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ON PUNCTURED BALLS IN MANIFOLDS

WOLFGANG HEIL

E. Brown showed that for any map f of a punctured disc B_n with n holes into a 2-manifold M that is an embedding of ∂B_n, there is an embedding g of a punctured disk B_k into M such that $g(\partial B_k)$ is a subcollection of $f(\partial B_n)$. In this paper E. Brown's approach is extended to show that a similar result holds for maps of punctured q-balls into certain q-manifolds ($q \geqq 3$).

Let $PC(q)$ denote the collection of (topological) q-manifolds M^q with the property that if h is an embedding of $S^{q-1}\times[0,1]$ into M^q that is null homotopic, then $h(S^{q-1}\times\frac{1}{2})$ bounds a topological q-cell in M^q.

Note that $PC(1)$ and $PC(2)$ consist of all 1-manifolds and 2-manifolds, respectively. It is well-known that $PC(3)$ consists of all 3-manifolds provided the Poincaré conjecture is true in dimension 3. Since the generalized Poincaré conjecture holds for dimensions $\geqq 5$, [2] we are led to conjecture that $PC(q)$ consists of all (topological) q-manifolds for $q \geqq 5$, particularly since, from the proposition below, if $h\colon S^{q-1}\to\partial M^q$ is an embedding such that $h(S^{q-1})$ is null-homotopic in M^q, then M^q is indeed a q-cell ($q \geqq 5$). However, C. McA. Gordon, whom I would like to thank most sincerely for providing the proof of the following proposition, informs me that C. T. C. Wall and John Morgan have counter examples for $q > 4$.

PROPOSITION. *Let $C \cong S^{q-1}$ be a boundary component of a compact q-manifold M. If $[C]=0$ in $\pi_{q-1}(M)$, then M is contractible.*

Proof. Let $q \geqq 3$. By the Whitehead and Hurewicz Theorems it suffices to show that $\pi_1(M)=1$ and $H_*(M)=0$. Now $\partial M = C$ since otherwise $[C]\neq 0$ in $H_{q-1}(M)$. Also, M is orientable since otherwise for the orientation cover M' of M we have $\partial M' = C'\cup C''$ (copies over C) and $[C']=0$ in $\pi_{q-1}(M')$, a contradiction.

There is a map $f\colon (B^q, S^{q-1})\to(M,\partial M)$ such that $f\,|\,S^{q-1}$ is a homeomorphism. Orient M so that f has degree 1. Then for the fundamental classes z_q, w_q in $H_q(B^q,S^{q-1})$, $H_q(M,\partial M)$, resp., we have $f^*(z_q)=w_q$ and a commutative diagram

$$\begin{array}{ccc} H^{q-k}(B^q,S^{q-1}) & \xleftarrow{f^*} & H^{q-k}(M,\partial M) \\ \Big\downarrow {\scriptstyle \cap z_q} & & \Big\downarrow {\scriptstyle \cap w_q} \\ H_k(B^q) & \xrightarrow{f_*} & H_k(M) \end{array}$$

By Lefschetz duality, the vertical maps are isomorphisms. Therefore $f_*(- \cap z_q)f^*$ is an isomorphism. It follows that f_* is onto and hence that $H_*(M) = 0$.

To show that $\pi_1(M) = 1$, let $p: \tilde{M} \to M$ be the universal covering. Then f lifts to $\tilde{f}: (B^q, S^{q-1}) \to (\tilde{M}, \partial\tilde{M})$. But $1 = deg(f) = \deg(p \circ \tilde{f}) = (\deg p)(\deg \tilde{f})$, hence $\deg p = \pm 1$ and $\pi_1(M) = 1$.

For $q \geqq 2$, $n \geqq 1$, let B^q_n be a punctured q-ball with $n - 1$ holes, i.e., B^q_n is obtained from S^q by removing the interiors of n mutually disjoint q-balls.

For a bicollared $S^{q-1} \subset M^q$ let $N \approx S^{q-1} \times I$ be a neighborhood of S^{q-1} and let $M' = \mathrm{cl}(M - N) \cup B' \cup B''$, where the boundaries of the q-balls B', B'' are attached to the boundary components $S^{q-1} \times 0$ and $S^{q-1} \times 1$ of $\mathrm{cl}(M - N)$. We say M' is obtained from M by *surgery along* S^{q-1}. Let X be the space obtained from M' by identifying B' and B'' under a homeomorphism. Note that X can be obtained from M^q by attaching a q-ball B to S^{q-1} along its boundary and $X - B = M' - (B' \cup B'') = M - S^{q-1}$.

