_{j,\varepsilon} \subset \bar{E}$ meeting $(\lambda_1, 0)$ such that $C_{j,\varepsilon} \subset \bar{\Omega}_{j,\varepsilon}$, $C_{j,\varepsilon}$ either meets $\partial\Omega_{j,\varepsilon}$ or meets $(\lambda_0, 0, 0)$, where λ_0 is a positive characteristic value of $\mathcal{A}(\lambda)$ and $M(\lambda_0, \mathcal{A})$ is odd. As the union of connected sets containing a common point is connected, there is a unique maximal closed connected subset C_{λ_1} of \bar{E} meeting $(\lambda_1, 0)$. Hence $\cup C_{j,\varepsilon} \subset C_{\lambda_1}$ and the theorem follows.

If we make the further assumption that F does not depend on w' then \hat{F} will be independent of λ and we no longer have to be concerned with the existence of bounded solutions of (2.1) with periods tending to zero.

THEOREM 2.4. *Let λ_1 be as in Theorem 2.2 and suppose $F = F(w)$. Then there exists a maximal subset C_{λ_1} of \bar{E} such that C_{λ_1} is closed and connected in both $R \times X^* \times \mathcal{C}$ and $R \times X^* \times H$ and either*

- (i) C_{λ_1} is unbounded in both $\mathbf{R} \times X^* \times \mathcal{C}$ and $\mathbf{R} \times X^* \times H$ or
(ii) C_{λ_1} meets a point $(\lambda_0, 0, 0)$ such that for some positive integer m , λ_0/m is a positive characteristic value of $A(\lambda)$, $\lambda_0 \notin S(\lambda_1)$ and $\mathcal{M}(\lambda_0/m)$ is odd.

Proof. It is readily verified that \mathcal{A}_1 , \mathcal{A}_2 and \mathcal{F} are completely continuous maps of $H \rightarrow H$ and $\mathbf{R} \times X^* \times H \rightarrow H$ (cf. [1], [2, pp. 155–156]). Moreover the algebraic multiplicity of λ_1 of $\mathcal{A}(\lambda)$ is same for $\mathcal{A}(\lambda)$ as map of $\mathcal{C} \rightarrow \mathcal{C}$ and of $H \rightarrow H$. Thus arguing as in the proof of Theorems 2.1 and 2.3 and applying Theorems 1.5 and 1.7 to (2.11), (2.12) on $\mathbf{R} \times X^* \times \mathcal{C}$ or $\mathbf{R} \times X^* \times H$ there exists a maximal connected set of nontrivial solutions of (2.20) whose closure, S , meets $(\lambda_1, 0, 0)$ and is either unbounded in both $\mathbf{R} \times X^* \times \mathcal{C}$ and $\mathbf{R} \times X^* \times H$ or meets a $(\lambda_0, 0, 0)$ where $\lambda_0 \in r(\mathcal{A})$ and $M(\lambda_0, \mathcal{A})$ is odd. Let C_{λ_1} be the maximal closed connected subset of S meeting $(\lambda_1, 0, 0)$ and containing those solutions (λ, μ^*, x) such that $\lambda \geq 0$. If C_{λ_1} meets a point of the form $(\hat{\lambda}, \mu_1^*, 0)$, $u_1^* \neq 0$, then C_{λ_1} must contain all points of the form $(\lambda, \mu_1^*, 0)$, $0 \leq \lambda$, and so C_{λ_1} satisfies (i). So suppose C_{λ_1} does not meet any points of the form $(\hat{\lambda}, \mu_1^*, 0)$. Then C_{λ_1} cannot meet any points of the form $(0, \mu^*, x)$. Indeed, suppose C_{λ_1} met a point $(0, \mu^*, x)$. If $x \neq 0$, then as $(0, \mu^*, x)$ is a solution of Eq. (2.20) we have $(x + \mu^*)'' \equiv 0$, which is impossible if $x \neq 0$. On the other hand $x = 0$ implies $\mu^* = 0$ and so $\lambda = 0$ is a bifurcation point of (2.11), (2.12). However as $\lambda = 0$ is not a characteristic value of the linearized part of (2.11), (2.12) it cannot be a bifurcation point. Therefore C_{λ_1} does not meet any points of the form $(0, \mu^*, x)$. By the complete continuity of the operators S is locally compact. Thus unless C_{λ_1} is unbounded there exists a $\delta > 0$ such that if $(\lambda, \mu^*, x) \in C_{\lambda_1}$, $\lambda > \delta$. Hence either C_{λ_1} is unbounded or $C_{\lambda_1} = S$ and C_{λ_1} satisfies (ii).

