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PAIRS OF SYMMETRIC BILINEAR FORMS IN CHARACTERISTIC 2

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The Grothendieck group of finite-length inner product modules over a PID is here shown to be a sum of countably many copies of the corresponding groups for the residue fields. It follows that nonsingular pairs of symmetric bilinear forms in characteristic 2 owe their extra complexity only to lack of a cancellation theorem: The invariants for isometry in other characteristics continue to determine classes in the Grothendieck group. This is also true for singular pairs.

1. Inner products on finite-length modules. Let R be a principal ideal domain, K its fraction field, and E the R-module K/R. If M is a finite-length R-module, then $M^* = \mathrm{Hom}_R(M, E)$ is abstractly isomorphic to M, and the canonical map $M \to (M^*)^*$ is an isomorphism [1, p. 94-97]. An R-bilinear function $B: M \times M \to E$ induces a homomorphism $m \mapsto B(m, -)$ from M to M^*, and every such homomorphism arises from a unique B. Since M and M^* have the same length, B is nondegenerate iff $M \to M^*$ is an isomorphism. Identifying M^{**} canonically with M, we see that B is symmetric iff $M \to M^*$ is self-adjoint. A nondegenerate symmetric B is called an *inner product* on M.

Clearly the (orthogonal) direct sum of two modules with inner products is again one. The set of isometry classes is thus a commutative semigroup, and we can form the associated Grothendieck group. If 2 is invertible in R, Theorem 1.3 of [6] implies that the semigroup has a cancellation theorem, so passage to the Grothendieck group changes nothing. In residue characteristic 2, the isometry classes are more complicated, but it turns out that the Grothendieck group still has the same structure:

THEOREM 1. *Let R be a principal ideal domain, and suppose that a specific generator p has been chosen for each prime ideal pR. The Grothendieck group of inner products on finite-length R-modules is then canonically isomorphic to*

$$\bigoplus_p \bigoplus_{n=1}^{\infty} WG(R/pR)\ ,$$

where $WG(R/pR)$ is the Grothendieck group of inner product spaces over R/pR.

Proof. The primary components of an inner product module are orthogonal to each other; for if x and y are in different components, $B(x, y)$ is annihilated by powers of two different primes and so vanishes. The semigroup and Grothendieck group thus decompose into direct summands corresponding to the primes. We may therefore restrict our attention to p-primary inner product modules, where p is a fixed prime of R.

Let M now be p-primary. For any submodule N, the orthogonal complement $N^{\perp}$ is by nondegeneracy isomorphic to $(M/N)^*$. Thus the lengths of N and $N^{\perp}$ add up to that of M, and hence $N = N^{\perp\perp}$. In particular, let $M(r) = \{x \in M \mid p^r x = 0\}$. We have $M(r) = (p^r M)^{\perp}$, since $B(x, p^r M) = 0$ iff $B(p^r x, M) = 0$ iff $p^r x = 0$. Consequently $M(r)^{\perp} = p^r M$.

Set $V_r(M) = M(r)/[M(r-1) + pM(r+1)]$; this is the usual roof-level vector space over R/pR whose dimension equals the number of R/p^rR-summands in M. Clearly $B(M(r), M(r))$ lies in the submodule $p^{-r}R/R$ of E. If $B(x, M(r)) \subseteq p^{-r+1}R/R$, then $p^{r-1}x \in M(r)^{\perp} = p^r M$, so $p^{r-1}x = p^r y$ for some y, and x is in $M(r-1) + pM(r+1)$. Hence B induces a nondegenerate symmetric bilinear form on $V_r(M)$ with values in the one-dimensional R/pR-space $p^{-r}R/p^{-r+1}R$. Having chosen a specific p, we have a canonical basis element $[p^{-r}]$ for this space, and so we can treat our form as having values in R/pR. Clearly an orthogonal sum of modules yields orthogonal sums on the V_r. Hence $M \mapsto \{V_r(M)\}$ induces a homomorphism from the Grothendieck group of p-primary inner product modules to $\bigoplus_{r=1}^{\infty} WG(R/pR))$.

