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TENSOR PRODUCTS OF FUNCTION RINGS UNDER COMPOSITION

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Let $C(X)$, $C(Y)$ be the rings of real-valued continuous functions on the completely regular Hausdorff spaces X, Y and let $T = C(X) \otimes C(Y)$ be the subring of $C(X \times Y)$ generated by functions of the form fg, where $f \in C(X)$ and $g \in C(Y)$. If P is a real polynomial, then $P \circ t \in T$ for every $t \in T$. If $G \circ t \in T$ for all $t \in T$ and if G is analytic, then G is a polynomial, provided that X and Y are both infinite (A. W. Hager, Math. Zeitschr. 92, (1966), 210–224, Prop. 3.). In this note I remove the condition of analyticity. Clearly the cardinality condition is necessary, for if either X or Y is finite, then $T = C(X \times Y)$ and $G \circ t \in T$ for every continuous G and for every $t \in T$.

It is convenient to admit a somewhat wider class of G's. Let $T^* = T + iT$, that is, the set of all functions $t_1 + it_2$ with $t_1, t_2 \in T$. (T^* is the tensor product of the complex-valued continuous function rings on X and Y). Define $K(X, Y)$ as the set of all continuous complex-valued functions G on R (the reals) with the property that $G \circ t \in T^*$ for all $t \in T$. Then the result is

THEOREM. *If X and Y are infinite completely regular Hausdorff spaces, then $K(X, Y)$ consists of all the polynomials with complex coefficients.*

It follows from the Theorem that if $G \circ t \in T$ for all $t \in T$, then G is a polynomial with real coefficients.

The proof of the Theorem, which is rather lengthy, will be broken up into a sequence of lemmas.

LEMMA 1. *Let φ and ψ be continuous mappings of X and Y onto X' and Y' respectively. Then $K(X, Y) \subset K(X', Y')$.*

Proof. Let $G \in K(X, Y)$, $t' \in T' = C(X') \otimes C(Y')$.
I must show that $G \circ t' \in T'^*$. Define t by

$$t(x, y) = t'(\varphi(x), \psi(y)) \qquad (x \in X,\ y \in Y).$$

Clearly $t \in T$, and by hypothesis $G \circ t \in T^*$. That is, there are continuous complex-valued functions $u_1, \cdots, u_n$ on X, $v_1, \cdots, v_n$ on Y, such that

$$(1) \qquad (G \circ t')(\varphi(x), \psi(y)) = \sum_{j=1}^{n} u_j(x) v_j(y) \qquad (x \in X,\ y \in Y).$$

If $y_0, y_1, \cdots, y_n$ are any elements of Y, then there exist complex $c_0, c_1, \cdots, c_n$ not all 0 such that

$$(2) \qquad \sum_{j=0}^{n} c_j (G \circ t')(\varphi(x), \psi(y_j)) = 0 \qquad (x \in X),$$

since (1) shows that the y-sections of $G \circ t$ are contained in an n-dimensional subspace of $C(X) + iC(X)$. Let $y'_0, \cdots, y'_n$ be any elements of Y', and let x' be any element of X'. Then, since φ and ψ are onto, there exist $y_0, \cdots, y_n$ and x such that

$$\varphi(x) = x', \quad \psi(y_j) = y'_j \qquad (j = 0, 1, \cdots, n).$$

Insert these values in (2) to get

$$\sum_{j=0}^{n} c_j (G \circ t')(x', y'_j) = 0.$$

This means that the y'-sections of $G \circ t'$ are contained in an n-dimensional subspace of $C(X') + iC(X')$. By Hager[1], this implies that $G \circ t' \in T'^*$. Hence $G \in K(X', Y')$.

LEMMA 2. *If $X' \approx X$, $Y' \approx Y$, then $K(X', Y') = K(X, Y)$.*

Proof. Immediate from Lemma 1.

LEMMA 3. *If the conclusion of the Theorem holds for all infinite subspaces X', Y' of R then the Theorem holds.*

Proof. Every infinite completely regular Hausdorff space can be mapped continuously onto an infinite subset of R. Apply Lemma 1 and the hypothesis.

LEMMA 4. *Suppose that X_0 and Y_0 are C-embedded in X and Y respectively. Then $K(X, Y) \subset K(X_0, Y_0)$.*

Proof. Let $G \in K(X, Y)$, $t_0 \in T_0 = C(X_0) \otimes C(Y_0)$. Then there is a $t \in T$ such that $t \mid (X_0 \times Y_0) = t_0$, obtained by extending each component of t_0. By assumption, $G \circ t \in T^*$. By restriction, $G \circ t_0 \in T_0^*$. Hence $G \in K(X_0, Y_0)$.