With a rather strong condition on the derivative of the nonlinear operators we can show the existence of an unbounded set of solutions C_λ . To this end we first show

LEMMA 2.5. *Let $T(\lambda, z) \in C^1$ be a completely continuous map of $\mathbf{R} \times X^* \times \mathcal{C} \rightarrow X^* \times \mathcal{C}$ such that $T(\lambda, z)$ has the same period as (λ, z) . Suppose $T(\lambda_0, z_0) = z_0$ and every $z \in N_\infty(I - T_z(\lambda_0, z_0))$ is $2\pi/m$ periodic where m is a positive integer. Then all solutions of*

$$(2.21) \quad T(\lambda, z) = z$$

in a neighborhood of (λ_0, z_0) are $2\pi/m$ periodic.

Proof. As $T(\lambda, z)$ is completely continuous so is the derivative

$T_z(\lambda_0, z_0) = T'$ [12, p. 51]. Thus by the Fredholm theory [11, pp. 179-190, 219], $\alpha(I - T') = p < \infty$, $\dim N_p(I - T') < \infty$ and $X^* \times \mathcal{C} = N_p(I - T') \oplus R_p(I - T')$. Let P be the canonical projection of $X^* \times \mathcal{C}$ onto $R_p(I - T')$. Then finding solutions of (2.21) is equivalent to solving

$$\begin{aligned} u - PT(\lambda, u + v) &= 0 \\ v - (I - P)T(\lambda, u + v) &= 0 \end{aligned}$$

where $u \in R_p(I - T')$ and $v \in N_p(I - T')$. By the implicit function theorem [5, pp. 265] near (λ_0, v_0) there exists a unique function $u = u(\lambda, v)$ such that $u(\lambda, v) \equiv PT(\lambda, u(\lambda, v) + v)$. Moreover $u(\lambda, v)$ is found by an iteration of $2\pi/m$ functions and so is itself $2\pi/m$ periodic. Thus all solutions of (2.21) near (λ_0, z_0) must be of the form $v + u(\lambda, v)$ and the result follows.

THEOREM 2.6. *Let $\lambda_1 < \dots < \lambda_K$ be the real characteristic values of $A(\lambda)$. Suppose there exist H consecutive positive characteristic values $\lambda_{k+1} < \dots < \lambda_{k+H}$, $1 \leq k \leq K$ such that $\sum_{j=1}^H M(\lambda_{k+j}, A)$ is odd, $p\lambda_j \notin [\lambda_{k+1}, \lambda_{k+H}]$ ($p=1, \dots; j \leq k$), $\lambda_{k+H} < 2\lambda_{k+1}$, $S(\lambda_{k+j}) = \{\lambda_{k+j}\}$ ($j=1, \dots, H$) and for $0 < \lambda_j \notin [\lambda_{k+1}, \lambda_{k+H}]$, $\mathcal{M}(\lambda_j)$ is even.*

(i) *If $F = F(w)$, \bar{E} contains a maximal set of solutions $C_{\lambda_{k+i}}$ such that, $1 \leq i \leq H$, $C_{\lambda_{k+i}}$ meets $(\lambda_{k+i}, 0, 0)$, $C_{\lambda_{k+i}}$ is closed and connected in both $\mathbf{R} \times X^* \times \mathcal{C}$ and $\mathbf{R} \times X \times H$ and either $C_{\lambda_{k+i}}$ is unbounded in both $\mathbf{R} \times X^* \times \mathcal{C}$ and $\mathbf{R} \times X^* \times H$ or meets a point $(\lambda_j, 0, 0)$ such that λ_j/m is a positive characteristic value of $A(\lambda)$ for some integer $m > 1$ and $\mathcal{M}(\lambda_j/m)$ is odd.*

(ii) *If in addition $F \in C^1$ and at any nontrivial solution $(\lambda_0, \mu_0^*, x_0)$ of (2.11), (2.12), the derivative with respects to (μ_0^*, x_0)*

$$\psi = \begin{bmatrix} I + A_{2m}^{-1} \left[\frac{1}{2\pi} \int_0^{2\pi} F'_1(x_0(s) + \mu_0^*) ds \right] & 0 \\ 0 & I - \mathcal{N}(\lambda_0) - \lambda_0 \mathcal{F}'(x_0 + \mu_0^*) \end{bmatrix}$$

is such that $(\mu^, x(s)) \in N_\infty(\Psi)$ implies $x(s)$ has the same period as $x_0(s)$, then $C_{\lambda_{k+j}}$ is unbounded.*