Thus far the argument is parallel to that in [6, § 1], where for $\text{char}(R/pR) \neq 2$ it is shown that the inner products $V_r(M)$ determine M up to isometry and can be arbitrarily prescribed. Obviously that implies the present theorem for such p. Henceforth, then, we assume that R/pR has characteristic 2.

Let p^n be the highest order occurring in M, so $M = M(n)$ and $V_n(M) \neq 0$. Let $[e_i]$ be a basis of $V_n(M)$ and set $N = \Sigma Re_i$, a free module over R/p^nR. If we write the values of B in $p^{-n}R/R$ as R/p^nR-multiples of $[p^{-n}]$, the R/p^nR-matrix $B(e_i, e_j)$ is invertible modulo p, since the form on $V_n(M)$ is nondegenerate. Hence the matrix itself is invertible, and B is nondegenerate on N. Consequently $M = N \oplus N^{\perp}$. Clearly $V_r(N) = 0$ and $V_r(N^{\perp}) \simeq V_r(M)$ for $r < n$. By induction, then, M can be written as an orthogonal sum of modules each of which carries a single one of the nonzero V_r-invariants of M.

Suppose now M is free over R/p^nR, and let $V_n(M)$ be the orthogonal sum of subspaces W_1 and W_2. Let $[e_i]$ be a basis of W_1, and set $M_1 = \Sigma Re_i$. As before $M = M_1 \oplus M_1^{\perp}$. The image of

$V_n(M_1)$ in $V_n(M)$ is clearly precisely W_1, and so $V_n(M_1^\perp) = W_1^\perp = W_2$. By induction then any orthogonal decomposition of $V_n(M)$ can be lifted to one of M.

Any inner product space over the field R/pR can be written as an orthogonal sum of one-dimensional summands and hyperbolic planes [3, p. 61]. Hence any M can be written as an orthogonal sum of pieces of the following types: either $M = Re \simeq R/p^nR$ where $B(e, e) = a/p^n$ and $a \not\equiv 0 \mod p$, or $M = Re + Rf \simeq R/p^nR \oplus R/p^nR$ where $B(e, e) = pa/p^n$ and $B(f, f) = pb/p^n$ and $B(e, f) = c/p^n$ with $c \equiv 1 \mod p$. In the second type we can find d with $cd \equiv 1 \mod p^n$ and replace f by df; we get then exactly $B(e, f) = 1/p^n$. It is clear that inner product modules of these types always are indecomposable (since $\operatorname{char}(R/pR) = 2$). Note also that for any n we can obviously construct M with $V_n(M)$ a hyperbolic plane or a prescribed one-dimensional space; thus the homomorphism to $\bigoplus_n WG(R/pR)$ is surjective.

Let M and N now be p-primary inner product modules with isometric V_r-spaces, and write them as orthogonal sums $\bigoplus M_r$ and $\bigoplus N_r$ where the M_r and N_r carry just one invariant each. Fix an isometry $V_r(M_r) \xrightarrow{\sim} V_r(N_r)$, choose an orthogonal decomposition of $V_r(M_r)$ into indecomposables, carry it over to $V_r(N_r)$, and form the corresponding decompositions of M_r and N_r. In this way we get M and N written as sums of indecomposable pieces with the same invariants. The proof will then be complete if we show that indecomposables with the same invariant are equivalent in the Grothendieck group.

Take first $M = Re$ with $B(e, e) = a/p^n$ and $M' = Re'$ with $B(e', e') = a'/p^n$, where we assume $a \equiv a' \mod p$. We show by induction on n that M and M' are equivalent in the Grothendieck group. If $n = 1$ or $a \equiv a' \mod p^n$, the result is trivial, so suppose $a' - a = p^r b$ with $b \not\equiv 0 \mod p$ and $1 \leqq r < n$. Let $N = Rf$ be cyclic of order p^{n-r} with $B(f, f) = b/p^{n-r}$. Inside the orthogonal sum $M \oplus N$ consider $e + f$ and $up^r e + f$, which for any u in R form a new basic generating set. We have $B(e + f,\ e + f) = (a/p^n) + (b/p^{n-r}) = a'/p^n$, so $e + f$ spans a copy of M'. The cross term $B(e + f,\ up^r e + f) = up^r a/p^n + b/p^{n-r}$ will vanish if we choose u so that $ua \equiv -b \mod p^{n-r}$; as $a \not\equiv 0 \mod p$ we can do this. The other summand has $B(up^r e + f, up^r e + f) = (u^2 p^r a + b)/p^{n-r}$. As $r \geqq 1$, we have $u^2 p^r a + b \equiv b \mod p$, so by induction this summand is equivalent to N in the Grothendieck group; hence the complementary summands M and M' are equivalent.