[1] Ibid. Prop. 1

LEMMA 5. *If X is an infinite subset of R, then there is a continuous mapping φ of X into R such that $\varphi[X]$ contains the terms of a convergent infinite sequence and its limit.*

Proof. If X is unbounded, let $p \in X$ and define

$$\varphi(x) = \frac{x-p}{1+x^2} \qquad (x \in X).$$

Then $\varphi[X]$ has the required property. If X is bounded, then it contains a countably infinite set $\{x_n\}$ such that $x_n \to q$ (perhaps not in X). Let $p \in X$ and define

$$\varphi(x) = (x-q)(x-p) \qquad (x \in X).$$

Clearly $\varphi(x_n) \to 0 = \varphi(p)$. Also the set $\{\varphi(x_n)\}$ is infinite. Hence $\varphi[X]$ has the required property.

LEMMA 6. *Let X_0 be any one infinite set $\{x_n\}_{n=0}^{\infty}$, with $x_n \to x_0$. If $K(X_0, X_0)$ consists of the complex polynomials, then the Theorem holds.*

Proof. Follows from Lemma 3, Lemma 5, Lemma 4, and the fact that X_0 is compact, hence C-embedded in $\varphi[X]$, and Lemma 2.

LEMMA 7. *Let $X_0 = \{j/n^2 : n \geqq 1, 0 \leqq j \leqq M_n\}$, where M_n is a sequence of positive integers satisfying $M_n \geqq n$ $(n \geqq 1)$. Let $G \in K(X_0, X_0)$, with $X_0 \subset Z(G)$, the zero-set of G. Then there exists an N such that*

$$\frac{M_n + 1}{n^2} \in Z(G) \qquad (n > N).$$

Proof. Define $t \in T_0 = C(X_0) \otimes C(X_0)$ by

$$t(x, y) = x + y \qquad (x \in X_0,\ y \in X_0).$$

Let $N = \operatorname{rank}(G \circ t)$, i.e., the dimension of the vector-space of y-sections of $G \circ t$. If $n > N$, there exist c_j $(j = 1, \cdots, N+1)$ (possibly depending on n) not all 0, such that

$$\sum_{j=1}^{N+1} c_j G\left(x + \frac{j}{n^2}\right) = 0 \qquad (x \in X_0).$$

(Note that the arguments

$$\frac{j}{n^2} \leqq \frac{N+1}{n^2} \leqq \frac{n}{n^2} \leqq \frac{M_n}{n^2}$$

are all in X_0). Let M be the largest j such that $c_j \neq 0$, so $1 \leqq M \leqq N+1$ and

$$(3) \qquad \sum_{j=1}^{M} c_j G\left(x + \frac{j}{n^2}\right) = 0 \qquad (x \in X_0).$$

Choose $x = (M_n + 1 - M)/n^2$. Since $M \leqq N + 1 < n + 1 \leqq M_n + 1$, $x > 0$. Since $M \geqq 1$, $x \leqq M_n/n^2$. Hence $x \in X_0$. Therefore, from (3),

$$(4) \qquad -c_M G\left(\frac{M_n + 1}{n^2}\right) = \sum_{j=1}^{M-1} c_j G\left(\frac{M_n + 1 - M + j}{n^2}\right).$$

Since $M_n + 1 - M + j \geqq n + 2 - M > n + 2 - (n+1) = 1$, and $M_n + 1 - M + j \leqq M_n + 1 - M + (M-1) = M_n$ for all j such that $1 \leqq j \leqq M - 1$, the arguments on the right in (4) are all in $X_0 \subset Z(G)$. Since $c_M \neq 0$,

$$G\left(\frac{M_n + 1}{n^2}\right) = 0 \qquad (n > N).$$

LEMMA 8. *Under the hypothesis of Lemma* 7, *but with* $M_n = n$ $(n \geqq 1)$, *there is an* $\alpha > 0$ *such that* $[0, \alpha] \subset Z(G)$.