Proof. The characteristic values $\lambda_{k+1}, \dots, \lambda_{k+H}$ are the only characteristic values of $\mathcal{N}(\lambda)$ in $[\lambda_{k+1}, \lambda_{k+H}]$ and by Lemma 2.2 $\sum_{j=1}^H M(\lambda_{k+j}, \mathcal{N})$ is odd. Thus arguing as in the proof of Theorem 2.4 there exists a maximal subset $C_{\lambda_{k+i}}$ of \bar{E} with the properties of (i). To complete the proof we suppose that F is as described in (ii) and $C_{\lambda_{k+i}}$ is bounded and meets a points of the form $(\lambda_1, 0, 0)$ where λ_1/m is a characteristic value of $A(\lambda)$ and m is an integer > 1 . As $\mathcal{N}(\lambda)$ and \mathcal{F} map $2\pi/h$ periodic functions into $2\pi/h$ periodic functions

it follows from Lemma 2.5 that $C_{\lambda_{k+i}}$ contains both 2π and $2\pi/m$ periodic solutions. By the connectedness of $C_{\lambda_{k+i}}$ there must exist a $2\pi/m$ periodic solution $(\lambda_0, \mu_0^*, x_0) \in C_{\lambda_{k+i}}$ such that every neighborhood of $(\lambda_0, \mu_0^*, x_0)$ contains 2π periodic solutions of (2.11), (2.12). But by our hypotheses and Lemma 3.5 this is impossible hence the result.

REMARK. Suppose (2.1) has the form

$$(2.22) \quad w'' + A_2 w + F(w, w', w'') = 0$$

where A_2 is an $n \times n$ matrix and $F(-w, w', -w'') = -F(w, w', w'')$ then if \mathcal{C} and H are replaced by \mathcal{C}_2 and H_2 proofs similar to those of Theorems 2.1, 2.3, and 2.4 will show that these theorems are also true for (2.22)

3. A counterexample. We give an example to show that unless some additional restrictions are imposed the system $x'' + A_1 x' + A_2 x + F(x) = 0$ may fail to have any periodic solutions even though $\lambda^2 A_2 + i\lambda A_1$ has only simple eigenvalues.

Consider the system

$$(3.1) \quad x'' + 3y' + 4x + y^3 = 0$$

$$(3.2) \quad y'' - 3x' + 4y - x^3 = 0.$$

The characteristic equation

$$\det \left[\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + i\beta \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} - \beta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] = 0$$

has four distinct roots $\pm 1, \pm 4$. Hence $\lambda = 1, 1/4$, is a simple characteristic value of $\lambda^2 A_2 + i\lambda A_1$. On the other hand the system (3.1), (3.2), has no nontrivial periodic solutions. Suppose $(x(t), y(t))$ is a periodic solution of (3.1), (3.2) of period a . Multiply (3.1) by y and (3.2) by x . Then integrate the resulting equations by parts over $[0, a]$, noting $\int_0^a x'x = \int_0^a y'y = y^2/2 \Big|_0^a = 0$. If we now subtract we obtain

$$\int_0^a y^4 + x^4 = 0.$$

This implies $y = x \equiv 0$ and hence there do not exist nontrivial periodic solutions of (3.1), (3.2).

After this work was completed, G. I. Ize's doctoral dissertation [9] was brought to our attention. In it he gives an alternate, analytic, definition of the algebraic multiplicity of a characteristic value and with his definition proves theorems similar to our Theorem 1.5.

REFERENCES

1. M. S. Berger, *On periodic solutions of second order Hamiltonian systems (I)*, J. Math. Anal. Appl., **29** (1970), 512-522.
2. M. S. Berger and M. S. Berger *Perspectives in Nonlinearity. An Introduction to Nonlinear Analysis*, Benjamin, New York, 1968.
3. E. A. Coddington and N. Levinson, *The Theory of Ordinary Differential Equations*, McGraw-Hill Book Co., New, York, 1955.
4. E. N. Dancer, *On the structure of solutions of non-linear eigenvalue problems*, Indiana U. Math. J., **23** (1974), 1069-1076.
5. J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1960.
6. R. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart, and Winston, New York, 1965.
7. P. R. Halmos, *Finite-Dimensional Vector Spaces*, Second Edition, D. Van Nostrand Company, Inc., Princeton, 1958.
8. ———, *Naive Set Theory*, D. Van Nostrand Company, Inc., Princeton, 1966.
9. G. I. Ize, *Bifurcation theory for Fredholm operators*, Doctoral Dissertation, New York University, April, 1974.
10. P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Analysis, **7** (1971), 487-513.
11. F. Riesz and B. Sz-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.
12. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, GITTL, Moscow, 1956; English trans., Holden-Day, San Francisco, Calif., 1964.
13. D. Westreich, *Periodic solutions of second order non-Lagrangian systems*, Israel J. Math., **16** (1973), 279-286.
14. ———, *Bifurcation of operator equations with unbounded linearized part*, Pacific J. Math., **57** (1975), 611-618.

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