Now take $M = M(a, b) = Re + Rf$ with $B(e, f) = 1/p^n$ and $B(e,e) = pa/p^n$ and $B(f, f) = pb/p^n$. We show that $M(a, b)$ is equivalent to $M(a, 0)$ in the Grothendieck group. The same argument then shows that $M(a, 0) \simeq M(0, a)$ is equivalent to $M(0, 0)$, so any two indecomposables of this type will be equivalent. If $b \equiv 0 \bmod p^{n-1}$ the result is trivial, so suppose $b = -p^{r-1}c$ with $c \not\equiv 0 \bmod p$ and $1 \leqq r < n$. Let $N = Re' \sim R/p^{n-r}R$ with $B(e', e') = c/p^{n-r}$. Inside the orthogonal sum $M(a, b) \oplus N$ we can for any u in R get a new basic generating set e, $f + e'$, and $g = p^r u(e - paf) + e'$. Obviously $B(e, e) = pa/p^n$ still, and $B(e, f + e') = 1/p^n$, while $B(f + e', f + e') = pb/p^n + c/p^{n-r} = 0$. Thus e and $f + e'$ span a copy of $M(a, 0)$. We have $B(e, g) = 0$ and $B(f + e', g) = [u(1 - p^2ab) + c]/p^{n-r}$; since $1 - p^2ab \equiv 1 \bmod p$, we can choose u to make the numerator $\equiv 0 \bmod p^{n-r}$, and the sum will then be orthogonal. Finally, we compute $B(g, g) = [-u^2ap^{r+1} + u^2a^2bp^{r+3} + c]/p^{n-r}$. The numerator is $\equiv c \bmod p$, so by the previous case this summand is equivalent to N in the Grothendieck group. Hence the other summand $M(a, 0)$ is equivalent to $M(a, b)$.

Porism 2. The indecomposable finite-length inner product modules are: cyclic $M = Re \simeq R/p^nR$ with $B(e, e) = a/p^n$ and $a \not\equiv 0 \bmod p$, and also (when char $(R/pR) = 2$) modules $M = Re + Rf \simeq (R/p^nR)^2$ with $B(e, f) = 1/p^n$ and $B(e, e) = a/p^{n-1}$ and $B(f, f) = b/p^{n-1}$.

REMARKS. (i) As in [6, 1.4] it is straightforward to check what happens under extension from R to a larger principal ideal domain S. If q is a prime factor of p in S, write $p = sq^e$ with $0 \neq \bar{s}$ in S/qS. Then if M has a p-invariant V_r, the extended inner product module $M \otimes_R S$ will have a q-invariant V_{er} given by $\bar{s}^r(V_r \otimes_{R/pR} S/qS)$. The scaling factor $\bar{s}^r$ occurs only because the multiplication identifying $p^{-r}R/p^{-r+1}R$ with R/pR depends on p; if we allow the invariants to be forms with values in $p^{-r}R/p^{-r+1}R$, as in [7], the construction of the V_r commutes precisely with base extension.

(ii) Karoubi in [2] studies a structure similar to that involved here, though he deals mainly with quadratic forms rather than inner products. The equivalence relation appropriate for his purpose is also much weaker than the one here. Its analogue in our situation would set M equivalent to $N^{\perp}/N$ whenever N is a submodule with $N \subseteq N^{\perp}$; this would collapse the infinite sum $\bigoplus_n WG(R/pR)$ to a single copy of the Witt group of R/pR.

(iii) A classification up to isometry will necessarily be more complicated when char $(R/pR) = 2$. Perhaps one can be found along the lines of that derived by Riehm in a related situation [4].