Proof. Define

$$\bar{M}_n = \sup\left\{M \colon G\left(\frac{j}{n^2}\right) = 0 \quad \text{for} \quad j = 0, 1, \cdots, M\right\}.$$

Note that $\bar{M}_n \geqq n$. Suppose that $\bar{\alpha} \equiv \underline{\lim}(\bar{M}_n/n^2) = 0$. Then there is an infinite sequence $n_1 < n_2 < \cdots$ such that

$$\frac{\bar{M}_{n_i}}{n_i^2} \to 0.$$

Define $L_n = \bar{M}_n$ if $n = n_i$ for some i, $L_n = n$ otherwise. Let

$$X' = \left\{\frac{j}{n^2} \colon 0 \leqq j \leqq L_n,\ n \geqq 1\right\}.$$

Then (i) $X' \approx X_0$, (ii) $X' \subset Z(G)$, (iii) X' is of the form prescribed in Lemma 7, since $L_n \geqq n$. By (i) and Lemma 2, $K(X_0, X_0) = K(X', X')$, so

$G \in K(X', X')$. Combining this with (ii), (iii), and Lemma 7, one finds that there is an N such that

$$\frac{L_n + 1}{n^2} \in Z(G) \qquad (n > N).$$

In particular, for $n = n_t > N$,

$$\frac{\bar{M}_n + 1}{n^2} \in Z(G).$$

This contradicts the definition of $\bar{M}_n$. Hence $\bar{\alpha} > 0$ ($\bar{\alpha} = +\infty$, possibly).

Clearly the set $B = \{j/n^2 \colon 0 \leqq j \leqq \bar{M}_n,\ n \geqq 1\}$ is dense in $[0, \bar{\alpha})$. Since $B \subset Z(G)$, there exists an $\alpha > 0$ such that $[0, \alpha] \subset \bar{B} \subset Z(G)$.

LEMMA 9. *Under the hypotheses of Lemma* 8, $G = 0$.

Proof. Let $\alpha = \sup\{a \colon [0, a] \subset Z(G)\}$. By Lemma 8, $\alpha > 0$. Suppose $\alpha < \infty$. Let $\xi \geqq 0$. For

$$t(x, y) = \alpha + \xi(x - y) \qquad (x, y \in X_0),$$

let $\operatorname{rank}(G \circ t) = M_\xi$. Define $N_\xi = 1 + \max(M_\xi, \xi M_\xi/\alpha)$. For $n \geqq N_\xi$, there exist c_j ($j = 0, 1, \cdots, M_\xi$) not all 0, such that

$$\sum_{j=0}^{M_\xi} c_j G\left(\alpha + \xi\left(x - \frac{j}{n^2}\right)\right) = 0 \qquad (x \in X_0). \tag{5}$$

(Note that for $0 \leqq j \leqq M_\xi$,

$$0 \leqq \frac{j}{n^2} \leqq \frac{M_\xi}{n^2} < \frac{N_\xi}{n^2} \leqq \frac{n}{n^2},$$

so $j/n^2 \in X_0$). If q is the least j such that $c_j \neq 0$, set $x = (q+1)/n^2$. Since $0 < q + 1 \leqq M_\xi + 1 \leqq N_\xi \leqq n$, $x \in X_0$. For $j = q + 1, \cdots, M_\xi$, one has $\alpha + \xi(x - j/n^2) \leqq \alpha$ and

$$\begin{aligned}\alpha + \xi\left(x - \frac{j}{n^2}\right) &\geqq \alpha + \xi\left(\frac{q+1}{n^2} - \frac{M_\xi}{n^2}\right)\\ &\geqq \alpha - \frac{\xi M_\xi}{n^2} \geqq \alpha - \frac{\xi M_\xi}{n} \geqq \alpha - \frac{\xi M_\xi}{N_\xi}\\ &\geqq \alpha - \frac{\alpha(N_\xi - 1)}{N_\xi} > 0.\end{aligned}$$

Hence $\alpha + \xi(x - j/n^2) \in Z(G)$, and from (5),

$$G\left(\alpha + \frac{\xi}{n^2}\right) = -\frac{1}{c_q} \sum_{j=q+1}^{M_\xi} c_j G\left(\alpha + \xi\left(x - \frac{j}{n^2}\right)\right) = 0.$$

Thus it has been proved that for each $\xi \geqq 0$, there is an N_ξ such that

$$G\left(\alpha + \frac{\xi}{n^2}\right) = 0 \qquad (n \geqq N_\xi).$$

For each $N = 1, 2, \cdots$, define

$$S_N = \left\{\xi \geqq 0 \colon n \geqq N \Rightarrow G\left(\alpha + \frac{\xi}{n^2}\right) = 0\right\}.$$