LEMMA. *Let S be a $(q-1)$-sphere in $X - B$. If $S \simeq 0$ in X, then $S \simeq 0$ in M'.*

Proof. Suppose S^{q-1} separates M into M_1 and M_2; then $M' = M_1' \cup M_2'$, where $M_1' = M_1 \cup B'$, $M_2' = M_2 \cup B''$. Let X_i' be obtained from M_i by collapsing S^{q-1} to a point. The projection $p: X \to X_1' \vee X_2'$ is a homotopy equivalence which sends S into X_1', say. This can be seen as follows: Identify a neighborhood of S^{q-1} with $N = S^{q-1} \times [-1, 1]$, where $S^{q-1} = S^{q-1} \times \{0\}$. Let w be the "centerpoint" of B and for $y \in S^{q-1}$ let $r(y)$ be the "radius" in B from y to w. In $X_1' \vee X_2'$ we identify $p(N) = (S^{q-1} \times I)/(S^{q-1} \times \{0\})$ with the cones over $S^{q-1} \times \{-1\}$ and $S^{q-1} \times \{1\}$ wedged together at their vertices to a vertex v. Let $g: X_1' \vee X_2' \to X$ be the map that is the identity outside $p(N)$ and which sends the join of x and v (for $x \in S^{q-1} \times \{-1\}$, respectively $S^{q-1} \times \{1\}$) linearly to $x \times [-1, 0] \cup r(x \times \{0\})$, resp. $x \times [0, 1] \cup r(x \times 0)$. Then it is clear that g is a homotopy inverse of p. But since X_1' is a retract of $X_1' \vee X_2'$ it follows that $S \simeq 0$ in X_1' already and hence in $M_1' \simeq X_1'$.

If S^{q-1} does not separate M, let $\tilde{X} \to X$ be the infinite cyclic covering of X determined by B: the q-ball B lifts to q-balls $\cdots B_{-1}, B_0, B_1, \cdots$ and each component of $\tilde{X} - \bigcup_{i=-\infty}^{\infty} B_i$ maps homeomorphically onto $X - B$. For each i, let X_i' be obtained from M' by collapsing B' and B'' to single points. There is a projection $\tilde{X} \to \vee_{i=-\infty}^{\infty} X_i'$ that is a homotopy equivalence and hence $\pi_{q-1}(X_j')$ injects into $\pi_{q-1}(\tilde{X})$, for each j. Let $\tilde{S}$ be a lift of S to $\tilde{X}$. Then $\tilde{S}$ lies in a component of $\tilde{X} - \cup B_i$ and is mapped into a factor X_j' of $\vee X_i'$. It follows that $\tilde{S} \simeq 0$ in X_j', hence $S \simeq 0$ in M'.

THEOREM. *Let $f: B^q_n \to M^q$ be a map such that $f \mid \partial B^q_n$ is a bicollared embedding, $f(\partial B^q_n) = S_1 \cup \cdots \cup S_n$. Suppose that the manifold M' obtained from M^q by surgery along S_i $(i = 2, \cdots, n)$ belongs to $PC(q)$. Then some subcollection of $\{S_1, \cdots S_n\}$ contains S_1 and bounds an embedded punctured q-ball in M.*

Proof. By Brown's result we can assume that $q \geqq 3$. Let X be obtained from M by attaching q-balls B_i to S_i $(i = 2, \cdots, n)$ along their boundaries. Then $X - \bigcup_{i=2}^n B_i = M' - \bigcup_{i=2}^n B'_i \cup \bigcup_{i=2}^n B''_i$, where B'_i, B''_i are the balls used for surgery on S_i. Now $S_1 \simeq 0$ in X. By the lemma, $S_1 \simeq 0$ in M'. Since $M' \in PC(q)$, S_1 bounds a q-ball B_* in M'. Let E be the component of $B_* - \bigcup_{i=2}^n (B'_i \cup B''_i)$ which has S_1 on its boundary. If for each $i = 2, \cdots, n$ only one of $\partial B'_i, \partial B''_i \subset \partial E$, then E is the desired punctured ball in M bounded by S_1 and some of the S_i's. In fact, this is the only case that can happen. For suppose for some i, $\partial B'_i$ and $\partial B''_i \subset \partial E$. Then let k be a simple arc in E from a point of $\partial B'_i$ to a point on $\partial B''_i$ such that k misses the other ∂B_j's and such that k corresponds to a simple closed curve in M that intersects S_i in one point and misses the other S_j's. In M, the intersection numbers $\#(k, S_i) = \pm 1$, but $\#(k, \Sigma_{j \neq i} S_j) = 0$, which is impossible since $S_i \sim \bigcup_{j \neq i} S_j$.

REFERENCES

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FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306