2. Nonsingular pairs. Let k be a field of characteristic 2, and R the principal ideal domain $k[\lambda]$. Let $T\colon E = k(\lambda)/k[\lambda] \to k$ denote the negative of the residue at ∞; that is, given f in $k(\lambda)$, expand it in a Laurent series in $1/\lambda$ and let $T(f)$ be the coefficient of $1/\lambda$.

LEMMA 3. *Let M and N be finite-length R-modules. Composition with T is a bijection from the space of R-bilinear $C: M\times N \to E$ to the space of k-bilinear $B: M \times N \to k$ satisfying $B(\lambda m, n) = B(m, \lambda n)$.*

Proof. Clearly $C \mapsto T\circ C$ is a k-linear map of one space into the other. If C is nonzero, $C(M, N)$ is a nonzero submodule of E and so contains some minimal submodule $p^{-1}R/R$; it is easy to see T is nonzero on $p^{-1}R/R$, so $T\circ C \neq 0$. To show now that the injection is bijective, we count dimensions. Formation of the space commutes with R-direct sums, so we may assume $M = Re \simeq R/p^rR$ and $N = Rf \simeq R/q^sR$. Trivially B, like C, must be zero if $p \neq q$; so suppose $p = q$ and $r \geqq s$. The possible C, given by $C(e, f) = a/p^s$ for arbitrary a, form a space of dimension $= \deg(p^s)$. But the adjointness condition shows the possible B are determined by the values $B(e, \lambda^j f)$ for $0 \leqq j < \deg(p^s)$.

In this correspondence, $C: M \times M \to E$ is an inner product iff $B = T\circ C$ is an inner product. Indeed, the transpose of C goes to the transpose of B, so by bijectivity C is symmetric iff B is; and each $C(m, M)$ is a submodule of E, so it would vanish if $T\circ C(m, M) = 0$. Now an R-module of finite length is a finite-dimensional k-space V and a prescribed map $S: V \to V$ giving the action of λ. If C is an inner product, S is self-adjoint for the inner product B, and conversely. Then also $A(v, w) = B(Sv, w)$ is another symmetric bilinear form on V; and since B is nondegenerate, every such form arises from a unique self-adjoint S. Thus:

PROPOSITION 4. *There are natural equivalences between*

(a) *pairs (A, B) of symmetric bilinear forms over k with B an inner product;*

(b) *pairs (S, B) with B an inner product and S self-adjoint with respect to B; and*

(c) *finite-length inner product modules over $k[\lambda]$.*

The results of § 1 now apply to pairs of forms. We can in fact push the application a bit farther: let us say that (A, B) is *nonsingular* if A-λB is nondegenerate over $k(\lambda)$. The proof of [6, 4.1] remains valid in characteristic 2 and shows that a nonsingular pair (A, B) can be canonically decomposed into one summand where B is nondegenerate and another where A is nondegenerate and the map giving B in terms of A is nilpotent. The summand with B degenerate we associate with the place $\lambda = \infty$ of $k(\lambda)$; Theorem 1 applies to it also once A and B are interchanged.

THEOREM 5. *Let k be a field. The Grothendieck group of nonsingular pairs of symmetric bilinear forms over k is naturally isomorphic to*

$$\bigoplus_p \bigoplus_{n=1}^{\infty} WG(\kappa(p)) ,$$

where p runs over all the k-places of $k(\lambda)$ and $WG(\kappa(p))$ is the Grothendieck group of inner product spaces over the residue field at p.

REMARKS. (i) In Theorem 1 we needed to choose specific generators of the prime ideals; here that can be done canonically by taking the monic irreducible polynomials (and $1/\lambda$ at ∞).

(ii) The behavior of the Grothendieck group under extension of k is as described in the remarks after Theorem 1.

(iii) The theorem is true in other characteristics [6, 4.2], but passage to the Grothendieck group is essential only when $\mathrm{char}(k)=2$.

(iv) The structure of inner products over k for $\mathrm{char}(k) = 2$ is studied in [3]. We might note that if k is perfect, any two inner products of the same rank are equivalent in $WG(k)$. The finite extensions $\kappa(p)$ are perfect if k is, so in that case two pairs of inner products are equivalent whenever the self-adjoint maps S are similar.