Clearly S_N is closed and $[0, \infty) = \cup_{N \geqq 1} S_N$. By the Baire category theorem, there is an interval $[u, v] \subset S_N$ for some $N \geqq 1$, with $0 \leqq u < v$. That is,

$$G\left(\alpha + \frac{\xi}{n^2}\right) = 0 \qquad (u \leqq \xi \leqq v,\ n \geqq N). \tag{6}$$

Thus the intervals $[\alpha + u/n^2, \alpha + v/n^2]$ are contained in $Z(G)$ for all $n \geqq N$. For sufficiently large n, these intervals overlap and fill out an interval $(\alpha, \beta]$, with $\beta > \alpha$. Hence $[0, \beta] \subset Z(G)$. This contradicts the definition of α, and shows that $\alpha = \infty$. Hence $G(x) = 0$ $(x \geqq 0)$. Finally, the function G_1 defined by $G_1(x) = G(1 - x)$ $(x \in R)$ belongs to $K(X_0, X_0)$ and $G_1(x) = 0$ $(x \in X_0)$. By what has just been proved, $G_1(x) = 0$ $(x \geqq 0)$, so $G(x) = 0$ $(x \leqq 1)$. Therefore $G = 0$. (There is an alternate proof that avoids the use of Baire category).

LEMMA 10. *Let $X_0 = \{j/n^2 \colon 0 \leqq j \leqq n, n \geqq 1\}$, and let $G \in K(X_0, X_0)$ satisfy, for some positive h and complex r,*

$$G(x + h) = rG(x) \qquad (x \in X_0).$$

Then G is a constant, and $r = 1$ unless that constant is 0.

Proof. The function G_1 defined by

$$G_1(x) = G(x + h) - rG(x) \qquad (x \in R)$$

belongs to $K(X_0, X_0)$, and $X_0 \subset Z(G_1)$. By Lemma 9, $G_1 = 0$, so

$$G(x+h) = rG(x) \qquad (x \in R).$$

Define $F(x) = G(hx)$ $(x \in R)$. Then $F \in K(X_0, X_0)$ and

$$F(x+1) = rF(x) \qquad (x \in R). \tag{7}$$

Let $N = \operatorname{rank}(F \circ t)$, where $t(x, y) = xy$ $(x, y \in X_0)$. Then the $N+1$ y-sections of $F \circ t$ at $y_j = 2^{-j}$ $(j = 0, 1, \cdots, N)$ are linearly dependent (note that $2^{-j} = 2^j/(2^j)^2 \in X_0$). Hence there exist $c_0, c_1, \cdots, c_N$ not all 0 such that

$$\sum_{j=0}^{N} c_j F(2^{-j}x) = 0 \qquad (x \in X_0). \tag{8}$$

As above, (8) holds for all $x \in R$, by Lemma 9. Let M be the least nonnegative integer for which an equation of the form (8) holds for all $x \in R$, with the sum running from 0 to M and the c_j not all 0. Then $c_M \neq 0$. If $M = 0$, then $F = 0$ and therefore $G = 0$. For $M > 0$, let q be the least j such that $c_j \neq 0$. Again, if $q = M$, then $G = 0$. Hence one may assume that $q < M$. Thus

$$\sum_{j=q}^{M} c_j F(2^{-j}x) = 0 \qquad (x \in R), \tag{9}$$

with $c_q \neq 0$, $c_M \neq 0$, $q < M$, and M minimal. Replace x by $2^M x + 2^M$. Then

$$\sum_{j=q}^{M} c_j F(2^{M-j}x + 2^{M-j}) = 0 \qquad (x \in R).$$

By (7),

$$\sum_{j=q}^{M} c_j r^{2^{M-j}} F(2^{M-j}x) = 0 \qquad (x \in R).$$

Replacing x by $2^{-M}x$, one gets

$$\sum_{j=q}^{M} c_j r^{2^{M-j}} F(2^{-j}x) = 0 \qquad (x \in R). \tag{10}$$

Combining (9) and (10), one has

$$\sum_{j=q}^{M-1} c_j (r - r^{2^{M-j}}) F(2^{-j}x) = 0 \qquad (x \in R). \tag{11}$$