We now show that equivalence indeed cannot be strengthened to isometry; this is true even over perfect fields, and does not depend on the failure of cancellation for single inner products.

PROPOSITION 6. *Let k be a field,* $\mathrm{char}(k) = 2$. *Then there are two indecomposable pairs of inner products over k which are equivalent in the Grothendieck group but not isometric.*

Proof. Let $p = \lambda - 1$. Let $M_0 = Re + Rf \simeq (R/p^3R)^2$ with $C(e, e) = 0 = C(f, f)$ and $C(e, f) = 1/p^3$. Let M_1 be similarly defined with $C(e, e) = 1/p^2 = C(f, f)$ and $C(e, f) = 1/p^3$. Each of these has as single invariant V_3 at p a hyperbolic plane, so the two are equivalent. Suppose there were a new basis $e' = re + sf$ and $f' = te + uf$ for M_1 giving an isometry to M_0. Then $0 = C(e', e') = (r^2 + s^2)/p^2$, so $r + s \equiv 0 \bmod p$. Similarly $0 = C(f', f')$ gives $t + u \equiv 0 \bmod p$. But then $ru - st \equiv ru - ru \equiv 0 \bmod p$, which is impossible since $ru - st$ is the determinant of the base change.

We can get a little more from this example. In M_0 clearly $B = T \circ C$ satisfies $B(x, x) = 0$, i.e., is alternating. In M_1 we have $C(re + sf,\ re + sf) = (r^2 + s^2)/p^2$; here $r^2 + s^2$ has only even-degree terms, and the same is true of its remainder when divided by $p^2 = \lambda^2 + 1$. Thus the values in E all have residue zero, and B again is alternating. Any two alternating forms of the same rank are isometric, so there is a map $M_0 \to M_1$ preserving B. Furthermore, the maps S_0, S_1 have the same invariant factors. Thus:

COROLLARY 7. *Let k be a field,* $\operatorname{char}(k) = 2$. *Then one can find a nondegenerate alternating bilinear form over k and two self-adjoint linear maps which are conjugate in the general linear group but not conjugate by a symplectic matrix.*

This complexity no longer exists, however, if one restricts attention to pairs of forms where *both* forms are alternating. Indeed, it is trivial to compute that the only indecomposable pairs (given by Porism 2) with this property are of the second type with $a = b = 0$, and so such pairs are determined by their module structure. The corresponding result for singular pairs—determination by the "Kronecker module" structure—is also true and drops out from the theorems in the next section. These results for pairs of alternating forms were recently proved in a different way by R. Scharlau [5].

Scharlau's result is proved in all characteristics; hence the phenomenon described in Corollary 7 cannot occur when $\operatorname{char}(k) \neq 2$, since then all skew-symmetric forms are alternating. This also, of course, can be proved by the methods used here. If B is skew-symmetric in Lemma 3, so is C; and then C is alternating if $1/2 \in R$. The decomposition arguments at the beginning of Theorem 1 show (with no hypothesis on characteristics) that a finite-length module with nondegenerate alternating form is a sum of pieces $Re + Rf = (R/p^nR)^2$ with $C(e, f) = 1/p^n$.

3. Singular pairs. We complete our study by allowing the possibility of singularity in A-λB. Throughout this section k will be a field of characteristic 2.

DEFINITION. Let $c_0, \cdots, c_{2m-1}$ be elements of k. A *basic singular pair* of symmetric bilinear forms over k is a $(2m + 1)$-dimensional space V having a basis $v_0, \cdots, v_m, w_0, \cdots, w_{m-1}$ and forms A, B given by

$$\begin{aligned} B(v_i, v_j) &= 0 & A(v_i, v_j) &= 0 \\ B(v_i, w_j) &= \delta_{ij} & A(v_i, w_j) &= \delta_{i,j+1} \\ B(w_i, w_j) &= c_{i+j} & A(w_i, w_j) &= c_{i+j+1}\,. \end{aligned}$$

PROPOSITION 8. (i) *A basic singular pair is singular and indecomposable.*

(ii) *Two basic singular pairs are isometric iff they have the same dimension and multiplication by a fixed nonzero square in k takes the constants of one to the constants of the other.*

(iii) *A basic singular pair has a basis $v_0 \cdots v_m, x_0 \cdots x_{m-1}$ satisfying*

$$\begin{aligned} B(v_i, v_j) &= 0 & A(v_i, v_j) &= 0 \\ B(v_i, x_j) &= \delta_{ij} & A(v_i, x_j) &= \delta_{i,j+1} \\ B(x_i, x_j) &= c_{2i}\delta_{ij} & A(x_i, x_j) &= c_{2i+1}\delta_{ij} . \end{aligned}$$

Proof. (i) It is easy to compute that $x \mapsto (A - \lambda B)(x, -)$ sends $\Sigma v_i \lambda^i$ to zero and $v_1 \cdots v_m, w_0 \cdots w_{m-1}$ to independent elements of the dual space; thus $\Sigma v_i \lambda^i$ is a basis for the radical of $A - \lambda B$ on $V \otimes k[\lambda]$. If we split V into an orthogonal sum, the rank one radical of $A - \lambda B$ must be contained in one piece, which then contains all the v_i; nothing outside $\Sigma k v_i$ is orthogonal to them all, so the other piece vanishes.

(ii) Replacing v_i, w_i by av_i and $(1/a)w_i$ changes c_i to $(1/a)^2 c_i$. Conversely, suppose φ: $V \rightarrow V'$ is an isometry. The generator of a rank one free module over $k[\lambda]$ is unique up to nonzero elements of k, so for some $0 \neq a$ we have $(\varphi \otimes \mathrm{id})(\Sigma v_i \lambda^i) = a \Sigma v'_i \lambda^i$; that is, $\varphi(v_i) = av'_i$. Then $\delta_{ij} = B(v_i, w_j) = B'(\varphi v_i, \varphi w_j) = B'(av'_i, \varphi w_j)$, and hence $\varphi(w_j)$ contains no w'-term except $(1/a)w'_j$. Write $\varphi(w_j) = (1/a)w'_j + \Sigma d_{jr} v'_r$. Then since $\operatorname{char}(k) = 2$ we get $c_{2j} = B(w_j, w_j) = (1/a)^2 B'(w'_j, w'_j) + \Sigma d_{jr}^2 B'(v'_r, v'_r) = (1/a)^2 c'_{2j}$; similarly $c_{2j+1} = A(w_j, w_j) = (1/a)^2 c'_{2j+1}$.

(iii) Set $x_i = w_i + \Sigma d_{ir} v_r$, where we take $d_{ir} = c_{i+r}$ if $i < r$ and $d_{ir} = 0$ otherwise.

In other characteristics it is possible to make all $c_i = 0$. Here the alternating basic singular pairs have that form, but there are others of the same dimension (and same "Kronecker module" type). But the proof given in [6, 3.1] for the decomposition theorem remains valid up to its last step; together with the uniqueness proof in [6, 3.3], it yields the following result.

THEOREM 9. *Any pair can be written as an orthogonal sum of a nonsingular pair and a collection of basic singular pairs. The nonsingular pair is unique up to isometry, and the number of basic singular summands of each dimension is uniquely determined.*

The decomposition is not unique, but here again the extra complexity of characteristic 2 cancels out in the Grothendieck group.

PROPOSITION 10. *Let V be a basic singular pair. The sum of two copies of V is isometric to the sum of one copy of V and one alternating basic singular pair.*

Proof. Let $v_0 \cdots w_{m-1}$ be the standard basis for one copy of V, and $v'_0 \cdots w'_{m-1}$ that for the other. Inside the direct sum, let Y be the subspace with basis elements $\{w_i\}$ and $\{v_i + v'_i\}$; this clearly is isometric to V. Let Y' be the subspace with basis elements $\{v'_i\}$ and $\{w_i + w'_i + \Sigma c_{i+r} v_r\}$. Straightforward computation shows that Y' is orthogonal to Y and is a basic singular pair with constants vanishing.

Combining these results, we get our final conclusion.

THEOREM 11. *The Grothendieck group of pairs of symmetric bilinear forms is naturally the direct sum of the group for nonsingular pairs and that for totally singular pairs. The latter is free abelian with one generator for each integer $m \geqq 0$, namely, the class of all basic singular pairs of dimension $2m + 1$.*

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