Because of the minimality of M, all the coefficients in (11) must be 0. Since $c_q \neq 0$,

$$r - r^{2^{M-q}} = 0.$$

Now $r = 0$ implies $G(x+h) = 0$ $(x \in R)$, that is, $G = 0$. Since $q < M$, $2^{M-q} \geqq 2$, so if $r \neq 0$, $r^m = 1$ with $m = 2^{M-q} - 1 \geqq 1$. It follows that

$$F(x+m) = r^m F(x) = F(x) \qquad (x \in R).$$

Thus F is periodic. Either F (hence G) is constant or it has a least positive period p. From (9),

$$\sum_{j=q}^{M} c_j F(2^{M-j}x) = 0 \qquad (x \in R).$$

Therefore

$$F(x) = -\frac{1}{c_M} \sum_{j=q}^{M-1} c_j F(2^{M-j}x) \qquad (x \in R).$$

Hence

$$\begin{aligned} F\left(x + \frac{p}{2}\right) &= -\frac{1}{c_M} \sum_{j=q}^{M-1} c_j F(2^{M-j}x + 2^{M-j-1}p) \\ &= -\frac{1}{c_M} \sum_{j=q}^{M-1} c_j F(2^{M-j}x) \\ &= F(x) \qquad (x \in R). \end{aligned}$$

This contradicts the fact that p is the minimal period. Hence F is a constant and so is G. If $G \neq 0$, then

$$G(x) = G(x+h) = rG(x)$$

implies that $r = 1$.

LEMMA 11. *Let* $X_0 = \{j/n^2 \colon 0 \leqq j \leqq n,\ n \geqq 1\}$, *and let* $G \in K(X_0, X_0)$. *Then G is a polynomial.*

Proof. Let $N = \operatorname{rank}(G \circ t)$, where

$$t(x, y) = x + y \qquad (x, y \in X_0).$$

Then, if one reasons as in Lemma 10, there is an $M \leqq N$ and $c_0, \cdots, c_M$, with $c_M = 1$, such that

$$\sum_{j=0}^{M} c_j G\left(x + \frac{j}{N^2}\right) = 0 \qquad (x \in X_0). \tag{12}$$

Equation (12) holds for all $x \in R$, by Lemma 9. Define $F(x) = G(x/N^2)$ $(x \in R)$. Then

$$\sum_{j=0}^{M} c_j F(x + j) = \sum_{j=0}^{M} c_j G\left(\frac{x}{N^2} + \frac{j}{N^2}\right) = 0 \qquad (x \in R). \tag{13}$$

One may assume that M is minimal for F in equation (13). Write

$$\varphi(z) = \sum_{j=0}^{M} c_j z^j.$$

Then, using the standard notation

$$(Ef)(x) = f(x + 1),$$

one has

$$(\varphi(E)F)(x) = 0 \qquad (x \in R).$$

Let r be any zero of $\varphi(z)$, so that $\varphi(z) = (z - r)\psi(z)$. Define

$$J(x) = (\psi(E)F)(x) \qquad (x \in R).$$

By the minimality of M, $J \neq 0$, and

$$\begin{aligned} J(x + 1) - rJ(x) &= (E - r)J(x) \\ &= (E - r)\psi(E)F(x) \\ &= \varphi(E)F(x) = 0 \qquad (x \in R). \end{aligned}$$

Since $J \in K(X_0, X_0)$ and $J \neq 0$, Lemma 10 yields $r = 1$. Thus all zeroes of $\varphi(z)$ are 1, and

$$\begin{aligned} &\varphi(z) = (z - 1)^M, \\ &(E - 1)^M F(x) = 0 \qquad (x \in R). \end{aligned}$$

Note that $M = 0$ implies $F = G = 0$. Let $P(x)$ be the polynomial of degree $\leqq M - 1$ which agrees with F at $x = 0, 1, 2, \cdots, M - 1$. Then

$$P(0) = F(0)$$
$$(E-1)P(0) = (E-1)F(0),$$
$$\cdots$$
$$(E-1)^{M-1}P(0) = (E-1)^{M-1}F(0).$$

Also, because $\deg P \leqq M - 1$,

$$(E-1)^M P(x) = 0 = (E-1)^M F(x) \qquad (x \in R).$$

Now

$$G_0(x) = (E-1)^{M-1}(P(x) - F(x)) \in K(X_0, X_0)$$

and

$$(E-1)G_0(x) = 0 \qquad (x \in R).$$

By Lemma 10, $G_0(x) = \text{constant} = G_0(0) = 0$. Thus

$$(E-1)^{M-1}P(x) = (E-1)^{M-1}F(x) \qquad (x \in R).$$

Continuing by induction, one obtains

$$(E-1)^{M-j}P(x) = (E-1)^{M-j}F(x) \qquad (x \in R)$$

for $j = 1, 2, \cdots, M$. Thus

$$F(x) = P(x) \qquad (x \in R).$$

Therefore F, hence G, is a polynomial.

Combination of Lemma 11 and Lemma 6 completes the proof of the Theorem.

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